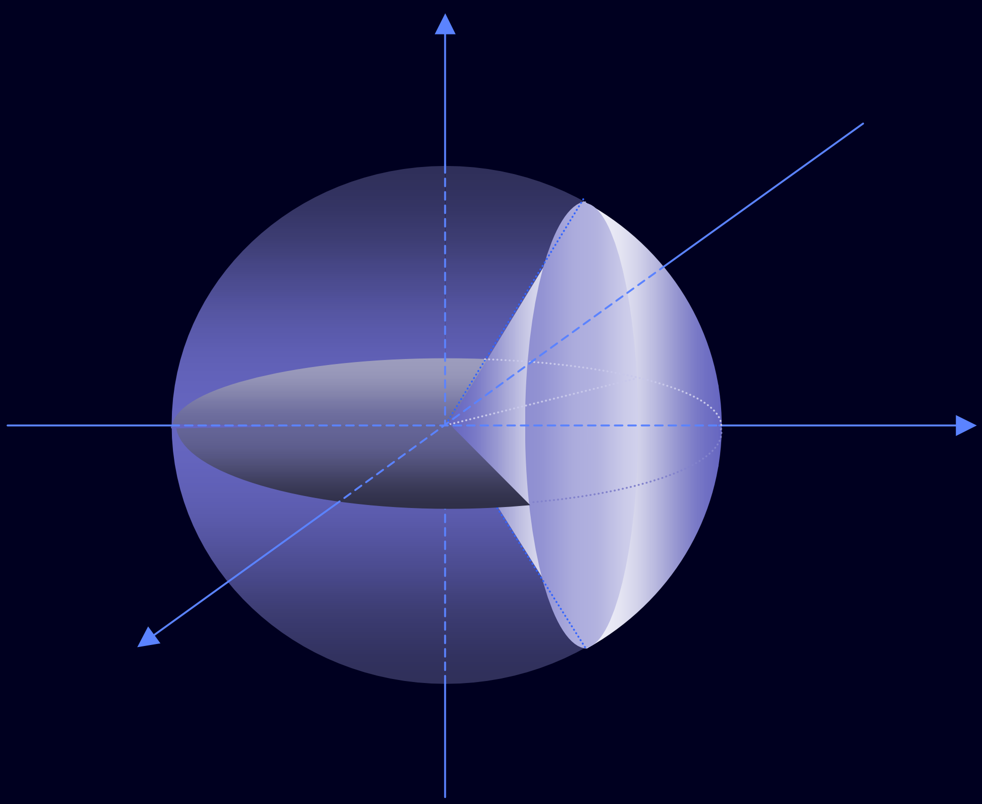
기본 함수

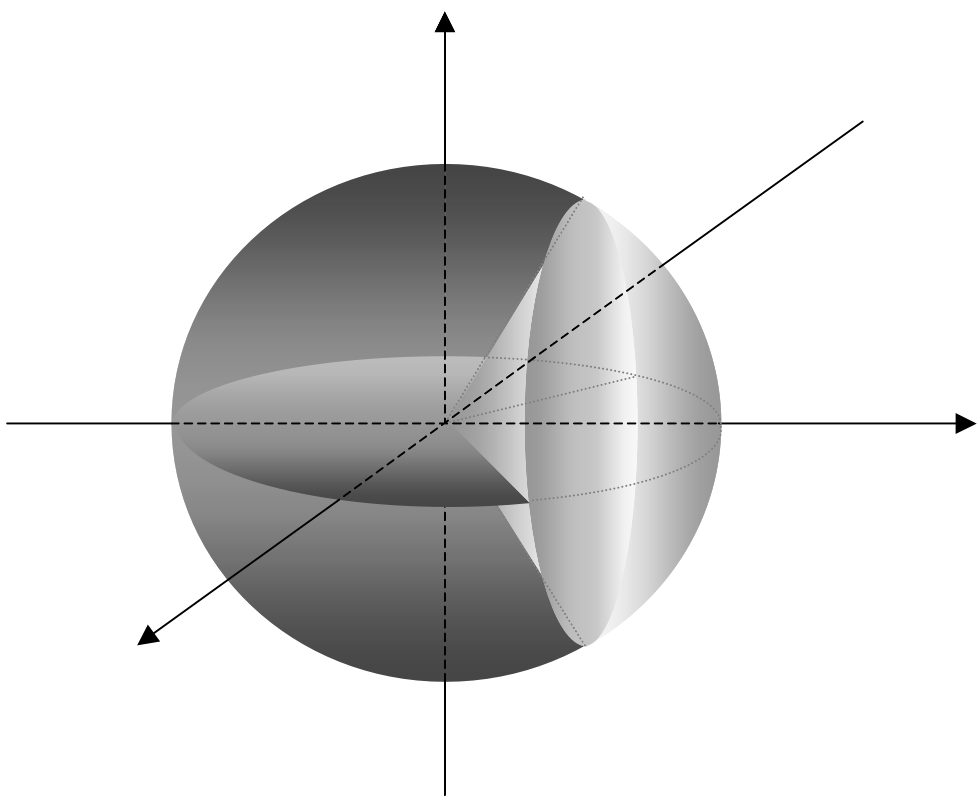
Basic Functions



김성렬

Seong R. Kim

Basic Functions



Seong R. KIM

Dear students,

Students need the best teacher, so you need examples, because examples are the best teacher. This is thus, a book of examples. All the examples here are fully worked, and explain **how** the basic but essential tools in math are made, together with **what** they are, **how** they work, and **how** to work with them. Such tools include numbers, formulas, identities, equations, laws, etc.

Doing math, we work with ideas and run ideas, because every thing in math is an idea. What idea though?

It's a concept. So we need to understand it if we want to use it. A number is an idea, for instance, and the same is true for a line or circle, too. And putting ideas together, we can build another, which becomes the base or an element of another, and each is connected. And that's the way your math grows.

So you get to build a circuit, and sometimes, need to fill the gap or repair the circuit so that you can get the sense of it. Your calculation thus, runs properly, and gets to the solution.

Getting the concept doing good examples, and gaining the fluency doing right examples, you can build the circuit strong.

This book is however, nothing but a bunch of examples until you get it powered. How then to get it powered, and make it run and work for you?

Read it closely at your pace, and then, do each example yourself. It is quite important to note that you do it in **your** own writing. Just watching someone doing it, you just only feel that you can do it, too, but can hardly do it when you need it. You do it, you get it. And you've got to do it over with different examples. It's a cliché, of course, but is always true that knowing is one thing and doing is another.

I've been helping students get conceptual understanding and fluency so that they can grow, take care of, and run their own math. The area covers the basics in algebra and geometry, and includes equations for unknowns and those for curves, functions and their graphs, and other math tools, which are the basic elements in calculus, which's been the core of my interest from my early age in high school.

Of my students, some are quite poor in math, and thus, are afraid of or even hate math, some require special education because of exceptional intelligence, some are smart enough, some are naïve and diligent, and many are clever but lazy. All the students are badly after though, one thing in common.

A Strong Foundation in Algebra

It is of course, the prime objective of my work, and I’m always happy and eager to help them achieve it. I can only help though. Your math grows as much as you grow and take care of it yourself. So does mine. What math then do students most often do or use in high schools or colleges?

Algebra and geometry.   What algebra then?

Elementary algebra, of course. Doing the algebra, you should be able to not only work with, but manipulate, that is, change, alter, and convert math expressions the way you want, so for instance, you can break apart and factor polynomials in many kinds. Also, you can handle properly ratios and rates, trigonometry, factors or divisors, constants and variables, powers and logarithms or exponents, equations and functions, along with their graphs, equalities and inequalities, identities and formulas, laws and rules, etc.

And if you want to handle them properly, you want to know their natures and how they mingle with each other.

So studying those stated above, you want to know **what** they are, **how** they work, and **how** to work with them or **what** to do with them. What then about the geometry?

Basically, the geometry is about shapes, and has much to do with positions and angles in algebra. Why algebra?

You often need to compute lengths, areas, or volumes in many situations. The shapes begin with points, lines, triangles, and circles, and move on to rectangles, squares, parallelograms or rhombuses, trapezoids, tetragons, other polygons as pentagons, polyhedrons as hexahedrons, etc.

Doing the geometry, too, we need to do the algebra stated above such as trigonometry. So it is analytic geometry, often called coordinate geometry, too. And doing it, we can specify positions using coordinates. So in that geometry, basically, we work with graphs. Putting a math idea in a graph, we can not only effectively think about it, but actually see it, too, and therefore, can not just easily, but efficiently work with it, too. What idea then is it?

The idea begins with a point, a line, a parabola, a circle, an ellipse, and a hyperbola, called a conic section or basic curve, and then, moves on to other curves, planes, surfaces, volumes, and other objects in various dimensional spaces, together with vectors. And using an angle, we can specify and work with an amount of turning or change in direction.

So learning, using, or applying those ideas or tools in math, we get to solve problems. And this book can help. It can help you build and run your own math engine in your brain, and thus, can help achieve your solid math skill.

It is however, not a magic book giving you a math skill of high caliber overnight. And it’s just a book of examples, and can have many mistakes. I’m a human, too. There’s no mojo making a math genius, and math is science, and is full of facts and ideas. And it is after all, not me, not your teacher, and not any book but you who actually make you put together some of those facts and ideas, and understand it.

Putting facts and ideas together, understanding it, and taking care of what you have learned doing examples, you grow your math. And this book is ready to help and can help you help yourself.

Solving a problem, we put it many different ways. For instance, while expanding or reducing it, or modifying or converting it, we keep searching for the solution, keep approaching the solution, and eventually, can get there.

It is often the case a problem itself is the solution. We can put a problem in many different ways, and eventually, can end up with the solution. How come then is the solution no other than the problem?

For instance, the solution to 3232 ÷ 101 is 32. And we can put it this way:

$3232 \div 101 = \frac{3232}{101} = \frac{32 \text{ x } 101}{101} = \frac{32}{1} = 32 \Rightarrow 3232 \div 101 = 32.$

And we can get this, too: 32 = 3232 ÷ 101. How?

$32 = \frac{32}{1} = \frac{32 \text{ x } 101}{101} = \frac{3232}{101} =$ 3232/101 = 3232 ÷ 101.

Too easy?

For another instance, the solution to $\boldsymbol{ax^2 + bx + c = 0}$ is

$\boldsymbol{x} = \frac{-\boldsymbol{b} \pm \sqrt{\boldsymbol{b}^2 - 4\boldsymbol{ac}}}{2\boldsymbol{a}}$, which is called the quadratic formula.

How come then is the solution no other than the problem?

We can put it this way:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow 2ax = -b \pm \sqrt{b^2 - 4ac}$$

$$\Rightarrow 2ax + b = \pm\sqrt{b^2 - 4ac} \Rightarrow (2ax + b)^2 = b^2 - 4ac$$

$$\Rightarrow 4a^2x^2 + 4abx + b^2 = b^2 - 4ac \Rightarrow 4a^2x^2 + 4abx = -4ac$$

$$\Rightarrow ax^2 + bx = -c \Rightarrow ax^2 + bx + c = 0.$$

And we can get this, too:

$$ax^2 + bx + c = 0 \quad \Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{How?}$$

$$ax^2 + bx + c = a(x^2 + \frac{b}{a}x) + c = a(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}) + c$$

$$= a(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}) - \frac{b^2}{4a} + c = a(x + \frac{b}{2a})^2 - \frac{b^2 - 4ac}{4a} = 0$$

$$\Rightarrow a(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a} \Rightarrow (x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\Rightarrow x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \Rightarrow x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

So when solving a problem, we expand or reduce it, modify or convert it, and keep searching for and approaching the solution. Then eventually, we can get there. What then do we do when doing the set of processes above?

We do algebra. You need a strong foundation in algebra. So if a problem is well defined, that is, if it makes sense, we should be able to get it solved the way as follows.

A problem $\Rightarrow$ **…** $\Rightarrow$ **…** $\Rightarrow$ **the solution**,

and thus, **the problem** $\Rightarrow$ **the solution**.

So solving a problem, we put it many different ways so that we can get to the solution. And that's the way math runs.

Many more examples are available for free, and you can get them downloaded at **www.runmath.com**. So visit the website, get them downloaded, and do the examples. The website is getting updated weekly or monthly.

May your math run very well.

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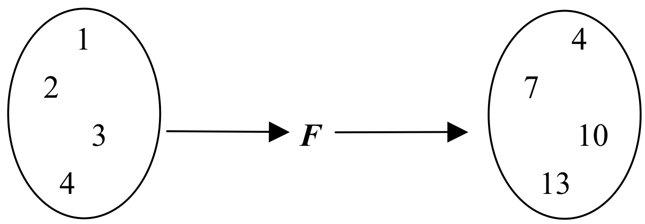
Contents

0.0. What is a function?



Though it looks like an equation, it is not. What then is it?

It's not just a math expression. We can call it an expression that creates a *correlation between data sets*. What data sets though?

The functions covered in this book are basic ones, which are thus, covered in high school math.
So such data sets are simple sets of numbers, usually real numbers.
And in elementary level as high school math, we normally use two sets of numbers.

Usually, we use as a function a math expression such as $\mathbf{2x}$, $\boldsymbol{x^2 + x + 2}$, $\boldsymbol{3^{x+2}}$, $\mathbf{log}\ \boldsymbol{x}$, $\mathbf{sin}\ \boldsymbol{x^2}$, etc. It can also, be as simple as 1, 3, -2, or 0, since a number is an expression, too.
How then does it create a correlation?

We can put a function this way, too:
A function is an expression *acting on a number set to produce another number set*.
How does a function then act on a number set?

A number from the set acted on is put into the variable in the function's expression, so a number gets produced. And the number produced belongs to the other set.
Suppose for instance, $\boldsymbol{X}$ is the set acted on, and is a set $\{1, 2, 3, 4\}$, the function is $\mathbf{3x + 1}$, and $\boldsymbol{Y}$ is the set produced, and is a set $\{4, 7, 10, 13\}$.

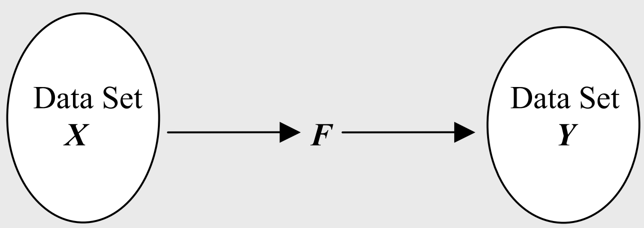
Then, if for instance, 1 from $\boldsymbol{X}$ is put into the expression $\mathbf{3x + 1}$, we get $3 \cdot 1 + 1 = 4$, so 4 gets produced. Thus, 4 belongs to $\boldsymbol{Y}$.
And the same is true, also, for all the other numbers in both sets.

That is to say that a function connects two number sets.
One is the set acted on, and the other is the set of all the numbers produced.

A function is therefore, not just an expression, but creates a *system*, where two sets of numbers are connected by the function, which is an expression as $\mathbf{2x}$ or $\boldsymbol{x} + \mathbf{1}$.

In short, a function is an expression connecting two number sets.
And assuming $\boldsymbol{F}$ is a function, we can put it the way as follows.

Fig. 0



Through the function $\boldsymbol{F}$, the data set $\boldsymbol{X}$ gets connected to the data set $\boldsymbol{Y}$.

Such two data sets are two sets of numbers. So a function creates a correlation between two number sets. What correlation, though?

A function creates a correlation by setting up two relationships.
One is a *dependence* relationship between the two sets.
And the other is a *mechanical* relationship between the numbers in one set and those in the other set. What mechanical relationship?

Basically, the mechanical relationship goes the way as follows.

A pair of two numbers gets generated *at a time*. And the two numbers in each pair get connected. So number-pairs keep getting made as the function keeps running.

In each pair, one is from one set, and the other belongs to the other set. And the two numbers get connected by the function, because the function uses one to make the other.

That is, the function connects two numbers by making a number-pair as (2, 7).

And a function can generate a sequence of pairs of numbers.
That is, a function can be taken as a machine generating pairs of numbers in a sequence.

And in high school math and calculus, all the numbers in both sets have to get connected. Thus, the function keeps running until all the numbers get connected. So?

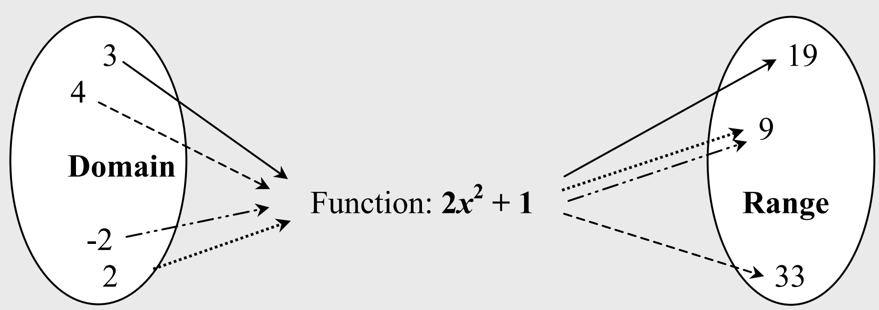
So eventually, the function connects the two sets.
What then about the dependence relationship?

Both sets can be the same or different. One can be bigger or smaller than the other. And it is often the case, both are identical. In nature though, the two sets are always different. How then are the two different?

One of the two is called a *domain*, and the other is called a *range*.
And the range is dependent upon the domain. How is it dependent, though?

We know a function is an expression acting on a number set, and producing another set of numbers. The *set acted on* is the *domain*, and the *set produced* is the *range*.

Fig. 1



A number called an *input* from the domain is put into the variable in the function, which therefore, produces a number called an *output*, which belongs thus, to the range.
And the same is true, also, for all the other numbers in both sets.

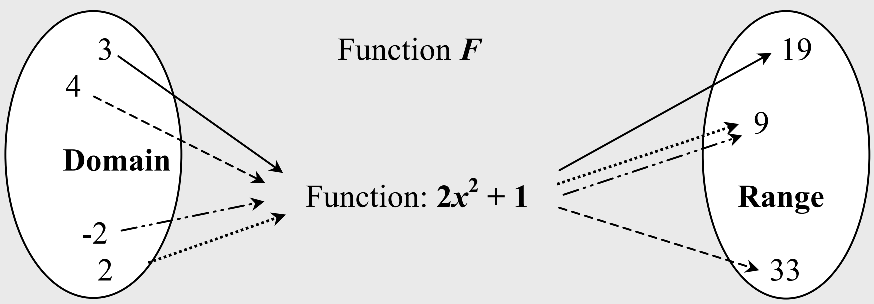
So a number in the range can get determined by a number in the domain.
And the same is true, also, for all the other numbers in the domain and the range.

That is, if no number is chosen in the domain, no number can be determined in the range, and if a number is chosen in the domain, one number has to be determined in the range.

So each number in the range is dependent upon at least one number in the domain. In sum, the range depends on the domain. So no domain, then no range.

And for the same reason, we can say that the domain causes the range. And each number in the range can be said to *correspond to* at least one number in the domain.

Fig. 2



So we can say that the numbers in the range correspond respectively to the numbers in the domain. And in the system that has the function ***F*** above, we can say

that 19 in the range corresponds to 3 in the domain,

that -2 gets chosen in the domain, enters the function, the function runs, then 9 gets determined in the range. And the same is true for 2 in the domain, too.

and also, that 4 in the domain is connected to 33 in the range, that is, the two get connected by means of the function $\mathbf{2x^2 + 1}$.

In sum, the domain and the range get connected via the function ***F***, which is $\mathbf{2x^2 + 1}$. And as others do in math, a function has its name.

Naming a function, we usually use two letters, together with brackets or parentheses. One is called a function *designator* (another word for a name), and the other is the variable used in the expression.

So for instance, if a function is called ***f***, and the expression is $\mathbf{2x + 1}$, we can name the function this way: $\boldsymbol{f(x)}$, read as ***f*** of ***x***.

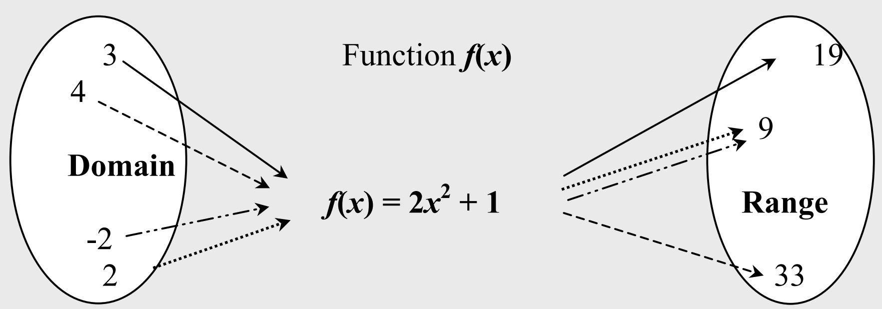
Then, we can set $\boldsymbol{f(x) = 2x + 1}$, which is the form we often use specifying a function.

And in $\boldsymbol{f(x)}$, we call $\boldsymbol{x}$ an *input variable*.

That's because it gets each of all the numbers in the domain, so each number is put *into* the variable $\boldsymbol{x}$. And for the same reason, each number in the domain is called *an input value*, often just called *an input* for short.

And for instance, putting an input 3 into $\boldsymbol{x}$ in the function $\boldsymbol{f}$, we do it this way: $\boldsymbol{f(3)}$, read as $\boldsymbol{f}$ of 3.
Then, the function $\boldsymbol{f}$ produces the output this way: $\boldsymbol{f(3) = 2{\cdot}3 + 1 = 6 + 1 = 7}$.

Fig. 3



And naming (or specifying) a function more specifically, we can use another variable that can get the value of the function. What value?

It is each and every number the function produces, and thus, is each and every output.

So for instance, if using $\boldsymbol{y}$ as the variable that gets the value of the function $\boldsymbol{f(x)}$, we can name the function $\boldsymbol{f}$ this way, too: $\boldsymbol{y = f(x)}$, and can specify the function $\boldsymbol{f}$ the way as follows. $\boldsymbol{y = f(x) = 2x + 1}$, which is a bit more specific than this: $\boldsymbol{f(x) = 2x + 1}$.

And we call $\boldsymbol{y}$ an *output variable*.
That's because it gets all the numbers (values) the function produces, and those numbers look getting *out* of the function.
And for the same reason, each of those numbers is called *an output value*, often just called *an output* for short. And of course, the output variable gets one output at a time, and the output belongs to the range.

So what is a function?

A function is a math expression connecting two number sets called a domain and a range. Connecting the two sets, the function generates a sequence of pairs of numbers as (0, 1), (3, 7), (8, 17), etc.

And for instance, in a pair (8, 17), 8 is called an input, and 17 is called the output for the input 8. In short, in (8, 17), 8 is an input, and 17 is an output.

And putting an input 8 into $\boldsymbol{x}$ in the function $\boldsymbol{y = f(x) = 2x + 1}$, we do it this way: $\boldsymbol{f(8)}$.

Then, the function $\boldsymbol{f}$ produces the output this way: $\boldsymbol{f(8) = 2 \cdot 8 + 1 = 16 + 1 = 17}$.
So in this case, we can call 17 the output for the input 8.

And the same is true for $\boldsymbol{f(8)}$, too, since we have $\boldsymbol{f(8) = 17}$.
So we can call $\boldsymbol{f(8)}$ the output for the input 8, too.

And we often say that the value of the function $\boldsymbol{f}$ is 17 when $\boldsymbol{x = 8}$, or that the value of the function $\boldsymbol{f}$ is 17 for $\boldsymbol{x = 8}$.

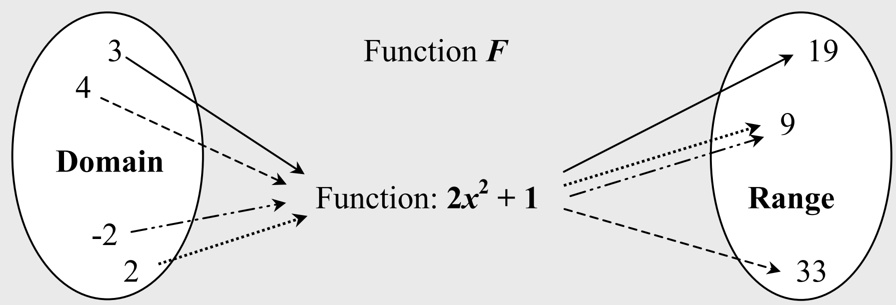
And we have $\boldsymbol{y = f(x)}$.

So the output variable $\boldsymbol{y}$ is said to get the value of $\boldsymbol{f}$ for each value of $\boldsymbol{x}$.

Thus, we can say that $\boldsymbol{y = 17}$ when $\boldsymbol{x = 8}$, that $\boldsymbol{y = 17}$ if $\boldsymbol{x = 8}$, or that $\boldsymbol{y = 17}$ for $\boldsymbol{x = 8}$.

0.1. How Functions Work 1

Fig. 0



So what is a function?

A function is an expression acting on a number set called a domain, and producing another set of numbers called a range.

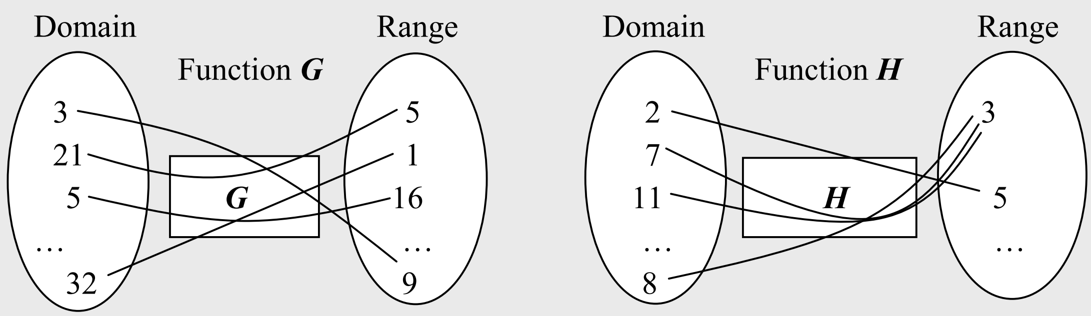
In short, a function is an expression that makes a range using a domain.

And we can notice that in the example above, some number in the range can correspond to more than one number in the domain. One number at a time though.

Each number in a range corresponds to *at least one* number in the domain.

Therefore, for some functions, one number in the range can correspond to many numbers or even all the numbers in the domain. So for instance, we can have the functions ***G*** and ***H*** in Fig. 1.

Fig. 1



That is because two or more numbers in the domain can cause the same number in the range. What then, about the numbers in the range?

As mentioned above, in the range, one number can correspond to many numbers in the domain. And if the range has one number only, that only number has to correspond to all the numbers in the domain.

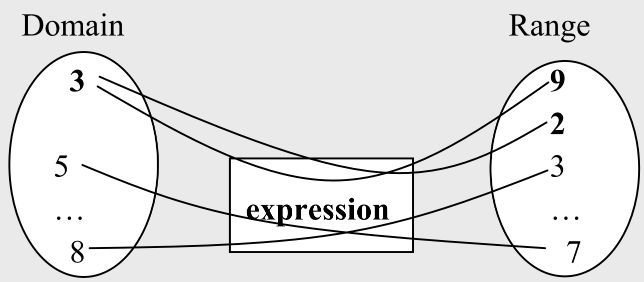
Note however, for any function, two or more numbers in a range cannot correspond to the same number in a domain.

In other words, ***no number*** in the domain ***can cause two or more*** numbers in the range.

So no matter what function it may be, it has to produce one number only at a time.

For instance, we do ***not*** have a function as in Fig. 9. If it were a function, the two numbers 9 and 2 in the range wouldn't be able to correspond to the number 3 in the domain.

Fig. 2 No function can do this:



Let's now, get back to the mechanical relationship.

We know that a machine basically repeats the same work.
A function is an expression made of math operations.

Every time a number is put into a function, the same group of operations repeats, and a number is produced.
So a function can be taken for a machine that performs a task over and over.

The task is a group of math operations as the ones in $\boldsymbol{x^2 - x + 1}$.
What are the operations though, in $\boldsymbol{x^2 - x + 1}$?

We can see three.
One is $\boldsymbol{x^2 = x \cdot x}$, which is a multiplication.
Another is $\boldsymbol{x^2 - x}$, which is a subtraction.
And the other is $(\boldsymbol{x^2 - x}) + \boldsymbol{1}$, which is an addition.

Doing this $\boldsymbol{x^2}$, we get a product. And using the product, we do this $\boldsymbol{x^2 - x}$.
Then, using the difference, we do this $(\boldsymbol{x^2 - x}) + \boldsymbol{1}$. And then, we get the sum.

The sum is a number that the expression $x^2 - x + 1$ produces.
That is, it's an output that the function $x^2 - x + 1$ produces for an input.

And every time the set of operations gets completed, an output gets produced.
So for instance,

if 1 gets into $x^2 - x + 1$, we get $1^2 - 1 + 1 = 1$,

if 2 gets into $x^2 - x + 1$, we get $2^2 - 2 + 1 = 3$.

and if 3 gets into $x^2 - x + 1$, we get $3^2 - 3 + 1 = 7$,
…

And we can get put it this way, too:
$1 \longrightarrow x^2 - x + 1 \longrightarrow 1$

So we get (1, 1), where 1 is an input, and 1 is the output.

$2 \longrightarrow x^2 - x + 1 \longrightarrow 3$

So we get (2, 3), where 1 is from the domain, and 3 belongs to the range.

$3 \longrightarrow x^2 - x + 1 \longrightarrow 7$ So we get (3, 7).

And assuming X is the domain, and Y is the range, and putting at once all the function executions for all the inputs in X, we can put it the way as follows.

$X \longrightarrow x^2 - x + 1 \longrightarrow Y$

And assuming F is the function $x^2 - x + 1$, we can simplify the idea the way as follows.

$X \xrightarrow{F} Y$.

Next, depending on the nature of the correspondence a function creates, the type of the function can get determined. One is *one-to-one*, and another is *many-to-one*.

What then do we mean by one-to-one?

Suppose no two numbers in the domain cause the same number in the range.

That is, every number in the domain causes a different number in the range.
In other words, every output is different.
Then, the function is one-to-one. Otherwise, it is many-to-one.

What then about one-to-many?

There is no such a function. That's because a function produces one number only at a time. In other words, in any function, no two numbers can be caused by a number.

So a function is either one-to-one or many-to-one.
And we can put the idea the way as follows, too:

If all the numbers in the range are used once only, it is one-to-one.
So every output is different. Thus, if $f(a) \neq f(b)$ where $a \neq b$, f is one-to-one.

If some or all the numbers in the range are used more than once, it is many-to-one.
So some or all the outputs can be the same. For instance:

If $f(2) = f(3) = 1$, f is many-to-one.
Also, if $g(x) = 2$ for all the values of x, g is many-to-one.
And we call such a function as the function g above a ***constant function***, since each and every output is constant, that is, the same.

And as mentioned earlier, in high school math and calculus, all the numbers in both the domain and the range have to get connected. And such a function is said to be *onto*, and thus, is called an onto function.

So more specifically, a function is either one-to-one and onto or many-to-one and onto.

Working with a function then do we need to see if a function is one-to-one or not?

Sometimes, we need to come up with a new function from a particular function. Coming up with the new function, we have to use as the domain the range of the function particular, and use as the range the domain of the particular function. And that's not it.

We need to maintain the existing correspondence between numbers in both sets, the domain and the range of the particular function.

For instance, if the particular function makes (1, 2), (4, 7), (6, -5) etc., the new function has to make (2, 1), (7, 4), (-5, 6), etc.

We call such a new function *an inverse function*.
And we call the new function stated above, the inverse of the particular function, and often just call it *the inverse* for short.

Then, the function particular has to be one-to-one. Why?

Taking the inverse of a function many-to-one, we cannot get a function.
Or rather, we cannot take the inverse of a function many-to-one. Why not though?

If a function is many-to-one, the inverse would be one-to-many, and thus, would not be a function. A function cannot be one-to-many.

Each number in a domain gets connected to one number only in the range.
So no two numbers in the domain can be connected to one same number in the range.
So what is a function?

A function is an expression that connects two number sets, called a domain and a range in a way that every time the function gets executed, a *pair* of numbers gets determined.

And number pairs get determined in either of the two ways as follows.

(2, 1), **(4, 3)**, **(7, 5)**,etc. Then, the function f is one-to-one, since $f(a) \neq f(b)$ if $a \neq b$.

(1, 3), **(5, 9)**, **(7, 3)**, etc. Then, the function f is many-to-one, since $f(1) = f(7)$.

The first one in each pair is from the domain, and the other belongs to the range.

And that's the way the two sets, that is, the domain and the range get connected.

But no function can generate number pairs the way as follows. **(1, 2)**, **(1, 9)**, **(3, 4)**, etc.

0.2. How Functions Work 2

Why is $\boldsymbol{f(x)}$ though, read as $\boldsymbol{f}$ of $\boldsymbol{x}$?

Saying $\boldsymbol{f}$ of $\boldsymbol{x}$, we are saying that $\boldsymbol{f}$ is a function of $\boldsymbol{x}$. So in short, we just say $\boldsymbol{f}$ of $\boldsymbol{x}$.
And saying $\boldsymbol{f}$ is a function of $\boldsymbol{x}$, we mean $\boldsymbol{f(x)}$ changes as $\boldsymbol{x}$ changes.
In short, $\boldsymbol{f}$ varies as $\boldsymbol{x}$ varies.

More specifically, as the value of $\boldsymbol{x}$ changes, the value of $\boldsymbol{f(x)}$ changes.
Suppose for instance, $\boldsymbol{y = f(x) = 2x + 5}$. Then:

If $\boldsymbol{x = 1}$, we get $\boldsymbol{y = f(1) = 2 \cdot 1 + 5 = 7}$, so we get $\boldsymbol{y = f(1) = 7}$. That is, $\boldsymbol{y = 7}$ if $\boldsymbol{x = 1}$.

If $\boldsymbol{x = 2}$, we get $\boldsymbol{y = f(2) = 2 \cdot 2 + 5 = 9}$, so we get $\boldsymbol{y = f(2) = 9}$. That is, $\boldsymbol{y = 9}$ if $\boldsymbol{x = 2}$.

So as the value of $\boldsymbol{x}$ changes, the value of $\boldsymbol{f(x)}$ changes, and so does the value of $\boldsymbol{y}$, of course. And we know $\boldsymbol{x}$ gets the inputs. And as the input changes, the output changes.

Saying thus, a function $\boldsymbol{f}$ of $\boldsymbol{x}$, or just $\boldsymbol{f}$ of $\boldsymbol{x}$, we are saying that $\boldsymbol{f}$ is a function of $\boldsymbol{x}$, and that $\boldsymbol{x}$ is the input variable.

And we mean that the value of $\boldsymbol{f}$ or the value of $\boldsymbol{f(x)}$ changes as the value of $\boldsymbol{x}$ changes.
Also of course, since $\boldsymbol{y = f(x)}$, the values of $\boldsymbol{y}$ changes as the value of $\boldsymbol{x}$ changes.

Suppose for another instance, $t = g(s) = 3s + 4$. Then:

If $s = 1$, we get $t = g(1) = 3{\cdot}1 + 4 = 7$, so we get $t = g(1) = 5$. That is, $t = 7$ if $s = 1$.

If $s = 2$, we get $t = g(2) = 3{\cdot}2 + 4 = 10$, so we get $t = g(2) = 10$. That is, $t = 10$ if $s = 2$.

So as s changes, the value of $g(s)$ changes, and so does the value of t, of course.
Thus, s gets the inputs, $g(s)$ gets the outputs, and so does t, since $t = g(s)$.

So saying a function g of s, or just g of s, we are saying that g is a function of s, and that s is the input variable.

And we mean that the value of g or $g(s)$ changes as the value of s changes.
And of course, since $t = g(s)$, the value of t changes as the value of s changes, too.

Specifying however, a function the way above, that is, with no domain specified, we just assume the largest possible domain.

Given for instance, a function this way: $y = f(x) = 2x + 5$, we are to assume that the domain is a set of all real numbers, because it is the largest possible domain.
Why is it the largest?

Every real number can be an input of the function f. In other words, the function f can be defined for every real number. So the largest domain is a set of all real numbers.

So just setting $y = f(x) = 2x + 5$, we mean $y = f(x) = 2x + 5$ for x real.

What if the domain of the function f above is a set of all numbers grater than 3?

Then, we can put it this way: $y = f(x) = 2x + 5$ for $x > 3$.

What then is the range of the function f above?

We have $x > 5$. So we can get $x > 5 \Rightarrow 2x > 10 \Rightarrow 2x + 1 > 11$.
Also, we have $y = f(x) = 2x + 1$ for $x > 5$. So we get $y = f(x) > 11 \Rightarrow y > 11$.
And we know y is the output variable, so it gets all the outputs that belong to the range.
The range is thus, a set of all numbers bigger than 11, and can be put this way: $y > 11$.

And normally, if asked to find a domain, we are to find the largest possible domain. What then is the domain if the function given is $\boldsymbol{y = g(x) = \sqrt{x+1}}$?

We know what's inside the square root sign is positive or 0.

So we get $\boldsymbol{x + 1 \geq 0 \Rightarrow x \geq -1}$, and thus, the (largest possible) domain is $\boldsymbol{x \geq -1}$.

If any number less than -1 is put into ($\boldsymbol{x}$ in) the function $\boldsymbol{g}$, we don't get a real number.
So for instance, we cannot make a function as $\boldsymbol{y = g(x) = \sqrt{x-2}}$ for $\boldsymbol{x > 1}$.
That's because if $\boldsymbol{x = 1.5}$, we get $\boldsymbol{g(1.5) = \sqrt{1.5-2} = \sqrt{-0.5}}$, which is not real.

What kind of functions then do we work with in high school math?

Based on expressions that functions have, we can classify or categorize functions.

An expression as $\boldsymbol{x^2 - x + 1}$ is said to be algebraic, so if a function has an algebraic expression, the function is called an algebraic function. And if a function is not an expression algebraic, the function is called a transcendental function.

For instance, expressions as $\mathbf{sin}\, \boldsymbol{x}$, $\boldsymbol{3^x}$, $\mathbf{log}\, \boldsymbol{x}$, $\boldsymbol{\sqrt{x+1}}$, and $\boldsymbol{\dfrac{x}{x^2+1}}$ are not algebraic, so if a function is such an expression, the function is said to be transcendental. In particular:

A function $\boldsymbol{y = f(x) = \sin x}$ for $\boldsymbol{x \geq \pi}$ is called a trigonometric function, and is called a sine function.

A function $\boldsymbol{y = g(x) = 3^x}$ for $\boldsymbol{x > -2}$ is called an exponential function.

A function $\boldsymbol{y = h(x) = \log x}$ for $\boldsymbol{x > 0}$ is called a log function.

A function $\boldsymbol{y = p(x) = \sqrt{x+1}}$ for $\boldsymbol{x \geq 2}$ is called a radical (or irrational) function.

A function $\boldsymbol{y = q(x) = \dfrac{x}{x^2+1}}$ for $\boldsymbol{x \geq 1}$ is called a fractional (or rational) function.

So in high school math, we work with some of those transcendental functions, together with algebraic ones.
And moving on to calculus in a college or university, we get to work with a function more advanced, of which the domain can have two or more sets of numbers.

For instance, we get to work with functions as $z = r(x, y) = x + y + 1$ for $x > 0$, and $y < 1$.

The function r above has not one but two input variables, and the two are x and y.
So the domain is not just one set, but is made of two sets.

One is for the input variable x, and is a set of all numbers greater than 0, that is, a set of all positive numbers.
And the other is for the other input variable y, and is a set of all numbers less than 1.
Of course, z is the output variable of the function r.
Every function has one output variable. So every range is just one set of numbers.

Of a function basic, the domain is made of one set of numbers.

And such a basic function can be called a 2-D function, because it has two data sets, since it has two variables, one is an input variable, and the other is an output variable.

So if the domain is made of two sets, the function can be said to be 3-D, because it has three data sets, since it has three variables, two are input variables, and the other is an output variable.

And if the domain is composed of three sets, the function can be called a 4-D function, because it has four data sets, since it has four variables, three are input variables, and the other is an output variable.   And so forth.

So a function has quite a few; a name, an expression, a domain, and a range.
That's not it though.

A function has its curve, too.
So we can put a function in a place called a graph. Putting thus, a function in a graph, we put the curve in a graph. Where do we put the graph, though?

We put a graph in a coordinate system.
If a function has two variables, an input variable and an output variable, we can say that we put the function in a coordinate system said to be 2-D. And we just call it a 2-D system, for short.
In a 2-D system, we usually put two axes perpendicular to each other, which means therefore, a plane.

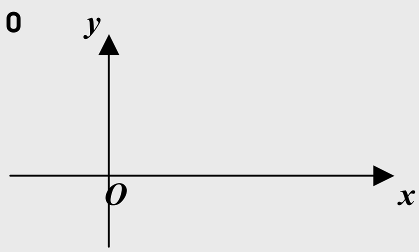
That's because we can use one axis for a length, and use the other for a width.
So usually, we just call such a 2-D system a coordinate plane.
What then about 3-D system?

We can call it a coordinate space, because in such a system, we put three axes perpendicular to each other. One is for a length, another is for a width, and the other is for a height. (The details are covered in the book **Function Transformations**.)

And we can name a coordinate system, too, and can do so using the variables used in the function to be put in the system.
So for instance, assuming we use $\boldsymbol{x}$ as the input variable, and use $\boldsymbol{y}$ as the output variable in a function $\boldsymbol{F}$, and putting the curve of $\boldsymbol{F}$ in a graph, we put the graph in the $\boldsymbol{x}$-$\boldsymbol{y}$ coordinate system, often called briefly the $\boldsymbol{x}$-$\boldsymbol{y}$ system, too, which is made of the $\boldsymbol{x}$-axis and the $\boldsymbol{y}$-axis perpendicular to each other, and thus, can be said to form a plane.

So we often just call the $\boldsymbol{x}$-$\boldsymbol{y}$ system the $\boldsymbol{x}$-$\boldsymbol{y}$ plane, too.

Fig. 0



In a function put the $\boldsymbol{x}$-$\boldsymbol{y}$ system, $\boldsymbol{x}$ is the input variable, and $\boldsymbol{y}$ is the output variable.
For instance, putting the function $\boldsymbol{F}$ in the $\boldsymbol{x}$-$\boldsymbol{y}$ system, we can set $\boldsymbol{y = F(x)}$.
And we can say that we can put the curve of $\boldsymbol{F}$ in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane or the $\boldsymbol{x}$-$\boldsymbol{y}$ system.
Why curve though?

Putting the curve of a function in a graph, we can actually see the function, and more importantly, we can see how the function behaves. So solving problems with functions, *we can get the solutions easier and quicker* putting the functions in their graphs.

(Details on graphs and curves are covered in the book **Function Transformations**.)
So a function is an expression acting on a number set called the domain, and producing another set called the range, and can actually show itself if it's put in a graph.

Can we have though, a function that is just a number as 1?
That is, can we have a function as $y = f(x) = 1$?

Yes, we can. Technically, a number itself can be an expression, too, since it expresses a value. And from a function's point of view, we can put the idea the way as follows.

As a range, we can use a set of one number only as $\{1\}$ or $\{0\}$. Why?

We can have such a function, and it is called a many-to-one function. So if the range has one number only, the only number corresponds to all the numbers in the domain.

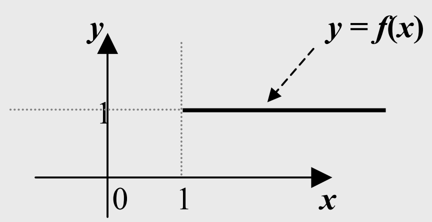
And we often call it a *constant function*.

In such a case, though the input changes, the output does not change. So in case of a constant function, the output is constant, and thus, the curve is a line parallel to the axis where all the inputs are placed. So for instance, being in the x-y plane, the line is parallel to the x-axis. And for instance, the functions below are constant functions.

$y = f(x) = 1$ for $x \geq 1$. The value of $f(x)$ is 1 if x is bigger than or equal to 1.

So for instance, we can get $f(1) = f(1.01) = f(2) = f(21) = 1$.

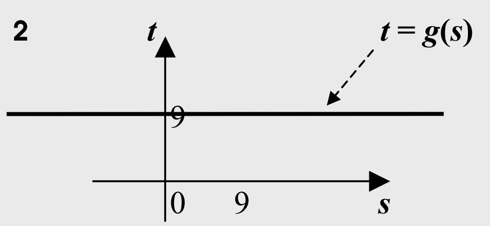
Fig. 1



The curve of f is a ray.

$t = g(s) = 9$ for s real. The value of $g(s)$ is 9 if s is a real number.

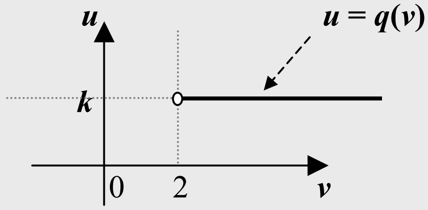
Fig. 2



The curve of g is a line.

$u = q(v) = k$ for $v > 2$, where k is a constant > 0. $q(v)$ is k if v is bigger than 2.

Fig. 3

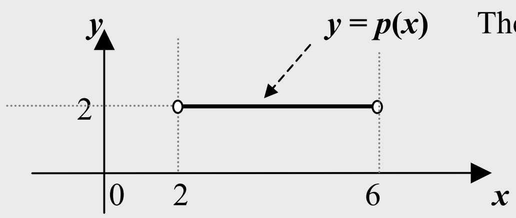


The curve of q is a ray.

Note that the point $(2, k)$ doesn't belong to the ray, which is the curve of q, of course. And of course, if $k < 0$, the ray will be below the x-axis.

$y = p(x) = 2$ for $2 < x < 6$. The value of $p(x)$ is 2 if x is between 2 and 6.

Fig. 4



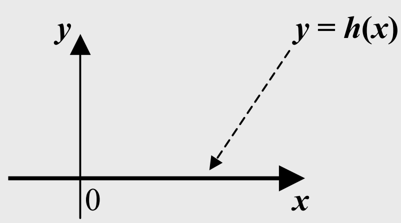
The curve of p is a line segment.

Note that the two points $(2, 2)$ and $(6, 2)$ do not belong to the curve of p.

$y = h(x) = 0$. In particular, we call such a function as h a zero function.

And of course, the curve of the zero function h is the x-axis itself.

Fig. 5



1. Why variables?

Working with functions, we usually work with variables, because functions are math expressions usually made of variables, together with numbers, of course.
Why though, are they called *variables*, and what's the use?

Doing math, we work with values. What values, though?

All kinds of values. Doing math, we do operations with values. Adding for instance, we do additions. Doing such operations, we get values, too. What values though?

Quite ambiguous, isn't it?
So we don't just do such operations. We use tools called *numbers* to specify values. So adding for instance, we add numbers together, and get the sum. So?

A number has a value. How does it get a value, though?

We assign a value to it.
So for instance, we assign seven to 7, which has thus, a value called seven. And that's it.
For another instance, 1 has a value called one, and is thus, of course, read as one.

And adding 7 to 1, we get the sum. Then, the sum gets a value called eight.
We have 8 that has a value called eight. So we can say that the sum is 8.

We have 2, also, so adding 2 and 7 together, we get the sum, which is 9. So this time, the sum is 9. Until we do another addition, the sum remains 9.

And we can give a name to the sum. So we can name it using a letter, and it can be called ***s***, for instance. Then, ***s*** can change its value every time the sum gets a new value.

Every time therefore, we add two numbers together, the sum gets a new value, so ***s*** gets the new value. Until however, we get the next sum, ***s*** has to keep the new value. That is to say that the value of ***s*** has to remain the same until it is assigned another new value.

In other words, the value of ***s*** has to be constant until we get the next sum.
In short, we say that ***s*** is *constant*.

And ***s*** is an object, so we just call ***s*** *a constant*.
So a constant is an object that keeps a value until it gets assigned a new value.

And we can make as many constants as we need. How then, do we make them?

We just assume those, or declare some letters constants, in a proper manner, of course.

So for instance, we can set up an operation this way: ***a* + *b* = *c*** where ***a***, ***b***, and ***c*** are constants. Or just setting ***a* + *b* = *c***, we normally assume that ***a***, ***b***, and ***c*** are constants.

Then, giving a value to each of ***a*** and ***b***, we get the sum, which is the value of ***c***.

So we have tools called *numbers* as 0, 1, 2, 3, etc., and tools called *constants* as ***a***, ***b***, etc.
Doing math therefore, we can conveniently use those tools doing operations as additions.
What is the difference though, between a number and a constant?

A number cannot change its value once it's been given a value.
So for instance, a number 7 has a value called seven forever, and thus, we cannot change the value of 7. That's because we do not give a new value to a number with a value.

A constant can however, change its value if we give a new value to it.
So we can give a new value to a constant with a value.
It has to keep though, a certain value until it gets assigned a new value.

So we can readily show a math operation in a general manner using such tools called constants. For instance, setting ***a* = 2*b*** where ***a*** and ***b*** are constants, we can say that the value of ***a*** is twice the value of ***b*** no matter what the value of ***b*** may be if ***b* ≠ 0**.

How then, do we know if ***c*** is a constant of a *particular kind*, for instance?

We can define it to be such a constant by declaring it a constant of the particular kind.

So for instance, declaring ***c*** a constant *integer*, we take ***c*** as an integer, and can use it as if it were an integer. And *at a time*, we can assign one number (value) only to a constant. Thus, we can assign one integer to ***c***, and can use ***c*** as the integer.

So for instance, ***c*** can get one of -2, 0, 1, 3, etc.
It has to however, keep the integer given until we assign another integer to it, so it cannot change its value by itself. So for instance:

Assuming now, ***c*** is given 3, we use ***c*** as 3.
And if we want to give –1 to ***c***, we give –1 to ***c***, and use ***c*** as –1, and of course, we do not use ***c*** as 3.
If however, we want to give 3 to ***c*** again, we give 3 to ***c***, and use ***c*** as 3, and of course, we do not use ***c*** as –1.

For another instance, declaring ***d*** a constant *real*, we can just declare ***d*** a constant.
So just declaring ***d*** a constant, we can use ***d*** as a real number,
and can assign at a time, one real number to ***d***, which has to keep the real number assigned until we assign another real number to it.

Suppose for instance, we get an equation $\boldsymbol{y = dx^2 + 1}$ where ***d*** is a nonzero constant.
Then, we use ***d*** as a nonzero real number, and ***d*** can be given one real number nonzero, but has to keep the real number given until it gets assigned another real number nonzero.

So the equation above can represent all the equations such as $\boldsymbol{y = x^2 + 1}$, $\boldsymbol{y = 2x^2 + 1}$, $\boldsymbol{y = -x^2 + 1}$, $\boldsymbol{y = \frac{x^2}{2} + 1}$, etc.

Suppose for another instance, we get an equation as follows.
$\boldsymbol{y = bx^2}$ where ***b*** is a constant integer
Then, ***b*** is an integer constant. So we use ***b*** as an integer, and ***b*** can get one integer, but has to keep the integer given until we give another integer to it.

Suppose for another instance, we get a statement as follows.

• <u>Assuming that ***a*** is constant, find the point where a line ***y* = 2*x* + *a*** meets the ***x***-axis</u>.

Then, we use ***a*** as a real number, and ***a*** can represent all real numbers, but has to keep a real number assigned until it gets assigned another real number.

Let's take one more example. Suppose we get a statement as follows.

Assuming ***d*** is a constant > 3, and ***A*** is a line ***y* = *x* + *d***,
find the point where ***A*** meets the ***x***-axis.

Then, we use ***d*** as a real number > 3, so ***d*** can be assigned one real number > 3, but has to keep the real number assigned until it gets assigned another real number > 3.

So a constant itself cannot change its value, but we can change the value of a constant if we need to.

And we have another kind in tools that work like numbers or constants, and thus can have values.

Such a tool is very different from a number, but seems quite close to a constant. It's certainly not a constant, of course. What then, is the tool?

It *looks* no other than a constant.
So like a constant, it can have one value at a time, and cannot have two or more values at a time. Unlike a constant though, it keeps changing its value. Or rather, it has to constantly change its value.

So its value varies continuously. And thus, we call it *a variable*.

So a variable varies its value continuously, and thus, its value keeps changing.
What values then, can a variable have?

A variable covers all values in a set. For instance, it can cover a set of all values ≥ 0.

And we use numbers to show values. So for instance, a variable can cover a set of all real numbers, a set of all nonzero real numbers, or a set of all real numbers less than 1.

And for instance, if a variable covers all real numbers positive, we only know the fact that the variable keeps changing its value covering all positive real numbers.

However, a variable cannot have two or more numbers at a time, and has to have one number only at each moment.

How then can we get a variable? That is, how can we make one?

Like other tools in math, defining a particular variable, we can make one.
So by a variable definition, we can make one and use it.

A variable varies covering all the values in a particular kind or set, which is a number set, of course. So we can define one by declaring it that way. So let's now make one.

Note:
We can use '$\in$' to show that particular objects *belong to* or *are elements of* other object. For instance, setting $\boldsymbol{a} \in \boldsymbol{B}$, we mean that $\boldsymbol{a}$ belongs to or is an element of $\boldsymbol{B}$, and also, setting $\boldsymbol{u}$ and $\boldsymbol{v} \in \boldsymbol{W}$, we mean that $\boldsymbol{u}$ and $\boldsymbol{v}$ both belong to or are elements of $\boldsymbol{W}$.

Suppose now, for instance, we get a statement as follows. $\underline{\boldsymbol{d} \text{ is real}}$.

Then, the statement can be a variable definition, and it can be assumed that $\boldsymbol{d}$ is a variable that can take any real number, and covers all real numbers.

So $\boldsymbol{d}$ keeps changing its value, and has one real number at each moment.

Suppose for another instance, we get an *equation* as follows. $\boldsymbol{y = x^2 + x + 1}$

Then, it is assumed that $\boldsymbol{x}$ is a variable that can take any real number, and covers all real numbers, because $\boldsymbol{x}$ can get any real number in this case. What then about $\boldsymbol{y}$?

It is a variable, too, and covers a set of some real numbers. And if no declaration is done to the letters as $\boldsymbol{x}$ and $\boldsymbol{y}$, we just assume that the letters are variables.
So $\boldsymbol{x}$ and $\boldsymbol{y}$ keep changing their values, and each has one real number unknown at a time.
Why unknown?

It's because a variable keeps changing its value. We only know what kind of values it can have at a time.

And we can notice that the values of the variables $\boldsymbol{x}$ and $\boldsymbol{y}$ are dependent upon each other. That is, the value of one variable determines the value of the other.

So for instance, when x is 1, that is, if $\boldsymbol{x} = \mathbf{1}$, we get $\boldsymbol{y} = \mathbf{1 + 1 + 1 = 3}$.
Also, when x is -2, that is, if $\boldsymbol{x} = \mathbf{-2}$, we get $\boldsymbol{y} = \mathbf{(-2)^2 + (-2) + 1 = 3}$, too.
In other words, if $\boldsymbol{y} = \mathbf{3}$, we get $\boldsymbol{x} = \mathbf{1}$ or $\mathbf{-2}$.

And in fact, the value of $\boldsymbol{y}$ is limited in this case. That's because we get

$\boldsymbol{y} = \boldsymbol{x}^2 + \boldsymbol{x} + \mathbf{1} = \boldsymbol{x}^2 + \boldsymbol{x} + \frac{1}{4} - \frac{1}{4} + \mathbf{1} = (\boldsymbol{x} + \frac{\mathbf{1}}{\mathbf{2}})^2 + \frac{3}{4} \geq \frac{3}{4}$. So we get $\boldsymbol{y} \geq \frac{3}{4}$.

Thus, we can say that $\boldsymbol{y}$ is a variable that covers all the numbers $\geq \frac{3}{4}$.
Also, we can limit the value a variable can take this way, too: $\boldsymbol{y} = \boldsymbol{x}^2 + \boldsymbol{x} + \mathbf{1}$ for $\boldsymbol{x} > \mathbf{1}$.

Then, it is assumed that $\boldsymbol{x}$ is a variable that can take any real number > 1.
What then about $\boldsymbol{y}$?

It is assumed that $\boldsymbol{y}$ is a variable, too, of course, but all the real numbers $\boldsymbol{y}$ can take are subject to the condition that $\boldsymbol{x} > \mathbf{1}$. So let's now find the set of all the numbers $\boldsymbol{y}$ can take.

To begin with, we have $\boldsymbol{y} = \boldsymbol{x}^2 + \boldsymbol{x} + \mathbf{1} = (\boldsymbol{x} + \frac{\mathbf{1}}{\mathbf{2}})^2 + \frac{3}{4}$. And we have $\boldsymbol{x} > \mathbf{1}$.

So we get $\boldsymbol{y} > (1 + \frac{1}{2})^2 + \frac{3}{4} = \frac{9}{4} + \frac{3}{4} = 3$ Thus, we get $\boldsymbol{y} > \mathbf{3}$.

So it is assumed that $\boldsymbol{y}$ is a variable that can take any real number > 3.

If defining a constant though, we need to declare it a constant. So for instance, if we just get $\boldsymbol{y = cx^2 + x + 1}$, we have to assume that not only $\boldsymbol{x}$ and $\boldsymbol{y}$ but $\boldsymbol{c}$ is a variable, too.

If however, we get $\boldsymbol{y = cx^2 + x + 1}$ where $\boldsymbol{c}$ is a constant, we just assume that $\boldsymbol{x}$ and $\boldsymbol{y}$ only are variables.

And for another instance, declaring $\boldsymbol{0 < c < 1}$ where $\boldsymbol{c}$ is constant, we mean that $\boldsymbol{c}$ is a constant that can be assigned at a time, one real number between 0 and 1.

Note that we can change the value of a constant if we want to, so a constant itself cannot change its value unlike a variable, and thus, a number as 7 *can be* technically *taken for* a constant, too, since 7 cannot change its value anyway. Normally however, we don't call a number a constant, of course.

• Let's next, define variables using set notation.

Setting $\boldsymbol{A = \{e|\ 1 < e < 2\}}$, and $\boldsymbol{c \in A}$, we mean that $\boldsymbol{c}$ is a variable that can take any real number between 1 and 2.   So it is just as good as setting $\boldsymbol{1 < c < 2}$.

For another instance, setting $\boldsymbol{A = \{e|\ 1 < e < 2\}}$, $\boldsymbol{B = \{t|\ 3 < t < 4\}}$, and $\boldsymbol{s \in A}$ or $\boldsymbol{B}$, we mean $\boldsymbol{s}$ is a variable that can take any number between 1 and 2 or between 3 and 4. And of course, $\boldsymbol{s}$ cannot take any number ≤ 1, any number from 2 to 3, and any number ≥ 4.

So for instance, $\boldsymbol{s}$ can take 1.5 or 3.8, but cannot take 2.
Also, it is just as good as setting $\boldsymbol{1 < s < 2}$ or $\boldsymbol{3 < s < 4}$.

• Let's this time, define constants using set notation.

Stating $\boldsymbol{A = \{e|\ 1 < e < 2\}}$, and $\boldsymbol{c}$ is a constant $\in \boldsymbol{A}$, we mean that $\boldsymbol{c}$ is a constant that can take at a time, one real number between 1 and 2.

So it is just as good as declaring or stating $\boldsymbol{1 < c < 2}$ where $\boldsymbol{c}$ is constant.

And next, stating <u>$\boldsymbol{A} = \{e|\ 1 < e < 2\}$, $\boldsymbol{B} = \{t|\ 3 < t < 4\}$, and $\boldsymbol{v}$ is a constant $\in$ $\boldsymbol{A}$ or $\boldsymbol{B}$</u>, we mean that $\boldsymbol{v}$ is a constant that can take at a time, one real number between 1 and 2 or between 3 and 4. And of course, $\boldsymbol{v}$ cannot take any number ≤ 1, any number from 2 to 3, and any number ≥ 4. So for instance, $\boldsymbol{v}$ can take 1.2 or 3.1, but cannot take 4 or 5.

Also, it is just as good as declaring <u>$\mathbf{1} < \boldsymbol{v} < \mathbf{2}$ or $\mathbf{3} < \boldsymbol{v} < \mathbf{4}$ where $\boldsymbol{v}$ is constant</u>.

Now, why do we frequently have to make and use, or work with variables?

Doing math, we often work with objects that change. What change, though?

Such an object changes by changing the value of itself.
Suppose for instance, an apple is falling.
Then, the position of the apple changes.

And we can put it the way as follows, too.

The distance from the position where the apple starts falling changes.
In short, the distance changes.
Then, what do we need to keep track of the distance?

We need an object that can change its value.
More specifically, we need an object that can indicate a distance, and can keep changing its own value. How then can we get the object?

We may want to begin with naming the object, and for instance, can name it ***d***.
Then, ***d*** is the distance, and keeps changing its value. How then do we call ***d***?

We call ***d*** a variable, and can have ***d*** get all real numbers it can get.

Since d is a distance, d has to be greater than or equal to 0. So we want d to be able to get any real number ≥ 0. Thus, we can simply declare that $d \geq 0$.

How then does d change?

The distance changes as time changes, so we need to keep track of the time, too. We don't just take the measurement of time though. We need an object that can indicate time, and can keep changing its own value. So what do we need?

We need another variable that indicates time, and for instance, can name it t.

Since t is time, t has to be greater than or equal to 0. So we want t to be able to get any real number ≥ 0. Thus, we can simply declare that $t \geq 0$, too.

So we now can use d as the distance, which keeps changing its value along the flow of time called t, which keeps changing, of course, its value, too.

So we now have two variables, and one is subject to the other.

Suppose now, we want to see where the apple is at a particular moment of time.

That is, we want to know a particular value of d after a particular amount of time t.

What then, do we need to find such a value of d?

We need a particular relation between the distance and time. That is, we want a specific correlation between d and t. In other words, we need the *connection* between the two. How then can we get the connection?

We can find it doing some experiments and collecting data. That is, taking distances after some amounts of time, we can come up with the connection.

How then can we come up with it?

We know that the distance d keeps changing its value as the time t changes its value.

Finding thus, the math expression that is expressed in terms of t, and produces a value of d for each value of t, we get the connection.

For instance, the expression can be $4.9t^2$ or $t^2 + 3t + 1$. What then is the connection?

The connection is of course, *the expression* in terms of t.
We know that *the expression* produces a value of d for each value of t.

So setting d = *the expression*, we get an equation, and we call in fact, the equation the *connective expression* between d and t.

And we call such an *expression* a function.
Also, we can say that the distance d is a function of time t.

That's because the value of d, that is, the value of the function changes as time t changes.

And in fact, the function is $\frac{1}{2}gt^2$, where g is a constant, and more specifically, is called the gravitational constant on earth, and is approximately 9.8 (m/sec^2).

So for each value of t, the value of $\frac{1}{2}gt^2$ is the value of the distance d.

And the connective expression between d and t is $d = \frac{1}{2}gt^2$

So we can now, define a function that specifies the correlation between the amount of distance d and the amount of time t.

And the inputs are the values of t, and the outputs are those of d.
What then is the domain?

The domain of the function is the set of all the values of t, of course.
How then can we get the domain?

Assuming that the apple is initially h meters away from the ground, we can say that the maximum distance the apple can travel is h meters. So finding t when $d = h$, we get

$d = \frac{1}{2}gt^2 \Rightarrow h = \frac{1}{2}gt^2 \Rightarrow t^2 = \frac{2h}{g} \Rightarrow t = \sqrt{\frac{2h}{g}}$, since we have $t \geq 0$.

So the domain is $0 \leq t \leq \sqrt{\frac{2h}{g}}$.

Thus, assuming the function is called $\boldsymbol{D}$, we can set $\boldsymbol{d = D(t) = \frac{1}{2}gt^2}$ for $\boldsymbol{0 \le t \le \sqrt{\frac{2h}{g}}}$.

And we know that the distance $\boldsymbol{d}$ can have values from 0 to $\boldsymbol{h}$, and is the output variable. So the range is $\boldsymbol{0 \le d \le h}$.

And we can apply such an idea as above to somewhere else.

For instance, we can use a variable as a speed, inventory, amount of time, price, height or altitude, depth, volume, force, area, etc.
In other words, we can take as a variable any object that has to change its value covering all the values defined. And we can keep track of its change. So a variable is a pretty handy tool for many uses, isn't it? We do math therefore, to make life easier.

And we want to note that a change does not just happen.
Where there is a change, there is a rate. What change though?

It can be a change in distance, a change in area, etc. What rate then?

It's a rate of change. What rate of change though?

It usually is a time rate of change, and for instance, we often use a time rate of change in distance, which is called a velocity, which has a magnitude called a speed.

For instance, light travels at approximately three hundred thousand kilometers a second, which is a magnitude of a rate of change, and the magnitude is the speed.
So the speed of light is about 300,000 km/sec.

A rate of change can be however, not only a velocity but something else, too.

It can be an acceleration, which is a rate of change in a rate of change, if you will.
And it can be called in fact, a time rate of change in velocity.
So a velocity can change as time changes, and we call such a rate an acceleration.
There can be thus, a rate of change in velocity. So a velocity can be a function of time, too.

And we will get to see and work with a lot of those in the math called calculus. Basically thus, calculus is about how functions work, and explains how things change in amount.

2. Formal Definitions

Defining an object in math, we produce a description of it. And the description has specifics on it, and is based on rules, principles, or theories relevant.

And we usually use a formal definition. What is a definition formal though?

Even if a definition is not formal, it is not informal.
What then is a formal definition?

Defining an object in math, we mainly use symbols, and use them as many as possible. So defining a function, too, we use symbols as many as possible. Thus, we minimize the use of natural language as English, and maximize the use of symbols.

And producing a definition the way above, we can say that we make a formal definition. Also, we can call a *formal* definition a *symbolic* definition, too, because of heavy use of symbols.
And the more formal it is, the easier it gets.

The worst thing in learning things in math is ambiguity, and we can reduce ambiguity by means of a formal definition. That is because it is concise and reduces the weaknesses in meaning or limitation of natural language, because we use symbols. So it can make a definition laconic, and remove room for ambiguity. We can therefore, alleviate the difficulty of understanding the definition.

And before we get into the discussion of formal definitions, we want to note and be familiar with the ideas as follows.

• We use '≡' to show that particular objects are identical to each other.
So for instance, $A \equiv B$ means that A and B are identical to each other.

• A symbol ∅ indicates an empty set,
which has no element in itself and can be put in {}.

• And we use '⊆' to indicate that a particular set is a subset of another set.
For instance, $S \subseteq T$ says that a set S is a subset of a set T.

Let's now make some function definitions, that is, define or make some functions.
We can make functions making formal definitions of those functions.
To begin with, we can define a function called G the way as follows.

Suppose that the function designator is G.

$X = \{s|\ 1 \leq s \leq 2\}$, which is the domain,
and $Y = \{t|\ 14 \leq t \leq 17\}$, which is the range.

And the expression is $3x + 11$.

Then, $y = G(x)$ for $x \in X$.

The set of statements in the box above can be a function definition, and there is nothing wrong with the definition, the definition of the function G. It is quite verbose, though.

So making or defining a function, we may want to use as many symbols as possible, and make use of words minimal. Then, we can make the definition clearer and succinct.

So let's now try defining the function G with less use of words.
That is, we are making a formal (symbolic) definition of the function G.

Then, we can produce the formal definition of G either of the ways as follows.

$\boldsymbol{X = \{s|\ 1 \le s \le 2\}}$, and $\boldsymbol{Y = \{t|\ 14 \le t \le 17\}}$.
$\boldsymbol{G: X \longrightarrow Y}$.
$\boldsymbol{y = G(x) = 3x + 11}$ for $\boldsymbol{x \in X}$.

$\boldsymbol{X = \{s|\ 1 \le s \le 2\}}$, and $\boldsymbol{Y = \{3s + 11|\ s \in X\}}$.
$\boldsymbol{G: X \longrightarrow Y}$, and $\boldsymbol{y = G(x)}$.

$\boldsymbol{X = \{s|\ 1 \le s \le 2\}}$, and $\boldsymbol{Y = \{3s + 11|\ 1 \le s \le 2\}}$.
$\boldsymbol{G: X \longrightarrow Y}$, and $\boldsymbol{y = G(x)}$.

So all the three definitions above are the same, and thus,
indicate one same function as follows. $\boldsymbol{y = G(x) = 3x + 11}$ for $\boldsymbol{1 \le x \le 2}$

"$\boldsymbol{y = G(x) = 3x + 11}$ for $\boldsymbol{1 \le x \le 2}$" is in fact, another formal definition of $\boldsymbol{G}$, and is the simplest.

Let's anyway now, take a look at what each component of the definition is about.

To begin with, the domain is $\boldsymbol{X = \{s|\ 1 \le s \le 2\}}$, which is telling us $\boldsymbol{X}$ is a set of all the real numbers that are greater than or equal to 1 and less than or equal to 2.

Next, the range is $\boldsymbol{Y = \{t|\ 14 \le t \le 17\}}$, which is saying that $\boldsymbol{Y}$ is made of all the real numbers ≥ 14 and ≤ 17. So using symbols, we can make communication simple and fast.

And describing a function, we can also say that a function is from a domain to a range.

So next, the expression below is saying that G is a function from X to Y.

$$G: X \longrightarrow Y$$

So we can readily and quickly see that the domain is X and the range is Y.
The arrow says that the values in X cause the values in Y.
That is to say that the function G produces the values in Y using the values of X.

So in a symbolic definition, we don't have to declare the domain and the range.
That is, we can still distinguish each by means of the direction of an arrow.

So the statement, "$G: X \longrightarrow Y$." is clearly saying that G is a function from X to Y.

> And next, the expression $y = G(x) = 3x + 11$ is saying that
> y is the output variable, gets each output, which belongs to Y,
> x is the input variable, gets each input from X,
> and the expression is $3x + 11$,
> together with the name of the function, which is $G(x)$ or just G.

What do we mean by though, $\{3s + 11|\ s \in X\}$ and $\{3s + 11|\ 1 \leq s \leq 2\}$?

First of all, both are the same sets, and each is saying that it is a set of all the values that can be made from the expression $3s + 11$ for $1 \leq s \leq 2$.

So it is a set of all the values that are made from the expression $3x + 11$ for $1 \leq x \leq 2$, and thus, the set is the range, and is $Y = \{e|\ 14 \leq e \leq 17\}$.

So $\{3s + 11|\ s \in X\}$, $\{3s + 11|\ 1 \leq s \leq 2\}$, and $\{e|\ 14 \leq e \leq 17\}$ are identical to each other.

That is, we have $\{e|\ 14 \leq e \leq 17\} \equiv \{3s + 11|\ 1 \leq s \leq 2\} \equiv \{3s + 11|\ s \in X\}$, because we have $X = \{e|\ 1 \leq e \leq 2\}$.

Also, we can see from the fact above, $3s + 11$ is in fact, equal to $3x + 11$, so we can say that we can choose any letter for a variable insofar as the consistency is maintained.

Now, from the definition of the function $\boldsymbol{G}$, we can say this:

The function is called $\boldsymbol{G}$, which is a function of $\boldsymbol{x}$,
and from the domain $\boldsymbol{X}$, each input gets into $\boldsymbol{x}$ in the expression $\mathbf{3}\boldsymbol{x} + \mathbf{11}$,
in which the operations get performed,
then the output for the input gets produced,
is put into $\boldsymbol{y}$,
and thus, is an element of the range $\boldsymbol{Y}$.

Besides, we can put the definition of $\boldsymbol{G}$ in any of the ways as follows, too.

$X = \{u|\ 1 \leq u \leq 2\}$, $Y = \{v|\ 14 \leq v \leq 17\}$, $X \xrightarrow{G} Y$, and $y = G(x) = 3x + 11$.

$X = \{s|\ 1 \leq s \leq 2\}$, $Y = \{3t + 11|\ t \in X\}$, $X \xrightarrow{G} Y$, and $y = G(x)$.

$X = \{p|\ 1 \leq p \leq 2\}$, $Y = \{3q + 11|\ 1 \leq q \leq 2\}$, $X \xrightarrow{G} Y$, and $y = G(x)$.

In the definitions above, we put the function designator above the arrow, and other than that, they are all the same as the previous ones.

That is, $G\colon X \longrightarrow Y$ is the same as $X \xrightarrow{G} Y$.

Now, let's have a look at another example.

$\boldsymbol{X}$ and $\boldsymbol{Y}$ are sets of all real numbers, $\boldsymbol{F}: \boldsymbol{X} \longrightarrow \boldsymbol{Y}$, and $\boldsymbol{y} = \boldsymbol{F}(\boldsymbol{x}) = \mathbf{2}\boldsymbol{x} + \mathbf{1}$.

$\boldsymbol{R}$ is a set of all real numbers, $\boldsymbol{X}$ and $\boldsymbol{Y} \subseteq \boldsymbol{R}$, $\boldsymbol{F}: \boldsymbol{X} \longrightarrow \boldsymbol{Y}$, and $\boldsymbol{y} = \boldsymbol{F}(\boldsymbol{x}) = \mathbf{2}\boldsymbol{x} + \mathbf{1}$.

$\boldsymbol{X}$ is a set of real numbers, $\boldsymbol{Y} = \{\mathbf{2}\boldsymbol{x} + \mathbf{1} | \boldsymbol{x} \in \boldsymbol{X}\}$, $\boldsymbol{F}: \boldsymbol{X} \longrightarrow \boldsymbol{Y}$, and $\boldsymbol{y} = \boldsymbol{F}(\boldsymbol{x})$.

$\boldsymbol{R}$ is a set of all real numbers, $\boldsymbol{X} \subseteq \boldsymbol{R}$, $\boldsymbol{Y} = \{\mathbf{2}\boldsymbol{s} + \mathbf{1} | \boldsymbol{s} \in \boldsymbol{X}\}$, $\boldsymbol{F}: \boldsymbol{X} \longrightarrow \boldsymbol{Y}$, and $\boldsymbol{y} = \boldsymbol{F}(\boldsymbol{x})$.

$\boldsymbol{R}$ is a set of all real numbers, $\boldsymbol{X} \subseteq \boldsymbol{R}$, $\boldsymbol{Y} = \{\boldsymbol{F}(\boldsymbol{x}) | \boldsymbol{x} \in \boldsymbol{X}\}$, $\boldsymbol{F}(\boldsymbol{x}) = \mathbf{2}\boldsymbol{x} + \mathbf{1}$, and $\boldsymbol{y} = \boldsymbol{F}(\boldsymbol{x})$.

Though looking different from each other, all the five definitions above are the same. In each of the definitions above, we can see these:

$\boldsymbol{F}$ is a function from $\boldsymbol{X}$ to $\boldsymbol{Y}$, so $\boldsymbol{X}$ is the domain of $\boldsymbol{F}$, and $\boldsymbol{Y}$ is the range.
The input variable is $\boldsymbol{x}$, and the output variable is $\boldsymbol{y}$, and the expression is $\mathbf{2}\boldsymbol{x} + \mathbf{1}$.

• Also, we often say that $\boldsymbol{y}$ is the *image* of $\boldsymbol{x}$ by the function $\boldsymbol{y} = \boldsymbol{F}(\boldsymbol{x})$.

So $\boldsymbol{Y}$ can be said to be the set of the images of all the elements in $\boldsymbol{X}$ by the function $\boldsymbol{F}$. And of course, the domain and range cannot be empty sets, that is, $\boldsymbol{X} \neq \varnothing$, and $\boldsymbol{Y} \neq \varnothing$.

Besides, we have another way where we can define a function.

We can express directly the expression, along with the arrow. So for instance, we can define the function $\boldsymbol{F}$ above in such a way as follows.

$\boldsymbol{X}$ is a set of all real numbers, $\boldsymbol{Y} = \{\boldsymbol{F}(\boldsymbol{x}) | \boldsymbol{x} \in \boldsymbol{X}\}$, $\boldsymbol{F}: \boldsymbol{x} \longrightarrow \mathbf{2}\boldsymbol{x} + \mathbf{1}$, and $\boldsymbol{y} = \boldsymbol{F}(\boldsymbol{x})$.

And we can put it this way, too:

$\boldsymbol{X}$ is a set of all real numbers, $\boldsymbol{Y} = \{\boldsymbol{F}(\boldsymbol{x}) | \boldsymbol{x} \in \boldsymbol{X}\}$, $\boldsymbol{x} \xrightarrow{F} \mathbf{2}\boldsymbol{x} + \mathbf{1}$, and $\boldsymbol{y} = \boldsymbol{F}(\boldsymbol{x})$.

That is, $(\boldsymbol{F}\!: \boldsymbol{x} \longrightarrow \mathbf{2}\boldsymbol{x} + \mathbf{1}) \equiv (\boldsymbol{x} \xrightarrow{F} \mathbf{2}\boldsymbol{x} + \mathbf{1})$.

So from the definition of $\boldsymbol{F}$ above, too, we can see $\boldsymbol{X}$ is the domain, and $\boldsymbol{Y}$ is the range, and for each value of $\boldsymbol{x}$, the output gets generated by the expression $\mathbf{2}\boldsymbol{x} + \mathbf{1}$, and therefore, we can set $\underline{\boldsymbol{y} = \boldsymbol{F}(\boldsymbol{x}) = \mathbf{2}\boldsymbol{x} + \mathbf{1}\text{ for }\boldsymbol{x}\text{ real}}$, or just $\underline{\boldsymbol{y} = \boldsymbol{F}(\boldsymbol{x}) = \mathbf{2}\boldsymbol{x} + \mathbf{1}}$, because in this case, it is assumed the domain is a set of all real numbers.

And there can be many other ways we can define a function.
For instance, we can sometimes use parameters, too.
And such a parameter can be called a medium.

Using a parameter to make a function, we parameterize a variable to produce a function. Then, the variable is the output variable in the function produced, and the parameter becomes the input variable.

So the function is a function of the parameter, and is called a parametric function.

Assuming for instance, $\boldsymbol{x} = \boldsymbol{t} + \mathbf{1}$ when $\boldsymbol{t} > \mathbf{0}$, and parameterizing $\boldsymbol{x}$ to make a function called $\boldsymbol{f}$, we can get $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{t}) = \boldsymbol{t} + \mathbf{1}$ for $\boldsymbol{t} > \mathbf{0}$.

Then, we can say $\boldsymbol{x}$ is parameterized, and $\boldsymbol{t}$ is the parameter, and $\boldsymbol{f}$ is called a parametric function, where $\boldsymbol{x}$ is the output variable, so $\boldsymbol{f}$ is a function of $\boldsymbol{t}$, which is thus, the input variable.

Parameterizing a variable to produce a function, we say we do parameterization.
Doing parameterization though, we usually parameterize two or more variables.
So we get two or more functions that are parametric.

And we use the same parameter as the input variable of each parametric function.
So the parameter can be called the common input variable of the functions.

That is, the parameter goes between the functions.
Thus, we can connect the functions by means of the parameter.

That is, all the output variables get connected through the medium called the parameter, and then, a new function or an equation gets produced.

In other words, we get a connective expression between the output variables by means of the parameter, and the expression is the expression of the new function or the equation we get.

Suppose for instance, $\boldsymbol{x = f(t) = t + 1}$ for $\boldsymbol{t > 0}$, and $\boldsymbol{y = g(t) = t^2}$ for $\boldsymbol{t > 0}$.

Then, $\boldsymbol{f}$ and $\boldsymbol{g}$ both are functions of $\boldsymbol{t}$, and can be called parametric functions, and $\boldsymbol{x}$ and $\boldsymbol{y}$ are output variables, so connecting the two output variables, we can come up with another function.

To begin with, from $\boldsymbol{x = f(t) = t + 1}$, we can get $\boldsymbol{t = x - 1}$ for $\boldsymbol{t > 0}$.

So we get $\boldsymbol{t = x - 1 > 0 \Rightarrow x > 1}$, and thus, we get $\boldsymbol{t = x - 1}$ for $\boldsymbol{x > 1}$.

And we have $\boldsymbol{y = g(t) = t^2}$, too, so we get $\boldsymbol{y = t^2 = (x - 1)^2 \Rightarrow y = (x - 1)^2}$ for $\boldsymbol{x > 1}$.

Naming the function above, $\boldsymbol{h(x)}$, we get a function $\boldsymbol{y = h(x) = (x - 1)^2}$ for $\boldsymbol{x > 1}$.

Connecting functions through a parameter, we say we do a parametric transformation, which is covered in the book, **Function Transformations**.

Note:
We can choose any letter for a variable insofar as the consistency is maintained. In other words, it doesn't matter what letter we take as the variable as long as the consistency is maintained.

So we can freely choose any letter provided we keep the consistency. Thus, for instance, "$\boldsymbol{y = g(x) = 8(3 - x)}$ where $\boldsymbol{x \geq 1}$." is the same as "$\boldsymbol{t = h(s) = 8(3 - s)}$ where $\boldsymbol{s \geq 1}$."

That is, the two functions $\boldsymbol{g(x)}$ and $\boldsymbol{h(s)}$ are the same functions.

In other words, though both have different names, both are the same, because both have the same domain, and the same expression.

The input variable $\boldsymbol{s}$ is no more than a container for one input value at a time, and covers all the values in the domain of the function $\boldsymbol{g}$. And the output variable $\boldsymbol{y}$ is a container for one output value at a time, and covers all the values in the range.

By the way, defining a function, we define a particular function. Defining a function particular, we make a function. Making a function, we can say we make a function definition, too.

And making such definitions, we can have two choices. One is a full definition, and the other is a short definition. It depends on how much detail we show on the function we define. And also, we can call a short definition a brief definition or a semi-definition, too.

A full definition is composed of the name, the domain and the expression. Quite often though, the expression is missing. Then, we can call it a short definition.

If the expression is unavailable or unnecessary for the moment, we can make a short definition using the name and domain only, and can call it the minimal definition.

So for instance, a function definition can be $\boldsymbol{y = f(x)}$ for $\boldsymbol{x > 2}$, or $\boldsymbol{t = g(s)}$ for $\boldsymbol{s}$ real.
And if we get $\boldsymbol{t = g(s)}$ only, it is assumed that the domain is a set of all real numbers.

3. How do we know if it is one–to–one?

Sometimes, we need to make sure if a function is one-to-one. Then, we need to give it a test to see if it produces a different output for each of all the inputs.

We cannot do so however, for each and every input, since the domain is usually an interval in a number line, and thus, has infinite inputs. So no matter how many numbers we may pick from the domain, the test is insufficient. How then can we do the test?

We cannot just pick several numbers at random from the domain and do the test either, because it doesn't confirm that a different output gets produced for each of all the inputs. What else then can we do?

In *math*, *arbitrary* means not just *random* but *comprehensive*, too.
So a value arbitrary in math has such generality, and can cover all the values in a particular kind or category. Thus, it can cover an interval in a number line.

What then do we need to work with doing such a test comprehensively and at random?

We have a tool that can cover all the values in an interval in a number line as the x-axis. What tool then can it be?

It is a constant. A constant can cover all values (numbers) in a kind or category, and thus, can cover an interval in a number line, or can cover even an entire number line itself. It can cover one value at a time, though. So what can we do with a constant?

We can use a constant as an input. How?

We need to use two constants, and let them be different, and belong to the domain so that each can cover the domain. Why two, though?

Showing that two outputs are different for the two constants, which are two inputs, we show that the function is one-to-one. How?

Since we have used constants that can cover all the values applicable, we have shown that no matter what two different inputs we may use, we get two different outputs, so it is a one-to-one function.
How can we show that the two outputs are different, though?

• Assuming for instance, $\boldsymbol{f(x) = 2x + 1}$, let's see now, if the function $\boldsymbol{f}$ is one-to-one.
Suppose first, $\boldsymbol{u}$ and $\boldsymbol{v}$ are constants, and $\boldsymbol{u \neq v}$.
Then, we get $\boldsymbol{f(u) - f(v) = (2u + 1) - (2v + 1) = 2(u - v)}$.
Next, we have $\boldsymbol{u \neq v}$, so we get $\boldsymbol{u - v \neq 0}$.
Thus, we get $\boldsymbol{f(u) - f(v) \neq 0 \Rightarrow f(u) \neq f(v)}$, so $\boldsymbol{f}$ is one-to-one.

• Assuming next, for another instance, $\boldsymbol{f(x) = 2x^2 + 1}$ for $\boldsymbol{x \geq 0}$, let's check to see if the function $\boldsymbol{f}$ is one-to-one.

Suppose first, $\boldsymbol{u}$ and $\boldsymbol{v}$ are constants, $\boldsymbol{u}$ and $\boldsymbol{v}$ both $\geq \boldsymbol{0}$, and $\boldsymbol{u \neq v}$.
Then, we get $\boldsymbol{f(u) - f(v) = (2u^2 + 1) - (2v^2 + 1) = 2(u^2 - v^2) = 2(u + v)(u - v)}$.

Next, we have $\boldsymbol{u \neq v}$, so we get $\boldsymbol{u - v \neq 0}$.
And we have $\boldsymbol{u}$ and $\boldsymbol{v}$ both $\geq \boldsymbol{0}$, too, together with $\boldsymbol{u \neq v}$.
So we get $\boldsymbol{u \neq -v}$, because $\boldsymbol{u}$ and $\boldsymbol{v}$ both cannot be 0 concurrently, and are not negative.
That is, we get $\boldsymbol{u + v \neq 0}$.

So in sum, we get $\boldsymbol{u - v \neq 0}$, and $\boldsymbol{u + v \neq 0}$.
Thus, we get $\boldsymbol{(u + v)(u - v) \neq 0 \Rightarrow f(u) - f(v) \neq 0 \Rightarrow f(u) \neq f(v)}$, so $\boldsymbol{f}$ is one-to-one.

• Suppose this time, $\boldsymbol{a}$ and $\boldsymbol{b}$ are constant, $\boldsymbol{a > 0}$, and $\boldsymbol{f(x) = ax^2 + b}$ for $\boldsymbol{x \geq 0}$, and let's check to see if the function $\boldsymbol{f}$ is one-to-one.

Then, assuming $\boldsymbol{u}$ and $\boldsymbol{v}$ are different, and using $\boldsymbol{u}$ and $\boldsymbol{v}$ as inputs, we need to show that $\boldsymbol{f(u) \neq f(v)}$.

Then, we can check to see if $\boldsymbol{f(u) - f(v) \neq 0}$ or $\boldsymbol{\dfrac{f(u)}{f(v)} \neq 1}$, where $\boldsymbol{f(v) \neq 0}$, of course.

So suppose now $\boldsymbol{u}$ and $\boldsymbol{v}$ are constants, $\boldsymbol{u}$ and $\boldsymbol{v}$ both $\boldsymbol{\geq 0}$, and $\boldsymbol{u \neq v}$.
It's because the domain is $\boldsymbol{x \geq 0}$, and we want to use two different inputs.

Then, first, $\boldsymbol{f(u) - f(v) = (au^2 + b) - (av^2 + b) = a(u^2 - v^2) = a(u + v)(u - v)}$.

Next, we have $\boldsymbol{a > 0}$, and $\boldsymbol{u \neq v}$, so we get $\boldsymbol{a \neq 0}$, and $\boldsymbol{u - v \neq 0}$.

And we have $\boldsymbol{u}$ and $\boldsymbol{v}$ both $\boldsymbol{\geq 0}$, too, together with $\boldsymbol{u \neq v}$.
Thus, we get $\boldsymbol{u \geq 0}$, and $\boldsymbol{0 \geq -v}$, since $\boldsymbol{u}$ and $\boldsymbol{v}$ are nonnegative.
So we get $\boldsymbol{u \neq -v}$, because $\boldsymbol{u}$ and $\boldsymbol{v}$ both cannot be 0 concurrently, since $\boldsymbol{u \neq v}$.
Thus, we get $\boldsymbol{u + v \neq 0}$. So we get $\boldsymbol{f(u) - f(v) \neq 0 \Rightarrow f(u) \neq f(v)}$, and thus, $\boldsymbol{f}$ is one-to-one.

• Suppose for another instance, $\boldsymbol{g(x) = (x - 1)(x - 2)(x - 3)}$, and let's check to see if the function $\boldsymbol{g}$ is one-to-one.

We can get $\boldsymbol{g(1) = (1 - 1)(1 - 2)(1 - 3) = 0}$ and $\boldsymbol{g(2) = (2 - 1)(2 - 2)(2 - 3) = 0}$.
So we get $\boldsymbol{g(1) = g(2)}$. Therefore, the function $\boldsymbol{g}$ is not one-to-one.

How can we just produce one example only?

This time, we have a little different situation.
It is not always the case where we have to show a function is one-to-one.
That is to say that if we can, we can also show that it is not the case.

Showing it's not the case, we can produce a particular example, called a *counterexample*, which can be called a counter evidence, too, of course.
Producing a disproof, we have only to show one counterexample where we can see it is not the case. So one counterexample is enough.

4.0. Composite Functions 1

Using a function, we can connect data sets. And basically, two sets get connected, one is the domain, and the other is the range. Sometimes though, we make a function putting together two functions, too. Putting the two together in fact, we combine the two.

And combining the two, we put one into the other. Or rather, we blend the two together. Then, we can get a new function. And the new function is called a composite function.

A composite function is a function, too. So we can make it blend into another function, and can get another composite function. So after all, if making a particular function, we can blend together not just two but thee or more functions, too. And the particular function is a blend of functions, and is called a composite function.

So making a composite function, we can say we blend or mix functions together. And thus, it can be called a blend or mix of functions.

As mentioned above however, blending more than two functions, we don't just mix them all at once.
We mix two at a time, and not three or more at once.
Making a composite function mixing together more than two functions, we mix them sequential manner.

We put them in a sequence, and then, begin with the first two. Then, we can get the blend of the two, and the blend is a composite function.
Next, mixing the composite function into the third one in the sequence, we can get another composite function.
And so forth until the last function, so eventually, we can get a mix of all the functions in the sequence.

Mixing two functions though, we don't just mix the two together.
We mix one into the other.
We cannot however, just mix one into the other. Why not?

We know a function has its domain and range.
And so to speak, taking a composite of two functions, we put the range of one into the domain of the other, if you will.
Thus, mixing functions together, we need to be careful with the sizes of the domains and the ranges. Whose domain and whose range though?

The range of the function that blends into. And the domain of the other function.
The range of the function that blends into has to be a subset of the domain of the other.
(A subset is a set smaller or equal.)
If the range is bigger, that is, the domain is smaller, they don't mix. Why not?

We will get to see why not shortly. Let's see first, how blending works.
Blending two functions, we plug one into the other. Then, we get another function, called a composite function.

And blending, we blend the expression of one function into the expression of the other.
Blending them, we plug the expression of one function into the input variable in the expression of the other function. Each and every input variable in the expression of the other function gets the expression. How?

Suppose for instance, a function $\boldsymbol{f(x)}$ blends into another function $\boldsymbol{g(x)}$, $\boldsymbol{2x}$ is the expression of $\boldsymbol{f(x)}$, and $\boldsymbol{x^2 + x}$ is the expression of the other function $\boldsymbol{g(x)}$.

Then, after blending, we get a new function where the expression is
$\boldsymbol{(2x)^2 + (2x) = 4x^2 + 2x}$. Why though?

What does the expression of a function stand for?

The expression of a function produces each and every output of the function.
So it stand for all the outputs of the function.
And the input variable in a function gets each and every input of the function.
So it stands for all the inputs in the function.

So in sum, an expression stands for a range, and an input variable stands for a domain. An expression is no other than a range, and an input variable is no other than a domain, if you will.
Thus, blending the expression of $\boldsymbol{f(x)}$ into the expression of $\boldsymbol{g(x)}$, we put the expression of $\boldsymbol{f}$ into the input variable, that is, $\boldsymbol{x}$ in $\boldsymbol{g(x)}$.

So the expression of $\boldsymbol{f}$ gets into each and every input variable in the expression of $\boldsymbol{g}$.

Thus, for instance, blending $\boldsymbol{x + 1}$ into $\boldsymbol{x^2 + 2x}$, we get $\boldsymbol{(x + 1)^2 + 2(x + 1) = x^2 + 4x + 3}$.

What can we notice then from the fact above?

We know the expression of a function produces each and every output of the function. So we can notice that each output of the function that blends into is used as each input of the other function.
That is, every output of the function that blends into is every input of the other function.

In other words, the range of the function blends into is the domain of the other.

Suppose now, blending $\boldsymbol{f}$ into $\boldsymbol{g}$, we get a composite function $\boldsymbol{h}$.
What then is the domain of the composite function $\boldsymbol{h}$?

We know the range of $\boldsymbol{f}$ is used as the domain of $\boldsymbol{g}$ during the process of making $\boldsymbol{h}$.
And of course, the range of $\boldsymbol{f}$ is made by the expression of $\boldsymbol{f}$ using the domain of $\boldsymbol{f}$.
That's because the expression gets inputs from the domain, and produces outputs, which belong to the range.
So the domain of the composite function $\boldsymbol{h}$ is the domain of $\boldsymbol{f}$, the function blends into.

And putting the ideas above into a diagram, we can put them this way:

$$\begin{tikzcd}
(\text{Domain of } f) \arrow[r, "f"] \arrow[rr, bend right, "h"'] & (\text{Range of } f \subseteq \text{Domain of } g) \arrow[r, "g"] & (\text{Range of } h \subseteq \text{Range of } g)
\end{tikzcd}$$

So we get (Domain of $\boldsymbol{f}$) $\xrightarrow{h}$ (Range of $\boldsymbol{h}$). And Range of $\boldsymbol{h} \subseteq$ Range of $\boldsymbol{g}$.

So the domain of $\boldsymbol{f}$ is the domain of $\boldsymbol{h}$, and the range of $\boldsymbol{h}$ is a subset of the range of $\boldsymbol{g}$.

What then, can be a problem when we make a composite function using two functions?

Making a composite function, we cannot just mix any functions, because the functions have to fit. Or rather, the number sets have to fit.

The range of the function blends into has to fit the domain of the other function.

That's because the range of the one blends into will be the domain of the other.

And the domain of the function blends into is the domain of the composite function.
And we will see shortly more concrete details on how it is the case.
Now, the range of the one blends into has to be a part of or the same as the domain of the other. So anyway, it is ideal that such a range is the same as such a domain.

We know domains and ranges are sets (of numbers). So we want to make sure that the range of the one blends into is a subset of the domain of the other.

(If $\boldsymbol{A}$ is a subset of a set $\boldsymbol{B}$, we get $\boldsymbol{A} \subseteq \boldsymbol{B}$, which means $\boldsymbol{A}$ is either $\boldsymbol{B}$ or a part of $\boldsymbol{B}$.)

Let's now make a case where the range of $\boldsymbol{f}$ is not a subset of the domain of $\boldsymbol{g}$, and see what will happen.

Suppose for instance, the domain of $\boldsymbol{g}$ is a set of all positive numbers, the range of $\boldsymbol{f}$ is a set of all positive numbers and 0, that is, a set of all numbers ≥ 0, and we want to make $\boldsymbol{f}$ blend into $\boldsymbol{g}$ to get a composite function called $\boldsymbol{h}$.

Then, the range of $\boldsymbol{f}$ has to be the domain of $\boldsymbol{g}$, but 0 cannot be an input of $\boldsymbol{g}$, because $\boldsymbol{g}$ is not defined for 0, since 0 is not in the domain of $\boldsymbol{g}$. So if in $\boldsymbol{g}$, 0 is used as an input, $\boldsymbol{g}$ crashes, if you will. What problem then, will the composite function $\boldsymbol{h}$ get?

The function $\boldsymbol{h}$ cannot hold for *some number*, which is of course, in the domain of $\boldsymbol{h}$.
How?

We know 0 is in the range of $\boldsymbol{f}$, so 0 is an output for *some number* in the domain of $\boldsymbol{f}$.
And we know the domain of $\boldsymbol{f}$ is the domain of the composite function $\boldsymbol{h}$. So the *some number* the function $\boldsymbol{h}$ cannot hold for is the *some number* in the domain of $\boldsymbol{f}$.
Thus, $\boldsymbol{h}$ will crash for the *some number*.

So the function $\boldsymbol{f}$ cannot blend into the other function $\boldsymbol{g}$.
That is to say that we cannot get the composite function $\boldsymbol{h}$.
Let's now make a bit more concrete case where we can see better how a composite function gets made.

Suppose for instance, mixing two functions $\boldsymbol{A}$ and $\boldsymbol{B}$, we make a new function $\boldsymbol{C}$.
Suppose specifically, we put $\boldsymbol{A}$ into $\boldsymbol{B}$ to get $\boldsymbol{C}$.

Then, $\boldsymbol{C}$ is a composite function where $\boldsymbol{A}$ blends into $\boldsymbol{B}$.
And one thing for sure now is that the domain of $\boldsymbol{C}$ is the domain of $\boldsymbol{A}$.

Suppose now, X_A is the domain of $\boldsymbol{A}$, Y_A is the range of $\boldsymbol{A}$, X_B is the domain of $\boldsymbol{B}$, Y_B is the range of $\boldsymbol{B}$, X_C is the domain of $\boldsymbol{C}$, and Y_C is the range of $\boldsymbol{C}$.

So $\boldsymbol{X}$ stands for a domain, and $\boldsymbol{Y}$ stands for a range. And $\boldsymbol{A}$ and $\boldsymbol{B}$ can be set as follows.

$X_A \xrightarrow{A} Y_A \qquad X_B \xrightarrow{B} Y_B$

Then first, putting $\boldsymbol{A}$ into $\boldsymbol{B}$, we need to have $Y_A \subseteq X_B$, which means Y_A is a subset of X_B.
That is, the range of $\boldsymbol{A}$ is equal to or a part of the domain of $\boldsymbol{B}$.

- So the mixing process goes as follows. $X_A \xrightarrow{A} (Y_A \subseteq X_B) \xrightarrow{B} Y_B$

Thus, all the inputs to be used in $\boldsymbol{B}$ are all the outputs made in $\boldsymbol{A}$.

That is, the range of $\boldsymbol{A}$ will be the domain of $\boldsymbol{B}$. So the mixing process continues as below, and then ends up with the function $\boldsymbol{C}$, which is the composite function.

$$\begin{tikzcd} X_A \arrow[r, "A"] \arrow[rr, bend right, "C"'] & Y_A \arrow[r, "B"] & Y_C \end{tikzcd} \quad \Rightarrow \quad X_C \xrightarrow{C} Y_C$$

In $\boldsymbol{C}$, we have $X_C = X_A$, of course.

Thus, all the inputs of $\boldsymbol{C}$ are from X_A, that is, the domain of $\boldsymbol{C}$ is the domain of $\boldsymbol{A}$.
What then about the range?

The range of $\boldsymbol{C}$ can be the range of $\boldsymbol{B}$.   What do we mean by 'can be' though?
It means that the range of $\boldsymbol{C}$ can be other than the range of $\boldsymbol{B}$, too.
In other words, it is not always the case where $\boldsymbol{Y_C} = \boldsymbol{Y_B}$.

That's because the range of $\boldsymbol{A}$ can be a subset of the domain of $\boldsymbol{B}$, and thus, can be smaller than the domain of $\boldsymbol{B}$.

And in that case, the range of $\boldsymbol{C}$ can also, be a subset of the range of $\boldsymbol{B}$.

4.1. Composite Functions 2

(Note that this section is for you, only if you really have to look at composite functions very closely.)

What if we want to get a new function combining three functions?

Two of the three get combined first, then a new function gets made, and is of course, a composite function.

And next, the new function gets put into the other, then another new function gets made, and is the composite function we want. So in the composite function, the domain is the domain of one of the two functions combined first.

Suppose for instance, mixing three functions ***A***, ***B***, and ***C***, we make a new function ***F***.
Suppose also, we begin with mixing ***A*** into ***B***.

Then first, putting ***A*** into ***B***, we get a new function called ***D***, for instance.
So ***D*** is a composite function, where ***A*** blends into ***B***.

In turn, putting the function ***D*** into the function ***C***, we get another new function, which is the function ***F*** stated above.
In short, putting ***A*** into ***B***, we get ***D***, then putting ***D*** into ***C***, we get ***F***.

Suppose now again, X_A is the domain of ***A***, Y_A is the range of ***A***, X_B is the domain of ***B***, Y_B is the range of ***B***, X_C is the domain of ***C***, and Y_C is the range of ***C***.

$$X_A \xrightarrow{A} Y_A \qquad X_B \xrightarrow{B} Y_B \qquad X_C \xrightarrow{C} Y_C$$

Then first, putting ***A*** into ***B***, we need to have $Y_A \subseteq X_B$. That is, Y_A is a subset of X_B.

In other words, the range of ***A*** is equal to or a part of the domain of ***B***.

$$\boldsymbol{X_A \xrightarrow{A} (Y_A \subseteq X_B) \xrightarrow{B} Y_B \qquad X_C \xrightarrow{C} Y_C}$$

Then, we get a new function ***D***, which is a composite function where ***A*** blends into ***B***.

$$\begin{tikzcd} \boldsymbol{X_A} \arrow[r, "\boldsymbol{A}"] \arrow[rr, bend right, "\boldsymbol{D}"'] & \boldsymbol{Y_A} \arrow[r, "\boldsymbol{B}"] & \boldsymbol{Y_D} \end{tikzcd} \quad \Rightarrow \quad \boldsymbol{X_D \xrightarrow{D} Y_D} \quad \text{In } \boldsymbol{D}\text{, we have } \boldsymbol{X_D = X_A}\text{, of course.}$$

Thus, the domain of ***D*** is the domain of ***A***. So ***D*** is a function from the domain of ***A*** to the range of ***D***, which can be the range of ***B***, too. Therefore, we get $\boldsymbol{X_D = X_A}$, and it can be the case where $\boldsymbol{Y_D = Y_B}$. And next, we need to blend ***D*** into ***C***.

$$\boldsymbol{X_A \xrightarrow{D} Y_D \qquad X_C \xrightarrow{C} Y_C}$$

So putting ***D*** into ***C***, we need to have $\boldsymbol{Y_D \subseteq X_C}$.

In other words, the range of ***D*** is a subset of the domain of ***C***.

$$\boldsymbol{X_A \xrightarrow{D} (Y_D \subseteq X_C) \xrightarrow{C} Y_C}$$

Then, we get the new function ***F***, which is a composite function where ***D*** blends into ***C***.

$$\begin{tikzcd} \boldsymbol{X_A} \arrow[r, "\boldsymbol{D}"] \arrow[rr, bend right, "\boldsymbol{F}"'] & \boldsymbol{Y_D} \arrow[r, "\boldsymbol{C}"] & \boldsymbol{Y_F} \end{tikzcd} \quad \Rightarrow \quad \boldsymbol{X_F \xrightarrow{F} Y_F} \quad \text{In } \boldsymbol{F}\text{, we have } \boldsymbol{X_F = X_A}\text{, of course.}$$

Thus, the domain of ***F*** is the domain of ***A***. So ***F*** is a function from the domain of ***A*** to the range of ***F***, which can be the range of ***C***, too. Therefore, we get $\boldsymbol{X_F = X_A}$, and it can be the case where $\boldsymbol{Y_F = Y_C}$.

Now, we know that the function ***D*** is a composite function where ***A*** blends into ***B***, and that the function ***F*** is a composite function where ***D*** blends into ***C***.
So the function ***F*** is the composite function where all the three functions ***A***, ***B***, and ***C*** blended together, and thus, in short, is a blend of ***A***, ***B***, and ***C***.

So we can make a new function blending two functions.
Blending the two functions though, we blend one into the other.
And the range of the function blends into is a subset of the domain of the other.

Then, we get a new function, which is called a composite function, which is a blend of two functions.

And repeating the same process with the composite function made and another function, we can get another composite function, which is a blend of three functions.

And of course, the range of the composite function made earlier is a subset of the domain of the other function. Also, the domain of the composite function newly made is the domain of the composite function made earlier.

So blending two functions the way above, we can get a composite function.
How then do we actually do such blending to make a composite function?

Defining a composite function, we make a composite function.
So let's make one now.

A composite function is a mix of two or more function, so such a definition begins with at least two function definitions. So for instance, it can go as follows.

$\boldsymbol{X_f}$, $\boldsymbol{Y_f}$, $\boldsymbol{X_g}$, and $\boldsymbol{Y_g}$ are sets of real numbers, where $\boldsymbol{Y_f \subseteq X_g}$,

$\boldsymbol{f}$: $\boldsymbol{X_f \longrightarrow Y_f}$, $\boldsymbol{g}$: $\boldsymbol{X_g \longrightarrow Y_g}$, $\boldsymbol{y = f(x)}$, $\boldsymbol{z = g(y)}$, and $\boldsymbol{z = h(x)}$.

(Note that a set of real numbers is a set of not all but some real numbers. So $\boldsymbol{X_f}$, $\boldsymbol{Y_f}$, $\boldsymbol{X_g}$, and $\boldsymbol{Y_g}$ can be different sets.)

The definition above is saying that $\boldsymbol{h}$ is a composite function where $\boldsymbol{f}$ blends into $\boldsymbol{g}$.
How?

Let's now take a closer look. So let's break apart a composite function, and see in detail how it gets made, and of course, how it works, too.

We have these:

X_f, Y_f, X_g, and Y_g are sets of real numbers, where $Y_f \subseteq X_g$,

$f: X_f \longrightarrow Y_f$, $g: X_g \longrightarrow Y_g$, $y = f(x)$, $z = g(y)$, and $z = h(x)$.

First, since Y_f is the range of f, all the numbers in Y_f are all the outputs of f.

And next, once f has blended into g, we get $X_g = Y_f$, that is, the domain of g changes to the range of f, that is, Y_f, so all the outputs of f are all the inputs of g.

More specifically, each time an output is made in the function f, the output is an input of the function g. So we can put it the way as follows.

We know y gets every output of f.
Also, since $y = f(x)$, $f(x)$ gets each output of f, too.

Next, we have $z = g(y)$, so the expression of g is an expression in terms of y as $2y + 1$.

So for instance, we can have $z = g(y) = 2y + 1$ for $y \in Y_f$.

And we have $y = f(x)$, too, so we can set $z = g(f(x))$ since $z = g(y)$.

That is, setting $z = g(f(x))$, read as g of f of x, we put $f(x)$ into y in $z = g(y)$.

And $g(f(x))$ means that each input of g is each output of f.

And since $y = f(x)$, the expression of f is an expression in terms of x as $x + 1$.
So for instance, we can have $y = f(x) = x + 1$ for $x \in X_f$.

Then, setting $z = g(f(x))$, we can put $(x + 1)$ into y in $z = g(y)$, since $f(x) = x + 1$.

So we get $z = g(y) = 2y + 1 \Rightarrow z = 2y + 1 = 2(x + 1) + 1 = 2x + 3 \Rightarrow z = 2x + 3$, which is an expression in terms of x, and can be used as an expression of a function.

Now, we have $z = h(x)$, which indicates the expression of h is expressed in terms of x.
And we have $z = 2x + 3$, too. So we can see that $z = h(x) = 2x + 3$.

And we know $\boldsymbol{x}$ gets its value in the domain of $\boldsymbol{f(x)}$, and thus, so does $\boldsymbol{x}$ in $z = \boldsymbol{h(x)}$.

That is, $\boldsymbol{x}$ in $z = \boldsymbol{h(x)}$ gets its value in the domain of $\boldsymbol{f}$, too, because $\boldsymbol{x}$ was from $(\boldsymbol{x} + \mathbf{1})$, which is the expression of $\boldsymbol{f}$. So the domain of $\boldsymbol{h}$ is the domain of $\boldsymbol{f}$, and thus, is X_f.

So putting threads together, we can say that $\boldsymbol{h}$ is a function that has the domain of $\boldsymbol{f}$. Thus, $\boldsymbol{h}$ is a composite function where $\boldsymbol{f}$ blends into $\boldsymbol{g}$, and is as follows.

$z = \boldsymbol{h(x)} = \mathbf{2}\boldsymbol{x} + \mathbf{3}$ for $\boldsymbol{x} \in X_f$.

And of course, $X_h = X_f$.   And we can put $\boldsymbol{h}$ the way below, too.

$\boldsymbol{h}$: $X_f \longrightarrow Y_h$, $\boldsymbol{y} = \boldsymbol{f(x)} = \boldsymbol{x} + \mathbf{1}$, $z = \boldsymbol{g(y)} = \mathbf{2}\boldsymbol{y} + \mathbf{1}$, and $z = \boldsymbol{h(x)}$.

Now, you may want to take a little break, and then go back to the beginning of this example, and go over the entire processes. A function is not an easy game, much less a composite function.

Thus, what's happening in $\boldsymbol{h}$ is as good as what's happening sequentially in $\boldsymbol{f}$ and $\boldsymbol{g}$.

It begins with $\boldsymbol{f}$, of course, which is the function blends into.   How?

Example is the best teacher, isn't it? So let's get back to the example. We have these:

$z = \boldsymbol{h(x)} = \mathbf{2}\boldsymbol{x} + \mathbf{3}$ for $\boldsymbol{x} \in X_f$.

$\boldsymbol{y} = \boldsymbol{f(x)} = \boldsymbol{x} + \mathbf{1}$ for $\boldsymbol{x} \in X_f$, and *after blending*, we get $z = \boldsymbol{g(y)} = \mathbf{2}\boldsymbol{y} + \mathbf{1}$ for $\boldsymbol{y} \in Y_f$.

So to begin with, suppose in $\boldsymbol{h}$, $\boldsymbol{x}$ gets 1 as an input from the domain X_f.

Then, z gets the output, which is $2 \cdot \mathbf{1} + 3 = \mathbf{5}$, which belongs to the range of $\boldsymbol{h}$, of course.

And suppose next, in $\boldsymbol{f}$, $\boldsymbol{x}$ gets 1 as an input from its domain, which is X_f.

Then, in $\boldsymbol{f}$, $\boldsymbol{y}$ gets the output, which is $\mathbf{1} + 1 = \mathbf{2}$, which belongs to the range Y_f.

And in turn, the output 2 is used as an input of $\boldsymbol{g}$, since the domain of $\boldsymbol{g}$ is the range of $\boldsymbol{f}$.

Thus, in $z = g(y)$, y gets 2, the output just has been made in $\boldsymbol{f}$, so z gets the output for 2, the value y has, and the output is $2 \cdot \mathbf{2} + 1 = \mathbf{5}$, which is an element in the range of $\boldsymbol{h}$.

So the major blending happens the way as follows.

The expression of $\boldsymbol{f}$ blends into the expression of $\boldsymbol{g}$ to result in the expression of $\boldsymbol{h}$.

That is, having ($y = x + 1$) blend into $z = 2y + 1$, we get $z = 2(x + 1) + 1 = 2x + 3$.

So we can put a bit differently, the definition of the composite function $\boldsymbol{h}$ the way below.

> X_f, Y_f, X_g, and Y_g are sets of real numbers, where $Y_f \subseteq X_g$,
> $f: X_f \longrightarrow Y_f$, $g: X_g \longrightarrow Y_g$, $y = f(x)$, $y = g(x)$, and $y = h(x) = g\{f(x)\}$.

The definition above is of course, no other than the definition stated earlier, which is

> X_f, Y_f, X_g, and Y_g are sets of real numbers, where $Y_f \subseteq X_g$,
> $f: X_f \longrightarrow Y_f$, $g: X_g \longrightarrow Y_g$, $y = f(x)$, $z = g(y)$, and $z = h(x)$.

Both look quite different though, because the variables are different, and we don't even see z in the first of the two definitions above. How then are both the same?

In functions, what matters regarding variables is not only their names but their roles, too. So we want to make sure *which has what value in kind.*

Thus, we can put the definition above the way below, too.

> X_f, Y_f, X_g, and Y_g are sets of real numbers, where $Y_f \subseteq X_g$,
> $f: X_f \longrightarrow Y_f$, $g: X_g \longrightarrow Y_g$, $y = f(x)$, $v = g(u)$, and $t = h(x) = g\{f(x)\}$.

What's important regarding variables in the definition above is made of two facts.

• One is that *after blending*, $\boldsymbol{u}$ gets values from $\boldsymbol{Y_f}$, which is the range of $\boldsymbol{f}$, so when blending, we can set $\boldsymbol{u = f(x)}$, and therefore, we get $\boldsymbol{g\{f(x)\}}$, which is $\boldsymbol{g(f)}$ for short.

So what $\boldsymbol{g(f)}$ is saying is
that we put the expression of $\boldsymbol{f}$ into every $\boldsymbol{u}$ in the expression of $\boldsymbol{g}$.

• And the other fact is that in $\boldsymbol{h(x)}$ and $\boldsymbol{f(x)}$ both,
the variable $\boldsymbol{x}$ gets values from $\boldsymbol{X_f}$, which is the domain of $\boldsymbol{f(x)}$.
So $\boldsymbol{h}$ and $\boldsymbol{f}$ share the same domain, which is the domain of $\boldsymbol{f}$.

That's because in $\boldsymbol{h(x)}$, $\boldsymbol{x}$ is from $\boldsymbol{f(x)}$, which is the function blends into.
And the first fact above explains it.

So we can try blending $\boldsymbol{f(x)}$ into $\boldsymbol{g(u)}$ to get the composite function $\boldsymbol{h(x)}$. Not quite clear?

A function is not an easy game, still less a composite function.
So let's see first, what's happening in the blending processes in the definition as follows.

$\boldsymbol{X_f}$, $\boldsymbol{Y_f}$, $\boldsymbol{X_g}$, and $\boldsymbol{Y_g}$ are sets of real numbers, where $\boldsymbol{Y_f \subseteq X_g}$,
$\boldsymbol{f: X_f \longrightarrow Y_f}$, $\boldsymbol{g: X_g \longrightarrow Y_g}$, $\boldsymbol{y = f(x)}$, $\boldsymbol{y = g(x)}$, and $\boldsymbol{y = h(x) = g\{f(x)\}}$.

To begin with, we had $\boldsymbol{y = f(x) = x + 1}$ for $\boldsymbol{x \in X_f}$, and $\boldsymbol{z = g(y) = 2y + 1}$ for $\boldsymbol{y \in Y_f}$.

And we now have $\boldsymbol{y = f(x)}$ for $\boldsymbol{x \in X_f}$, and *after blending*, $\boldsymbol{y = g(x)}$ for $\boldsymbol{x \in Y_f}$.

So we get $\boldsymbol{y = f(x) = x + 1}$ for $\boldsymbol{x \in X_f}$, and $\boldsymbol{y = g(x) = 2x + 1}$ for $\boldsymbol{x \in Y_f}$.

And that's not it, we have $\boldsymbol{y = h(x) = g\{f(x)\}}$, too. And this is the place where the major blending begins. So the question is, what do we mean by $\boldsymbol{g\{f(x)\}}$?

Putting $\boldsymbol{f(x)}$ into $\boldsymbol{x}$ in $\boldsymbol{g(x)}$, we get $\boldsymbol{g(f(x))}$, and making it more legible, we can put it this way $\boldsymbol{g\{f(x)\}}$. And $\boldsymbol{g\{f(x)\}}$ means that each input of $\boldsymbol{g}$ is each output of $\boldsymbol{f}$.

Now, we have $\boldsymbol{y = f(x) = x + 1}$, and $\boldsymbol{y = g(x) = 2x + 1}$.

So putting $\boldsymbol{f(x)}$ into $\boldsymbol{x}$ in $\boldsymbol{y = g(x)}$, we get $\boldsymbol{y = g\{f(x)\} = 2f(x) + 1}$.

And we have $\boldsymbol{f(x) = x + 1}$, so we get $\boldsymbol{y = 2f(x) + 1 = 2(x + 1) + 1 = 2x + 3}$, which is an expression in terms of $\boldsymbol{x}$, but is certainly not the expression of the function $\boldsymbol{y = g(x)}$. What expression then is it?

We have not only $\boldsymbol{y = g(x)}$. But we have $\boldsymbol{y = h(x)}$, too.

So we get $\boldsymbol{y = h(x) = 2x + 3}$ for $\boldsymbol{x \in X_f}$. How is that $\boldsymbol{x \in X_f}$, though?

That's because $\boldsymbol{x}$ is from $(\boldsymbol{x + 1})$, which is the expression of $\boldsymbol{f}$, of which the domain is $\boldsymbol{X_f}$.

Now, let's see next, what's happening in the blending procedure in the definition below.

$\boldsymbol{X_f}$, $\boldsymbol{Y_f}$, $\boldsymbol{X_g}$, and $\boldsymbol{Y_g}$ are sets of real numbers, where $\boldsymbol{Y_f \subseteq X_g}$,
$\boldsymbol{f: X_f \longrightarrow Y_f}$, $\boldsymbol{g: X_g \longrightarrow Y_g}$, $\boldsymbol{y = f(x)}$, $\boldsymbol{v = g(u)}$, and $\boldsymbol{t = h(x) = g\{f(x)\}}$.

To begin with, we had
$\boldsymbol{y = f(x) = x + 1}$ for $\boldsymbol{x \in X_f}$, and $\boldsymbol{z = g(y) = 2y + 1}$ for $\boldsymbol{y \in Y_f}$.

And we now have $\boldsymbol{y = f(x)}$ for $\boldsymbol{x \in X_f}$, and *after blending*, $\boldsymbol{v = g(u)}$ for $\boldsymbol{u \in Y_f}$, so we get

$\boldsymbol{y = f(x) = x + 1}$ for $\boldsymbol{x \in X_f}$, and $\boldsymbol{v = g(u) = 2u + 1}$ for $\boldsymbol{u \in Y_f}$.

And of course, we have $\boldsymbol{t = h(x) = g\{f(x)\}}$, too.
So again, this is the place where the major blending happens.

So the question is again, what do we mean by $\boldsymbol{g\{f(x)\}}$?

Putting $f(x)$ into u in $g(u)$, we get $g\{f(x)\}$, and $g\{f(x)\}$ means that each input of g is each output of f.
Now, we have $y = f(x) = x + 1$, and $v = g(u) = 2u + 1$.

So putting $f(x)$ into u in $v = g(u)$, we get $v = g\{f(x)\} = 2f(x) + 1$.

And we have $f(x) = x + 1$, so we get $v = 2f(x) + 1 = 2(x + 1) + 1 = 2x + 3$, which is an expression in terms of x, which is clearly not the expression of the function $v = g(u)$. What expression is it then?

We have $t = h(x)$, too. So we get $t = h(x) = 2x + 3$ for $x \in X_f$.

And x in $h(x)$ is from $f(x)$, of which the domain is X_f, so in $h(x)$, we have $x \in X_f$.

And t gets outputs of the function h, and thus, is just the output variable in h.
What if we like to change to w, for instance, the name of the input variable in $f(x)$?

We can do so, and yet, we want to change the name of the input variable in the function $h(x)$, too. Then, the definition above simply changes its look, and will be the one below.

> X_f, Y_f, X_g, and Y_g are sets of real numbers, where $Y_f \subseteq X_g$,
> $f: X_f \longrightarrow Y_f$, $g: X_g \longrightarrow Y_g$, $y = f(w)$, $t = g(u)$, and $t = h(w) = g\{f(w)\}$.

So the name of the input variable in the composite function is the same as that of the input variable in the function blends into.

That's because getting the expression of the composite function, we put into each and every input variable in the expression of the other function, the expression of the function blends into.
In short, the input variable of a composite function is that of the function that blends into.
That is, both names have to be the same.
So a composite function is a blend of functions.
It depends on however, the way functions get blended together.
That is, the blending sequence matters.

4.2. Composite Functions 3

(Note again that this section is for you, only if you really have to look at composite functions very closely.)

How then can we specify the blending sequence making a composite function?

Defining a composite function, we have another way of doing it.

We can use an operator as + or – when making a composite function.

As the operator, we use a dot •, which is normally a bit bigger than the tiny one · often used in a multiplication as in $\mathbf{5 \cdot 3 = 15}$. We can call it a composite-function-operator.

> So for instance, a composite function $g\{f(x)\}$ can be put in $g \bullet f(x)$, which is simply read as g dot f of x. And thus, we can set $g \bullet f(x) = g\{f(x)\}$.
>
> Or briefly, we can put it this way, too: $g \bullet f = g(f)$.

So we can use such an operator making a definition of a composite function.

And for instance, we can define a composite function in such a way as follows.

> X_f, Y_f, X_g, Y_g, and W are sets of real numbers, where $Y_f \subseteq X_g$,
>
> f: $X_f \longrightarrow Y_f$, g: $X_g \longrightarrow Y_g$, and $g \bullet f : X_f \longrightarrow W$.

(And of course, it can be the case where $\boldsymbol{W} = \boldsymbol{Y_g}$.)

A bit more specifically, we can put it the way below, too.

> $\boldsymbol{X_f}$, $\boldsymbol{Y_f}$, $\boldsymbol{X_g}$, and $\boldsymbol{Y_g}$ are sets of real numbers, where $\boldsymbol{Y_f} \subseteq \boldsymbol{X_g}$,
>
> $\boldsymbol{f}$: $\boldsymbol{X_f} \longrightarrow \boldsymbol{Y_f}$, $\boldsymbol{g}$: $\boldsymbol{X_g} \longrightarrow \boldsymbol{Y_g}$, $\boldsymbol{y = f(x)}$, and $\boldsymbol{g \bullet f : x \longrightarrow g\{f(x)\}}$ or just $\boldsymbol{g(f)}$.

And we can notice that $\boldsymbol{g \bullet f}$ can be taken for a function designator, that is, a name.

So specifying the input variable in the composite function $\boldsymbol{g \bullet f}$, we can put it this way, too: $\boldsymbol{(g \bullet f)(x)}$, which is however, no other than $\boldsymbol{g \bullet f(x)}$.

It's a bit bulky though, to carry around with, so we often give a single name to it, and for instance, we can set $\boldsymbol{h = g \bullet f}$. So in that case, we can also set $\boldsymbol{h(x) = (g \bullet f)(x)}$.

And for instance, we can define a composite function in such a way as follows.

> $\boldsymbol{X_f}$, $\boldsymbol{Y_f}$, $\boldsymbol{X_g}$, $\boldsymbol{Y_g}$, and $\boldsymbol{Y_h}$ are sets of real numbers, where $\boldsymbol{Y_f} \subseteq \boldsymbol{X_g}$,
>
> $\boldsymbol{f}$: $\boldsymbol{X_f} \longrightarrow \boldsymbol{Y_f}$, $\boldsymbol{g}$: $\boldsymbol{X_g} \longrightarrow \boldsymbol{Y_g}$, $\boldsymbol{h}$: $\boldsymbol{X_f} \longrightarrow \boldsymbol{Y_h}$, and $\boldsymbol{h = g \bullet f}$.

So we can see that

$$\boldsymbol{X_f \xrightarrow{f} Y_f \xrightarrow{g} Y_h \quad \Rightarrow \quad X_f \xrightarrow{g \bullet f} Y_h}.$$

And of course, it can be the case where $\boldsymbol{Y_h = Y_g}$.
And it is worth to note that it is usually the case where $\boldsymbol{g \bullet f \neq f \bullet g}$.

That is because first

In $\boldsymbol{g \bullet f}$, the function blends into is $\boldsymbol{f}$, so the domain is the domain of $\boldsymbol{f}$.

In $\boldsymbol{f \bullet g}$ though, the function blends into is $\boldsymbol{g}$, so the domain is the domain of $\boldsymbol{g}$.

And next, even if the domains are identical for both functions, the expressions can be still different. So it is usually the case where $\boldsymbol{g \bullet f \neq f \bullet g}$.

Why usually, though? Can it be then the case where $\boldsymbol{g \bullet f = f \bullet g}$, too?

Yes, it can. Suppose for instance, $\boldsymbol{y = f(x) = x}$ for $\boldsymbol{x}$ real, and $\boldsymbol{y = g(x) = x^2 + 1}$ for $\boldsymbol{x}$ real.

Then, we get $\boldsymbol{g \bullet f(x) = x^2 + 1}$, and also, $\boldsymbol{f \bullet g(x) = x^2 + 1}$. How?

We have $\boldsymbol{f(x) = x}$, so we can just set $\boldsymbol{f = x}$.
And we have $\boldsymbol{g \bullet f(x) = g(f)}$, and $\boldsymbol{g(x) = x^2 + 1}$, too.

So we get $\boldsymbol{g \bullet f(x) = g(f) = f^2 + 1 = x^2 + 1}$, since $\boldsymbol{f = x}$. And thus, $\boldsymbol{g \bullet f(x) = x^2 + 1}$ for $\boldsymbol{x}$ real.

And by the same token, we ge $\boldsymbol{f \bullet g(x) = f(g)}$, where $\boldsymbol{f(x) = x}$, and $\boldsymbol{g = x^2 + 1}$.

So we get $\boldsymbol{f \bullet g(x) = f(g) = g = x^2 + 1}$. And thus, $\boldsymbol{g \bullet f(x) = x^2 + 1}$ for $\boldsymbol{x}$ real.

Now, the function $\boldsymbol{y = f(x) = x}$ is an identity function. So we can notice that if either of the two functions is an identity function, we can get $\boldsymbol{g \bullet f = f \bullet g}$.

Suppose this time, $\boldsymbol{y = f(x) = x + 1}$, and $\boldsymbol{y = g(x) = x^2}$.

Then, we get $\boldsymbol{g \bullet f(x) = g(f) = f^2 = (x + 1)^2}$, but we get $\boldsymbol{f \bullet g(x) = f(g) = g + 1 = x^2 + 1}$.

• Now, let's take a look at how some examples work.

First, assuming $\boldsymbol{y = f(x) = x + 1}$ for $\boldsymbol{1 \leq x \leq 2}$, and $\boldsymbol{y = g(x) = 2x + 1}$ for $\boldsymbol{2 \leq x \leq 3}$, let's find $\boldsymbol{y = h(x) = (g \bullet f)(x)}$.

Since $\boldsymbol{h}$ is a function, too, finding $\boldsymbol{h}$, we want to get the expression, along with the domain.

Now, to begin with, assuming $\boldsymbol{X}$ is a domain, and $\boldsymbol{Y}$ is a range, we can put it this way:

$$\begin{tikzcd} X_f \arrow[r, "f"] \arrow[rr, bend right, "h"'] & Y_f \arrow[r, "g"] & Y_h \end{tikzcd} \Rightarrow X_f \xrightarrow{g\bullet f} Y_h$$. And of course, in $\boldsymbol{h}$, $X_h = X_f$, the domain of $\boldsymbol{f}$.

So the domain of $\boldsymbol{h}$ is $\mathbf{1 \le x \le 2}$. And next, we want to check to see if the range of the function blends into is a subset of the domain of the other function. That is, the range of the function $\boldsymbol{f}$ has to be equal to or a part of the domain of the function $\boldsymbol{g}$.

So checking it, we get $\mathbf{1 \le x \le 2 \Rightarrow 2 \le x + 1 \le 3 \Rightarrow 2 \le y \le 3}$, which is the range of $\boldsymbol{f}$.

We know domains and ranges are sets of numbers, and the domain of $\boldsymbol{g}$ is $\mathbf{2 \le x \le 3}$.

So the range of $\boldsymbol{f}$ is a set of all numbers from 2 and 3, and so is the domain of $\boldsymbol{g}$.

The range of $\boldsymbol{f}$ is thus, equal to the domain of $\boldsymbol{g}$, so $\boldsymbol{h}$ can be defined in the domain of $\boldsymbol{f}$.

Now, we have $y = f(x) = x + 1$, $y = g(x) = 2x + 1$, and $y = h(x) = (g\bullet f)(x) = g(f)$.

So we ge $h(x) = g(f) = 2f + 1$, since $g(x) = 2x + 1$.

Thus, $h(x) = 2f + 1 = 2(x + 1) + 1 = 2x + 3$, since $f(x) = x + 1 \Rightarrow f = x + 1$.

Next, the domain of $\boldsymbol{h}$ is the domain of $\boldsymbol{f}$, so the domain is $\mathbf{1 \le x \le 2}$.

And we can see the domain has no problem with the expression where $\mathbf{2x + 3}$.

Next, let's see if the range of $\boldsymbol{h}$ is the same as the range of $\boldsymbol{g}$.

We have $y = g(x) = 2x + 1$ for $2 \le x \le 3$, and $y = h(x) = 2x + 3$ for $1 \le x \le 2$.

So beginning with the range of $\boldsymbol{g}$, we get $2 \le x \le 3 \Rightarrow 4 \le 2x \le 6 \Rightarrow 5 \le 2x + 1 \le 7$.

And next, finding the range of $\boldsymbol{h}$, we get $1 \le x \le 2 \Rightarrow 2 \le 2x \le 4 \Rightarrow 5 \le 2x + 3 \le 7$.

So both are the same, and in this case, we can say that the composite function is a function from the domain of the function blends into to the range of the other function.

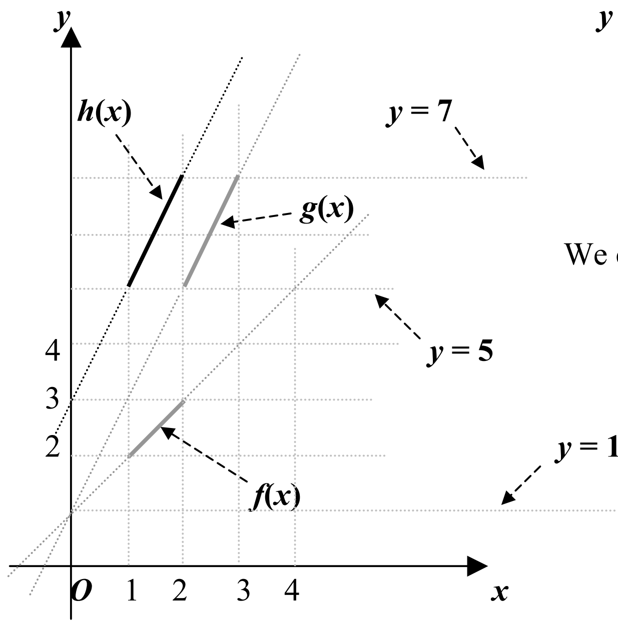
Let's now put in a graph all the functions $\boldsymbol{f}$, $\boldsymbol{g}$, and $\boldsymbol{h}$.

Fig. 5.0

$y = f(x) = x + 1$ for $1 \le x \le 2$

$y = g(x) = 2x + 1$ for $2 \le x \le 3$

$y = h(x) = (g\bullet f)(x) = 2x + 3$ for $1 \le x \le 2$



We can actually see $\boldsymbol{h}$ has all the inputs of $\boldsymbol{f}$ and all the outputs of $\boldsymbol{g}$.

Now, let's for next instance, find $y = h(x) = (g\bullet f)(x)$ assuming $y = f(x) = 2x$ for $1 \le x \le 2$, and $y = g(x) = 2x + 1$ for $2 \le x \le 4$. Then first, we may want to put it this way:

$$\begin{tikzcd} X_f \arrow[r, "f"] \arrow[rr, bend right, "h"'] & Y_f \arrow[r, "g"] & Y_h \end{tikzcd} \quad \Rightarrow \quad X_f \xrightarrow{g\bullet f} Y_h.$$ And of course, we have $X_h = X_f$.

So the domain of $\boldsymbol{h}$ is $1 \le x \le 2$. Next, checking to see if the range of $\boldsymbol{f}$ is a subset of the domain of $\boldsymbol{g}$, we get $1 \le x \le 2 \Rightarrow 2 \le 2x \le 4 \Rightarrow 2 \le y \le 4$, which is the range of $\boldsymbol{f}$.

We know the domain of $\boldsymbol{g}$ is $2 \le x \le 4$.

So the range of $\boldsymbol{f}$ equals the domain of $\boldsymbol{g}$, and thus, $\boldsymbol{h}$ can be defined in the domain of $\boldsymbol{f}$.

Now, we have $y = f(x) = 2x$, $y = g(x) = 2x + 1$, and $y = h(x) = (g \bullet f)(x) = g(f)$.

So we get $h(x) = g(f) = 2f + 1 = 2(2x) + 1 = 4x + 1$, which has no problem with the domain given. Thus, we get $y = h(x) = 4x + 1$ for $1 \le x \le 2$.

Next, let's see if the range of h is the same as the range of g.

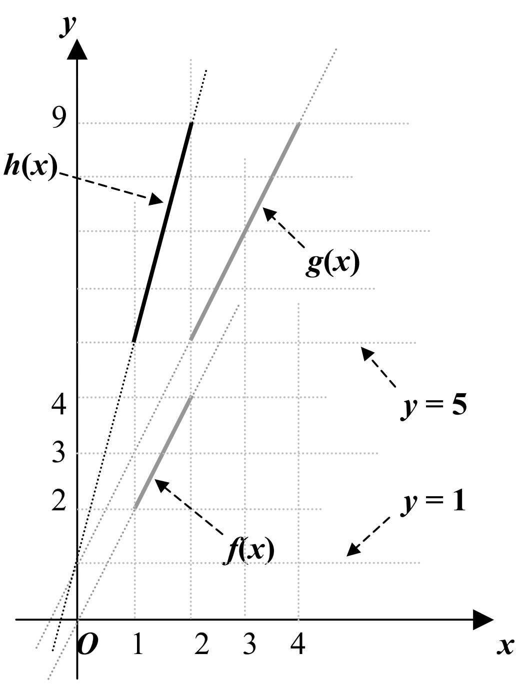
Beginning with the range of g, we get $2 \le x \le 4 \Rightarrow 4 \le 2x \le 8 \Rightarrow 5 \le 2x + 1 \le 9$.

And next, finding the range of h, we get $1 \le x \le 2 \Rightarrow 4 \le 4x \le 8 \Rightarrow 5 \le 4x + 1 \le 9$.

So in this case, too, h is a composite function from the domain of the function blends into to the range of the other function.

Let's now put in a graph all the functions f, g, and h.

Fig. 5.1



$y = f(x) = 2x$ for $1 \le x \le 2$

$y = g(x) = 2x + 1$ for $2 \le x \le 4$

$y = h(x) = 4x + 1$ for $1 \le x \le 2$

In the function h, all the inputs are those of f, and all the outputs are those of g.

Now, let's for next instance, assuming $y = f(x) = x + 1$ for $1 \le x \le 2$, and $y = g(x) = x^2$ for $2 \le x \le 3$, find $y = h(x) = (g \bullet f)(x)$. To begin with, we can put it this way:

$$\begin{tikzcd} X_f \arrow[r, "f"] \arrow[rr, bend right, "h"'] & Y_f \arrow[r, "g"] & Y_h \end{tikzcd} \quad \Rightarrow \quad X_f \xrightarrow{g\bullet f} Y_h.$$ In $\boldsymbol{h}$, we have $\boldsymbol{X_h = X_f}$, the domain of $\boldsymbol{f}$.

So the domain of $\boldsymbol{h}$ is $\mathbf{1 \le x \le 2}$. And next, checking to see if the range of $\boldsymbol{f}$ is a subset of the domain of $\boldsymbol{g}$, we get $\mathbf{1 \le x \le 2 \Rightarrow 2 \le x + 1 \le 3 \Rightarrow 2 \le y \le 3}$, which is the range in $\boldsymbol{f}$.

We know the domain in $\boldsymbol{g}$ is $\mathbf{2 \le x \le 3}$.

So the range in $\boldsymbol{f}$ equals the domain in $\boldsymbol{g}$, and thus, $\boldsymbol{h}$ can be defined in the domain of $\boldsymbol{f}$.

Now, we have $\boldsymbol{y = f(x) = x + 1}$, $\boldsymbol{y = g(x) = x^2}$, and $\boldsymbol{y = h(x) = (g\bullet f)(x) = g(f)}$.

So we get $\boldsymbol{h(x) = g(f) = f^2 = (x + 1)^2}$, which has no problem with the domain $\mathbf{1 \le x \le 2}$.

Thus, we get $\boldsymbol{y = h(x) = (x + 1)^2}$ for $\mathbf{1 \le x \le 2}$. Let's next, check the ranges of $\boldsymbol{h}$ and $\boldsymbol{g}$.

Beginning with the range of $\boldsymbol{g}$, we get $\mathbf{2 \le x \le 3 \Rightarrow 4 \le x^2 \le 9}$.

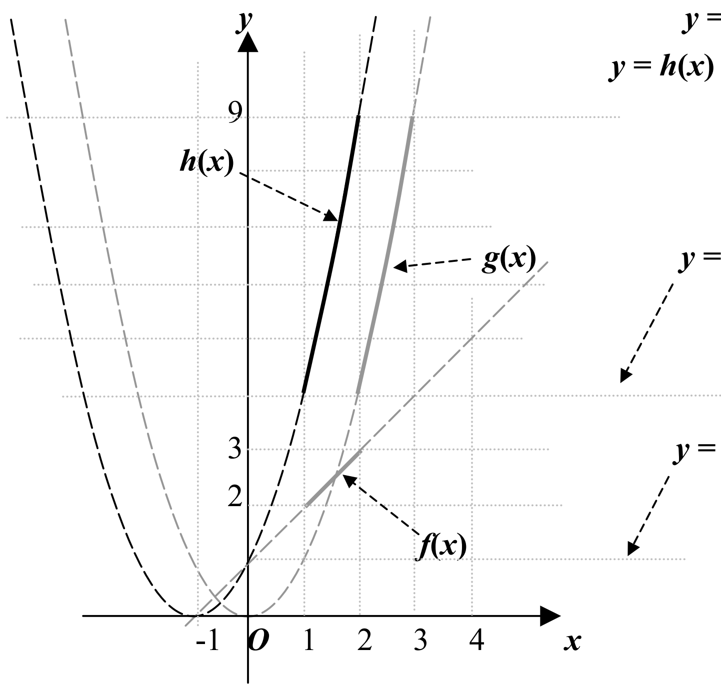
And next, finding the range of $\boldsymbol{h}$, we get $\mathbf{1 \le x \le 2 \Rightarrow 2 \le x + 1 \le 3 \Rightarrow 4 \le (x + 1)^2 \le 9}$.

So $\boldsymbol{h}$ is a composite function from the domain of $\boldsymbol{f}$ to the range of $\boldsymbol{g}$.

Let's now have a look at the graph where all the functions $\boldsymbol{f}$, $\boldsymbol{g}$, and $\boldsymbol{h}$ are put.

Fig. 5.2

$\boldsymbol{y = f(x) = x + 1}$ for $\mathbf{1 \le x \le 2}$
$\boldsymbol{y = g(x) = x^2}$ for $\mathbf{2 \le x \le 3}$
$\boldsymbol{y = h(x) = (x + 1)^2}$ for $\mathbf{1 \le x \le 2}$



Now, let's for next instance, find $\boldsymbol{y = h(x) = (g \bullet f)(x)}$ assuming $\boldsymbol{y = f(x) = x^2}$ for $\boldsymbol{x \geq 0}$, and $\boldsymbol{y = g(x) = \sqrt{x}}$ for $\boldsymbol{x \geq 0}$. Then first, we can put it this way:

$$\begin{tikzcd} X_f \arrow[r, "f"] \arrow[rr, bend right, "h"'] & Y_f \arrow[r, "g"] & Y_h \end{tikzcd} \quad \Rightarrow \quad X_f \xrightarrow{g \bullet f} Y_h$$

In $\boldsymbol{h}$, we have $\boldsymbol{X_h = X_f}$, the domain of $\boldsymbol{f}$.

So the domain of $\boldsymbol{h}$ is $\boldsymbol{x \geq 0}$. And next, checking to see if the range of $\boldsymbol{f}$ is a subset of the domain of $\boldsymbol{g}$, we get $\boldsymbol{x \geq 0 \Rightarrow x^2 \geq 0 \Rightarrow y \geq 0}$, which is the range of $\boldsymbol{f}$.

We know the domain in $\boldsymbol{g}$ is $\boldsymbol{x \geq 0}$.
So the range of $\boldsymbol{f}$ equals the domain of $\boldsymbol{g}$, and thus, $\boldsymbol{h}$ can be defined in the domain of $\boldsymbol{f}$.

Now, we have $\boldsymbol{y = f(x) = x^2}$, $\boldsymbol{y = g(x) = \sqrt{x}}$, and $\boldsymbol{y = h(x) = (g \bullet f)(x) = g(f)}$.

So we get $\boldsymbol{h(x) = g(f) = \sqrt{f} = \sqrt{x^2} = x}$, which has no problem with the domain $\boldsymbol{x \geq 0}$.

Thus, we get $\boldsymbol{y = h(x) = x}$ for $\boldsymbol{x \geq 0}$. And let's next, check the ranges of $\boldsymbol{h}$ and $\boldsymbol{g}$.

Beginning with the range of $\boldsymbol{g}$, we get $\boldsymbol{x \geq 0 \Rightarrow \sqrt{x} \geq 0}$.

Next, finding the range of $\boldsymbol{h}$, we simply get $\boldsymbol{x \geq 0}$.

So in this case, too, $\boldsymbol{h}$ is a function from the domain of $\boldsymbol{f}$ to the range of $\boldsymbol{g}$.

Now, let's for next instance, find $\boldsymbol{y = h(x) = (g \bullet f)(x)}$ assuming $\boldsymbol{y = f(x) = x^2}$ for $\boldsymbol{x \geq 0}$, and $\boldsymbol{y = g(x) = \sqrt{x - 4}}$ for $\boldsymbol{x \geq 4}$. To begin with, we have

$$\begin{tikzcd} X_f \arrow[r, "f"] \arrow[rr, bend right, "h"'] & Y_f \arrow[r, "g"] & Y_h \end{tikzcd} \quad \Rightarrow \quad X_f \xrightarrow{g \bullet f} Y_h.$$

And of course, we have $\boldsymbol{X_h = X_f}$.

So the domain of $\boldsymbol{h}$ is $\boldsymbol{x \geq 0}$. Next, checking to see if the range of $\boldsymbol{f}$ is a subset of the domain of $\boldsymbol{g}$, we get $\boldsymbol{x \geq 0 \Rightarrow x^2 \geq 0 \Rightarrow y \geq 0}$, which is the range of $\boldsymbol{f}$.

However, the domain of $\boldsymbol{g}$ is $\boldsymbol{x \geq 4}$.

So the range of $\boldsymbol{f}$ is not a subset of the domain of $\boldsymbol{g}$, and on the contrary, the domain of $\boldsymbol{g}$ is a part of the range of $\boldsymbol{f}$. And thus, $\boldsymbol{h}$ cannot be defined in the domain of $\boldsymbol{f}$.

Let's see though, what will happen in this case.

We have $\boldsymbol{y = f(x) = x^2}$, $\boldsymbol{y = g(x) = \sqrt{x-4}}$, and $\boldsymbol{y = h(x) = (g \bullet f)(x) = g(f)}$.

So we get $\boldsymbol{h(x) = g(f) = \sqrt{f-4} = \sqrt{x^2-4}}$, which has however, a lot of problem with the domain of $\boldsymbol{f}$, which is $\boldsymbol{X_f}$.

That is because the domain is $\boldsymbol{x \geq 0}$, but if $\boldsymbol{x = 1}$, for instance, what's inside the square root is -3, which is not allowed.

And therefore, we cannot get a composite function that can hold for the domain of the function blends into. And the function blends into is $\boldsymbol{f}$, of course, in this case.

Let's see now what can be the domain of $\boldsymbol{h}$. Finding it, we get

$\boldsymbol{x^2 - 4 \geq 0 \Rightarrow (x+2)(x-2) \geq 0 \Rightarrow x \geq 2}$ or $\boldsymbol{x \leq -2}$.

So the maximum domain of $\boldsymbol{h}$ can be $\boldsymbol{\{x|\ x \geq 2}$ or $\boldsymbol{x \leq -2\}}$ or simply, $\boldsymbol{x \geq 2}$ or $\boldsymbol{x \leq -2}$.

What do we mean by the maximum domain, though?

The domain of $\boldsymbol{h}$ can be either of $\boldsymbol{x \geq 2}$, $\boldsymbol{x \geq 3}$, $\boldsymbol{x \geq 5}$, and such, each of which is smaller than the maximum domain above.

We know both functions $\boldsymbol{f}$ and $\boldsymbol{g}$ can be defined for $\boldsymbol{x \geq 4}$, for instance, and also, the function $\boldsymbol{h}$ can be defined for $\boldsymbol{x \geq 4}$.

So if the domain $\boldsymbol{X_f}$ is $\boldsymbol{x \geq 4}$, we can get a composite function that can be defined in the domain of the function blends into.

And the composite function is $y = h(x) = \sqrt{x^2 - 4}$ for $x \in X_f$, which is $x \geq 4$.

And let's this time, find $y = h(x) = (f \bullet g)(x)$ using the two functions above.

Then first, we have

$$\begin{tikzcd} X_g \arrow[r, "g"] \arrow[rr, bend right, "h"'] & Y_g \arrow[r, "f"] & Y_h \end{tikzcd} \Rightarrow \quad X_g \xrightarrow{f \bullet g} Y_h.$$ In h, we have $X_h = X_g$, the domain of g.

So the domain of h is $x \geq 4$. And next, checking to see if the range of g is a subset of the domain of f, we get $x \geq 4 \Rightarrow x - 4 \geq 0 \Rightarrow \sqrt{x-4} \geq 0 \Rightarrow y \geq 0$, which is the range of g, and equals the domain of f. So h can be defined in the domain of g.

Let's see though, what will happen in this case.

We have $y = g(x) = \sqrt{x-4}$, $y = f(x) = x^2$, and $y = h(x) = (f \bullet g)(x) = f(g)$.

So we get $h(x) = f(g) = g^2 = x - 4$, which has no problem with the domain X_g, $x \geq 4$.

Thus, we get $y = h(x) = x - 4$ for $x \geq 4$. Let's next, check the ranges of f and h.

Beginning with the range of h, we get $x \geq 4 \Rightarrow x - 4 \geq 0 \Rightarrow y \geq 0$.

Next, getting the range of f, we get $x \geq 0 \Rightarrow x^2 \geq 0 \Rightarrow y \geq 0$.

So h can be called a function from the domain of g to the range of f.

Now, let's for next instance, find $y = h(x) = (g \bullet f)(x)$ assuming $y = g(x) = x^2$ for $0 \leq x \leq 2$, and $y = f(x) = \sqrt{1 - x^2}$ for $0 \leq x \leq 1$. To begin with, we have

$$\begin{tikzcd} X_f \arrow[r, "f"] \arrow[rr, bend right, "h"'] & Y_f \arrow[r, "g"] & Y_h \end{tikzcd} \Rightarrow \quad X_f \xrightarrow{g \bullet f} Y_h.$$ In h, we have $X_h = X_f$, of course.

So the domain of h is $0 \leq x \leq 1$.

Next, checking to see if the range of f is a subset of the domain of g, we get

$$0 \le x \le 1 \Rightarrow 0 \le x^2 \le 1 \Rightarrow -1 \le -x^2 \le 0 \Rightarrow 0 \le 1 - x^2 \le 1 \Rightarrow 0 \le \sqrt{1-x^2} \le 1 \Rightarrow 0 \le y \le 1,$$

which is the range of f. And the domain of g is $0 \le x \le 2$.

So the range of f is a subset of the domain of g,
and thus, h can be defined in the domain of f.

Let's see though, what will happen in this case.

We have $y = f(x) = \sqrt{1-x^2}$, $y = g(x) = x^2$, and $y = h(x) = (g \bullet f)(x) = g(f)$.

So we get $h(x) = g(f) = 1 - x^2$, which has no problem with the domain X_f, $0 \le x \le 1$.

Let's next, check the ranges of g and h.

Beginning with the range of g, we get $0 \le x \le 2 \Rightarrow 0 \le x^2 \le 4 \Rightarrow 0 \le y \le 4$.

Next, finding the range in h, we get

$$0 \le x \le 1 \Rightarrow 0 \le x^2 \le 1 \Rightarrow -1 \le -x^2 \le 0 \Rightarrow 0 \le 1 - x^2 \le 1 \Rightarrow 0 \le y \le 1,$$

which is not the range in g. So the range in h is just a part of the range in g.

That's because the range of the function blends into is a part of the domain of the other function.

We know that the range of the function blends into is used as the domain of the other function.

The function blends into is f, where the range is $0 \le y \le 1$, and the other function is g, where the domain is $0 \le x \le 2$.

Therefore, h is not a composite function from the domain of the function blends into to the range of the other function.

However, $\boldsymbol{h}$ is a function working perfectly for the domain of the function blends into.

So we can still get the composite function that is perfectly defined in the domain of the function blends into.
Thus, in this case, too, we can get a composite function working nicely.

For one more instance, let's find $\boldsymbol{y = q(x) = (h \bullet (g \bullet f))(x)}$ using the functions as follows.

$\boldsymbol{y = f(x) = x + 1}$ for $\boldsymbol{1 \le x \le 2}$, $\quad \boldsymbol{y = g(x) = 2x + 1}$ for $\boldsymbol{2 \le x \le 3}$, and
$\boldsymbol{y = h(x) = 9 - x}$ for $\boldsymbol{5 \le x \le 7}$.

Without showing variables, we can put $\boldsymbol{q}$ briefly this way, too: $\boldsymbol{q = h \bullet g \bullet f}$, which is thus, equal to $\boldsymbol{q = h \bullet (g \bullet f)}$, and is not equal to $\boldsymbol{q = (h \bullet g) \bullet f}$, of course. And $\boldsymbol{q}$ is a blend of $\boldsymbol{f}$, $\boldsymbol{g}$, and $\boldsymbol{h}$, but is not just a blend but a mixture made in the sequential manner as follows.

Now, finding $\boldsymbol{q = (h \bullet (g \bullet f))(x)}$, we want to do $\boldsymbol{g \bullet f}$, first.

So assuming $\boldsymbol{p = g \bullet f}$, we want to first, combine $\boldsymbol{f}$ and $\boldsymbol{g}$ to get $\boldsymbol{p}$, which is the blend of $\boldsymbol{f}$ and $\boldsymbol{g}$, which is more specifically, a composite function where $\boldsymbol{f}$ blends into $\boldsymbol{g}$.

Next, the function $\boldsymbol{p}$ blends into the other function $\boldsymbol{h}$, then we get $\boldsymbol{q}$, which is a composite function, where $\boldsymbol{p}$ blends into $\boldsymbol{h}$. So to begin with, we have

$$\begin{tikzcd} X_f \arrow[r, "f"] \arrow[rr, bend right, "p"'] & Y_f \arrow[r, "g"] & Y_p \end{tikzcd} \quad \Rightarrow \quad X_f \xrightarrow{g \bullet f} Y_p.$$

And in $\boldsymbol{p}$, we have $\boldsymbol{X_p = X_f}$, of course.

So the domain of $\boldsymbol{p}$ is the domain of $\boldsymbol{f}$, and therefore, is $\boldsymbol{1 \le x \le 2}$.

Now, we have $\boldsymbol{y = f(x) = x + 1}$, $\boldsymbol{y = g(x) = 2x + 1}$, and $\boldsymbol{y = p(x) = (g \bullet f)(x) = g(f)}$.

So we get $\boldsymbol{p(x) = g(f) = 2f + 1 = 2(x + 1) + 1 = 2x + 3}$, which has no problem with the domain where $\boldsymbol{1 \le x \le 2}$.

And next, we want to blend $\boldsymbol{p}$ into the function $\boldsymbol{h}$. Prior to such blending though, we want to make sure that the range of $\boldsymbol{p}$ fits the domain of $\boldsymbol{h}$, which is $\boldsymbol{5 \le x \le 7}$.

That is, the range of $\boldsymbol{p}$ needs to be equal to or a part of the domain of $\boldsymbol{h}$.

So finding the range of $\boldsymbol{p}$, we get

$\mathbf{1 \le \boldsymbol{x} \le 2 \Rightarrow 2 \le 2\boldsymbol{x} \le 4 \Rightarrow 5 \le 2\boldsymbol{x} + 3 \le 7 \Rightarrow 5 \le \boldsymbol{y} \le 7}$, which equals the domain of $\boldsymbol{h}$.

So now we can move on to this: $\boldsymbol{y = q(x) = (h \bullet p)(x)}$, because $\boldsymbol{p = g \bullet f}$.

Then, we have

$$\begin{tikzcd} X_p \arrow[r, "p"] \arrow[rr, bend right, "q"'] & Y_p \arrow[r, "h"] & Y_q \end{tikzcd} \quad \Rightarrow \quad X_f \xrightarrow{h \bullet p} Y_h.$$ And we have $\boldsymbol{X_q = X_p = X_f}$, of course.

So the domain of $\boldsymbol{q}$ is $\mathbf{1 \le \boldsymbol{x} \le 2}$.

Now, we have $\boldsymbol{y = p(x) = 2x + 3}$, $\boldsymbol{y = h(x) = 9 - x}$, and $\boldsymbol{y = q(x) = (h \bullet p)(x) = h(p)}$.

So we get $\boldsymbol{q(x) = h(p) = 9 - p = 9 - (2x + 3) = -2x + 6}$, which has no problem with the domain where $\mathbf{1 \le \boldsymbol{x} \le 2}$.

Therefore, we get $\boldsymbol{y = q(x) = -2x + 6}$ for $\mathbf{1 \le \boldsymbol{x} \le 2}$.

Next, let's check the ranges in $\boldsymbol{q}$ and $\boldsymbol{h}$.

Then, beginning with the range of $\boldsymbol{h}$, we get

$\mathbf{1 \le \boldsymbol{x} \le 2 \Rightarrow 2 \le 2\boldsymbol{x} \le 4 \Rightarrow -4 \le -2\boldsymbol{x} \le -2 \Rightarrow 2 \le -2\boldsymbol{x} + 6 \le 4 \Rightarrow 2 \le \boldsymbol{y} \le 4}$.

Next, the domain of $\boldsymbol{h}$ is $\mathbf{5 \le \boldsymbol{x} \le 7}$, so finding the range of $\boldsymbol{h}$, we get

$\mathbf{5 \le \boldsymbol{x} \le 7 \Rightarrow -7 \le -\boldsymbol{x} \le -5 \Rightarrow 2 \le 9 - \boldsymbol{x} \le 4 \Rightarrow 2 \le \boldsymbol{y} \le 4}$, which equals the range of $\boldsymbol{q}$.

So the function $\boldsymbol{q}$ is $\boldsymbol{y = q(x) = -2x + 6}$ for $\mathbf{1 \le \boldsymbol{x} \le 2}$, which is a composite function from the domain of $\boldsymbol{f}$ to the range of $\boldsymbol{h}$.

Therfore, the composite function $\boldsymbol{q}$ has all the inputs of $\boldsymbol{f}$,
and produces all the outputs of $\boldsymbol{h}$.

So $\boldsymbol{q}$ is a blend of three functions $\boldsymbol{f}$, $\boldsymbol{g}$, and $\boldsymbol{h}$.

And the blending sequence matters.

So more specifically, $\boldsymbol{q}$ is a composite function, where $\boldsymbol{f}$ blends into $\boldsymbol{g}$, then the blend ($\boldsymbol{g} \bullet \boldsymbol{f}$) blends into the other function $\boldsymbol{h}$.

Let's now put in a graph all the functions $\boldsymbol{f}$, $\boldsymbol{g}$, $\boldsymbol{h}$, $\boldsymbol{p}$, and $\boldsymbol{q}$.

$$\boldsymbol{y = f(x) = x + 1} \text{ for } \boldsymbol{1 \le x \le 2}$$

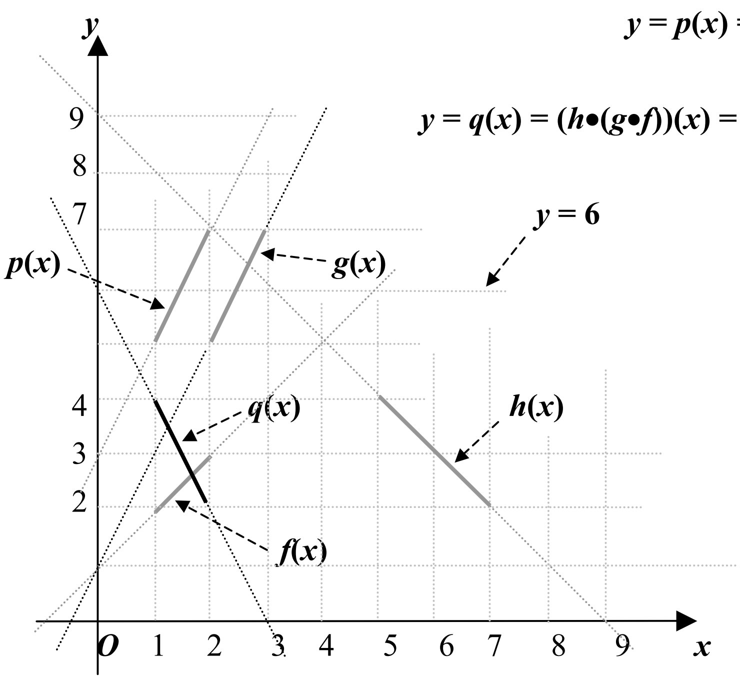
$$\boldsymbol{y = g(x) = 2x + 1} \text{ for } \boldsymbol{2 \le x \le 3}$$

$$\boldsymbol{y = p(x) = (g \bullet f)(x) = 2x + 3} \text{ for } \boldsymbol{1 \le x \le 2}$$

$$\boldsymbol{y = h(x) = 9 - x} \text{ for } \boldsymbol{5 \le x \le 7}$$

$$\boldsymbol{y = q(x) = (h \bullet (g \bullet f))(x) = (h \bullet p)(x) = -2x + 6} \text{ for } \boldsymbol{1 \le x \le 2}$$

Fig. 5.3



5.0. Inverse Functions 1

Why is it inverse, though?

As mentioned earlier on one-to-one functions, we sometimes need to come up with a new function from a function given.

Coming up with such a function, for instance, from a function ***A***, we need to use as the domain the range of the function ***A***, and use as the range the domain of the function ***A***.

So a domain becomes a range, and a range becomes a domain. There's one more to it.

We need to keep the existing correspondence between the data in the domain and range of the function ***A***. That is, the number pairs have to remain. Only roles are reversed.

For instance, if the function ***A*** makes (1, 2), (4, 7), (6, -5), etc., the new function has to make (2, 1), (7, 4), (-5, 6), etc.

We call such a new function an inverse function.
And taking the inverse function of a function given, we call it the inverse of the given function, and often just call it *the inverse* for short.

Suppose for instance, that we are given a function ***f*** of which the domain is ***X***, and the range is ***Y***, and that ***f*** is one-to-one.

Then, we can come up with a new function of which the domain is ***Y*** and the range is ***X***. And the new function will have a new expression other than that of ***f***.

That's because the new expression has to maintain the existing number pairs, but the two numbers in each pair have to exchange their positions.

So for instance, assuming ***g*** is the new function, and ***f*****(4) = 7**, we need to get ***g*****(7) = 4**.

Thus, the dependence relationship is inverted. So we call such a new function the inverse of the function given, and just call it the inverse, too, for short.

Therefore, ***g*** is called the inverse of ***f***, and we say that such a function as ***f*** is invertible.

So if a function is invertible, it can have its inverse, and if not invertible, it can have no inverse.

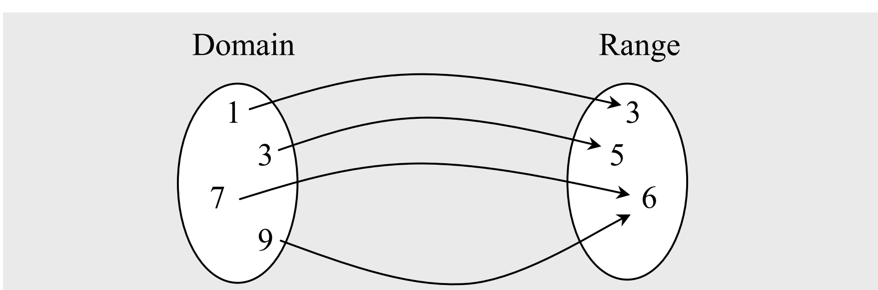
Finding or making the inverse of a function, we say we take the inverse of the function. And a function has to be, of course, invertible if we want to take the inverse of the function.   What function then, can be invertible?

If it is invertible, it is one-to-one, and if one-to-one, it is invertible.
So only functions one-to-one can be invertible. And we should be able to get the inverse of a function one-to-one.   Why one-to-one, though?

Of a function, <u>each input has to get paired with one output only</u>, which is a rule of a function. That is, each element in a domain gets paired with one element only in the range, and thus, is *not* allowed to get paired with another element in the range.

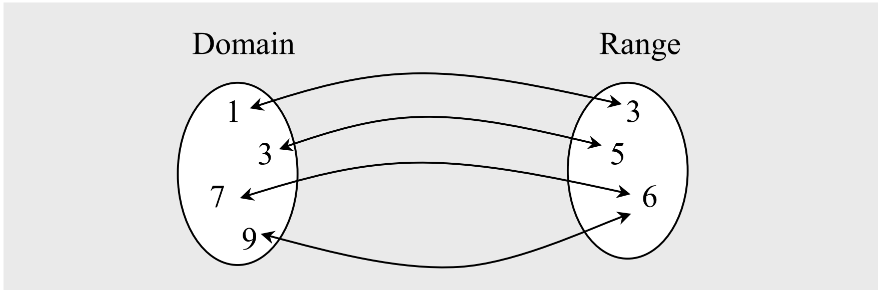
So for instance, assuming ***h*****(1) = 3**, we cannot get ***h*****(1) = 5** or any number other than 3 if the function ***h*** is invertible, of course.

Fig. 0



Let's this time, look at both sets the other way around.

Fig. 1



Then, unlike an input, we can see that an output can get paired with more than one input, which is in fact, another rule of a function.

That is to say that each element in the range can get paired with more than one element in the domain.
So for instance, even though $\boldsymbol{h(7) = 6}$, we can get $\boldsymbol{h(9) = 6}$, too.
That is to say that a function can make number pairs the way as follows. (7, 6) and (9, 6)

Thus, an element in the range gets paired with an element in the domain, and yet, can still get paired with another element or even more in the domain. So what?

- If a function has the inverse, the function has to be one-to-one.

Otherwise, the inverse would not be a function. If the function we take the inverse of is many-to-one, the inverse would be one-to-many, but no function is one-to-many.

Suppose for instance, $\boldsymbol{k}$ were the inverse of $\boldsymbol{h}$ above.

Then, we would get not only $\boldsymbol{k(6) = 7}$ but $\boldsymbol{k(6) = 9}$, too, which is however, not allowed.
No function makes number pairs this way: (6, 7) and (6, 9). So $\boldsymbol{k}$ cannot be a function.

That is to say that $\boldsymbol{h}$ cannot have the inverse. That is because $\boldsymbol{h}$ is not one-to-one.

So only functions one-to-one can be invertible, and such a function can have the inverse.
How then can we get the inverse function?

Suppose again, $\boldsymbol{f}$ is a function from $\boldsymbol{X}$ to $\boldsymbol{Y}$, and is one-to-one, and $\boldsymbol{g}$ is the inverse of $\boldsymbol{f}$.

Then, of course, $\boldsymbol{g}$ is one-to-one, too, since $\boldsymbol{f}$ is one-to-one.
And the domain of $\boldsymbol{g}$ is the range of $\boldsymbol{f}$, and the range of $\boldsymbol{g}$ is the domain of $\boldsymbol{f}$.

That is, the roles of the data sets get reversed. So $\boldsymbol{Y}$ is the domain of $\boldsymbol{g}$, and $\boldsymbol{X}$ is the range of $\boldsymbol{g}$, which is thus, a function from $\boldsymbol{Y}$ to $\boldsymbol{X}$. That is only one thing, though.

What's really about the inverse is the expression of the inverse.

Let's now call the function $\boldsymbol{f}$ the original function.

Then, what matters is <u>how the inverse produces all the inputs</u> of the original function <u>using all the corresponding outputs</u> of the original.

So what we want to know is the way the input changes as the output changes.
How then can we find the way?

Assuming for instance, $\boldsymbol{y = f(x) = x + 1}$, we can say that $\boldsymbol{y}$ changes as $\boldsymbol{x}$ changes.
How then does $\boldsymbol{y}$ change? In other words, what is the way $\boldsymbol{y}$ changes as $\boldsymbol{x}$ changes?

It is $\boldsymbol{x + 1}$. It is saying that for each and every value of $\boldsymbol{x}$, the $\boldsymbol{y}$-value is the sum of 1 and the $\boldsymbol{x}$-value. So the expression of $\boldsymbol{f}$ shows the way $\boldsymbol{y}$ changes as $\boldsymbol{x}$ changes.

That is to say that the expression of $\boldsymbol{f}$ shows the way the output changes as the input changes. What then do we need to get finding the inverse of $\boldsymbol{f}$?

Finding the inverse, we want to find the way the input changes as the output changes.
And the way is the expression of the inverse. How then can we find the expression?

I can get straight to the point, but I'm going to beat around the bush, because I want you to reason.

Finding the inverse, we don't just find it, of course. We can try finding it using the expression of the original.
We have $\boldsymbol{y = f(x) = x + 1}$, so we get $\boldsymbol{y = x + 1}$.

Then, we can say that $\boldsymbol{y}$ is an expression in terms of $\boldsymbol{x}$.
What then do we mean by the expression?

The expression shows how the $\boldsymbol{y}$-value gets made for each and every $\boldsymbol{x}$-value.
What then do we need to find the way the $\boldsymbol{x}$-value gets made for each and every $\boldsymbol{y}$-value?

We want to get an expression in terms of $\boldsymbol{y}$.
It's because the expression shows how the $\boldsymbol{x}$-value gets made for each and every $\boldsymbol{y}$-value.
How then can we get the expression?

The expression of the inverse is normally different from that of the original.
In what case then, the expression of the inverse is the same as that of the original?

If the original is an identity function, the expression of the inverse is that of the original, which is, of course, quite trivial. So it's not worth a talk. What if the original is not an identity function? How then can we get the expression of the inverse?

The expression shows the way the $\boldsymbol{x}$-value gets made as the $\boldsymbol{y}$-value changes. So?

The expression of the original is the way the $\boldsymbol{y}$-value gets made as the $\boldsymbol{x}$-value changes.
That is to say that the expression of the original connects the $\boldsymbol{y}$-value and the $\boldsymbol{x}$-value.
So we can try finding the expression of the inverse using the expression of the original.
How though?

We have $\boldsymbol{y = f(x)}$, which can be taken as an equation for $\boldsymbol{x}$, because $\boldsymbol{f(x)}$ is an expression in terms of $\boldsymbol{x}$ as $\boldsymbol{2x}$, $\boldsymbol{x^2 - 1}$, $\boldsymbol{x + 3}$, etc. Then, we have $\boldsymbol{y = 2x}$, $\boldsymbol{y = x^2 - 1}$, $\boldsymbol{y = x + 3}$, etc.

What then do we get solving for $\boldsymbol{x}$, the equation $\boldsymbol{y = f(x)}$?

We can get an equation, where the input variable $\boldsymbol{x}$ is set equal to an expression expressed in terms of $\boldsymbol{y}$, that is, the output variable.

So the expression in term of $\boldsymbol{y}$ shows the way the $\boldsymbol{x}$-value gets made as the $\boldsymbol{y}$-value changes.
And the expression is the very expression of the inverse.

We need to note however, that we normally put a function in the $\boldsymbol{x}$-$\boldsymbol{y}$ system.

Putting a function in the $\boldsymbol{x}$-$\boldsymbol{y}$ system, we take $\boldsymbol{x}$ as the input variable, and take $\boldsymbol{y}$ as the output variable.

So once we've got the equation stated above, we need to swap the two variables.
That is, we want to replace $\boldsymbol{x}$ with $\boldsymbol{y}$, and replace $\boldsymbol{y}$ with $\boldsymbol{x}$.

Then, we produce a function definition that describes the inverse.
A function definition is a description of a function as $y = p(x) = x^2 + 3x + 1$ for $x > 2$.

Let's now find the inverse of the function f assuming the inverse is g.

We have $y = f(x) = x + 1$. So solving for x, the equation $y = x + 1$, we get $x = y - 1$, which is an expression in terms of y, and is the very expression of the inverse g.

• So *for now*, we can put g the way as follows. $x = g(y) = y - 1$.

Then, in the inverse function g above, y gets inputs, and thus, is the input variable, and x gets outputs, and thus, is the output variable.

By convention though, we normally take x as an input variable, and take y as an output variable. That is, we normally put a function in the x-y system.

• Producing thus, the inverse g, we want to put it this way: $y = g(x) = x - 1$.

Then, of course, the inverse function g produces all the inputs of the original function f using as inputs, all the corresponding outputs of the original f.

Suppose now, for another simple instance, of f, each output is twice each input.

Then, of the inverse g, each output has to be half each input.

So the expression of $f(x)$ is $2x$, and thus, the expression of $g(x)$ is $\frac{1}{2}x$.

Solving in fact, for x first, the equation $y = 2x$, we get $x = \frac{1}{2}y$.

Next, swapping x and y, we get $y = \frac{1}{2}x$, which is the expression of the inverse.

That is, the inverse is $y = g(x) = \frac{1}{2}x$.

Let's find, for another instance, the inverse of $y = f(x) = 2x + 1$.

Then, solving for $\boldsymbol{x}$ first, the equation $\boldsymbol{y = 2x = 1}$, we get $\boldsymbol{2x = 1 - y \Rightarrow x =}\ \dfrac{1-\boldsymbol{y}}{2}$.

Then, swapping the variables, we get $\boldsymbol{y} =\ \dfrac{1-\boldsymbol{x}}{2}$.

Assuming thus, the inverse is $\boldsymbol{g}$, we get $\boldsymbol{y = g(x)} =\ \dfrac{1-\boldsymbol{x}}{2}$.

What then, about this case: $\boldsymbol{y = f(x) = \sqrt{x+4}+1}$ for $\boldsymbol{x > 5}$?

Solving for $\boldsymbol{x}$, the equation $\boldsymbol{y = f(x)}$, we get:$\boldsymbol{x = y^2 - 2y - 3}$.
Then, swapping $\boldsymbol{x}$ and $\boldsymbol{y}$, we get $\boldsymbol{y = x^2 - 2x - 3}$, which is the very expression of the inverse function $\boldsymbol{y = g(x)}$.

We don't just produce however, the definition of the inverse the way as follows.
$\boldsymbol{y = g(x) = x^2 - 2x - 3}$. Why not?

That's because the inverse is a function one-to-one.
And if not specified, the domain is assumed to be the largest set of numbers that can be the values of the input variable. So in the case above, the domain is assumed to be a set of all real numbers. Thus, the function $\boldsymbol{g}$ produced the way above is not one-to-one.

So we need to find and specify the domain of the inverse.

Finding in fact, the range of $\boldsymbol{f}$, we get $\boldsymbol{y \geq 4}$. So the domain of the inverse is $\boldsymbol{x \geq 4}$.

Thus, the inverse is $\boldsymbol{y = g(x) = x^2 - 2x - 3}$ for $\boldsymbol{x \geq 4}$, which is one-to-one.
What then about this case: $\boldsymbol{y = f(x) = x^2 - 2x - 3}$ for $\boldsymbol{x > 3}$?

We've got to solve if for $\boldsymbol{x}$, too, and then, swap the variables in the equation we get as the solution. That's not it though.
We need to specify the domain of the inverse, and determine its expression considering the domain, too.

So in reality, what really matters is your caliber in algebra.
You need to have a strong foundation on algebra. What algebra?

Elementary algebra, together with geometry analytic, called coordinate geometry, too.

And the algebra needs to include polynomial factorizations, which covers how to manipulate expressions, that is, how to change, alter, or convert expressions so that you can actually get to the very solution you need to put down on your paper.

5.1. Inverse Functions 2

So what is an inverse function?

Answering the question, we produce the definition for inverse functions.
So the definition describes an inverse function, and says what it is.
What then is the definition?

Assuming $\boldsymbol{f}$ is the original function, which is the function we take the inverse of, we can put the *definition for inverse functions* the way as follows:

> $\boldsymbol{X}$ and $\boldsymbol{Y}$ are sets of real numbers.
> $\boldsymbol{f}$: $\boldsymbol{X} \longrightarrow \boldsymbol{Y}$, and $\boldsymbol{f}$ is one-to-one.
> $\boldsymbol{g}$: $\boldsymbol{Y} \longrightarrow \boldsymbol{X}$, $\boldsymbol{y = f(x) \Leftrightarrow x = f^{-1}(y)}$, and $\boldsymbol{g = f^{-1}}$.

Quite often though, we just use as the definition, the expression below:

> $\boldsymbol{y = f(x) \Leftrightarrow x = f^{-1}(y)}$, and $\boldsymbol{g = f^{-1}}$.

And defining quickly a particular inverse, we can use a short definition, and can put it this way: $\boldsymbol{g = f^{-1}}$, read as the inverse of $\boldsymbol{f}$, or often just $\boldsymbol{f}$ inverse.

What then can we do with the definition above?

We have some theorems on inverse functions, which can be of much help when we find inverse functions, and do problems with such functions. Understanding those theorems, we need to know the definition. And in particular, understanding inverse functions, we can better understand a log function, which is an inverse of an exponential function.

We are now going to cover some theorems on inverse functions. Math begins with definitions, so we may want to begin with the *definition for inverse functions* as follows.

$\boldsymbol{X}$ and $\boldsymbol{Y}$ are sets of real numbers.
$\boldsymbol{f}: \boldsymbol{X} \longrightarrow \boldsymbol{Y}$, and $\boldsymbol{f}$ is one-to-one.
$\boldsymbol{g}: \boldsymbol{Y} \longrightarrow \boldsymbol{X}$, $\boldsymbol{y = f(x) \Leftrightarrow x = f^{-1}(y)}$, and $\boldsymbol{g = f^{-1}}$.

Usually though, as the definition, we just use $\boldsymbol{y = f(x) \Leftrightarrow x = f^{-1}(y)}$.

And the theorems below can be of much help doing problems with inverse functions.

0. $\boldsymbol{(f^{-1})^{-1} = f}$

1. $\boldsymbol{f^{-1}\{f(x)\} = x}$, and in short, $\boldsymbol{f^{-1}(f) = x}$.

2. $\boldsymbol{f\{f^{-1}(y)\} = y}$, and in short, $\boldsymbol{f(f^{-1}) = y}$.

3. Assuming $\boldsymbol{y = f(x) = x}$, we get $\boldsymbol{f^{-1}(f) = f(f^{-1})}$.

4. If $\boldsymbol{I_x}$ is an identity function of $\boldsymbol{x}$, and $\boldsymbol{I_y}$ is an identity function of $\boldsymbol{y}$, we get $\boldsymbol{I_x = f^{-1}\bullet f}$, $\boldsymbol{I_y = f\bullet f^{-1}}$, and $\boldsymbol{(g\bullet f = I_x \text{ and } f\bullet g = I_y) \Leftrightarrow g = f^{-1}}$.

5. If $\boldsymbol{z = h(y)}$ is one-to-one, we get $\boldsymbol{(h\bullet f)^{-1} = f^{-1}\bullet h^{-1}}$.

Let's see now, how the theorems can hold, and begin with the theorem 0, $\boldsymbol{(f^{-1})^{-1} = f}$.

First, putting $\boldsymbol{(f^{-1})^{-1} = f}$ more specifically, we get $\boldsymbol{(f^{-1})^{-1}(x) = f(x)}$.

Next, by the definition where $\boldsymbol{y = f(x) \Leftrightarrow x = f^{-1}(y)}$, we get

$\boldsymbol{y = f(x) \Leftrightarrow x = f^{-1}(y) \Leftrightarrow y = (f^{-1})^{-1}(x)}$. Thus, $\boldsymbol{f(x) = (f^{-1})^{-1}(x)}$. What's going on?

Looking at the statement that $\boldsymbol{y = f(x) \Leftrightarrow x = f^{-1}(y) \Leftrightarrow y = (f^{-1})^{-1}(x)}$, we can see

$\boldsymbol{y = f(x)}$ and also, $\boldsymbol{y = (f^{-1})^{-1}(x)}$.

So we get $\boldsymbol{y = f(x) = (f^{-1})^{-1}(x) \Rightarrow f(x) = (f^{-1})^{-1}(x)}$. And in short, $\boldsymbol{f = (f^{-1})^{-1}}$.

So the theorem that $\boldsymbol{(f^{-1})^{-1} = f}$ is saying a fact that the inverse of the inverse is the original function. The fact sounds self-evident. Nevertheless, the proof itself can still be quite confusing to many students. Simple is not necessarily easy unless understood. We can hardly see things not only too far but too close, too.

So let's this time, take a closer look in a bit different angle.
To begin with, the definition for inverse functions is $\boldsymbol{y = f(x) \Leftrightarrow x = f^{-1}(y)}$.

So by the definition, we get $\boldsymbol{x = f^{-1}(y) \Leftrightarrow y = (f^{-1})^{-1}(x)}$. How?

We have used just the definition that $\boldsymbol{y = f(x) \Leftrightarrow x = f^{-1}(y)}$. Still foggy?

First, taking the inverse of $\boldsymbol{f}$, we use a notation where $\boldsymbol{f^{-1}}$, read as $\boldsymbol{f}$ inverse.

Next, the statement that $\boldsymbol{y = f(x) \Rightarrow x = f^{-1}(y)}$ is saying that taking the inverse of a function $\boldsymbol{y = f(x)}$, we get $\boldsymbol{x = f^{-1}(y)}$, which is thus, the inverse of $\boldsymbol{f}$.

And the statement that $\boldsymbol{y = f(x) \Leftarrow x = f^{-1}(y)}$ is saying that taking the original function of the inverse $\boldsymbol{x = f^{-1}(y)}$, we get $\boldsymbol{y = f(x)}$, which is therefore, the original function.

Also, of course, taking the inverse of the inverse, we get the original function back.
That is, $\boldsymbol{(f^{-1})^{-1} = f}$, which is the original.
And we can put the idea this way: $\boldsymbol{x = f^{-1}(y) \Leftrightarrow y = (f^{-1})^{-1}(x)}$, where $\boldsymbol{(f^{-1})^{-1} = f}$.

So we get $\boldsymbol{x = f^{-1}(y) \Leftrightarrow y = f(x)}$.
Specifically though, what can we mean by this: $\boldsymbol{y = f(x) \Rightarrow x = f^{-1}(y)}$?

First, of the inverse function $\boldsymbol{f^{-1}}$, all the inputs are all the outputs of $\boldsymbol{f}$.
And we know we use $\boldsymbol{y}$ as the output variable in $\boldsymbol{f}$, and $\boldsymbol{y}$ gets every output of $\boldsymbol{f}$.
So we use $\boldsymbol{y}$ as the input variable in $\boldsymbol{f^{-1}}$.

Next, all the outputs of $\boldsymbol{f}^{-1}$ are all the inputs of the original function $\boldsymbol{f}$.
And we know we use $\boldsymbol{x}$ as the input variable in $\boldsymbol{f}$, and $\boldsymbol{x}$ gets every input of $\boldsymbol{f}$.
So we use $\boldsymbol{x}$ as the output variable in $\boldsymbol{f}^{-1}$.

(Functions are not easy, much less the inverses. It is often the case though, once understood, it's quite obvious.)

• And moving now, on to the theorem 1, we have $\boldsymbol{f^{-1}\{f(x)\} = x}$. How?

To begin with, we have the definition where $\boldsymbol{y = f(x) \Leftrightarrow x = f^{-1}(y)}$.

So we get $\boldsymbol{f^{-1}(y) = f^{-1}\{f(x)\}}$ if putting $\boldsymbol{f(x)}$ into $\boldsymbol{y}$ in $\boldsymbol{f^{-1}(y)}$, since $\boldsymbol{y = f(x)}$.

Thus, we get $\boldsymbol{x = f^{-1}(y) = f^{-1}\{f(x)\} \Rightarrow x = f^{-1}\{f(x)\}}$, which is, for short, $\boldsymbol{x = f^{-1}(f)}$.

• And moving next, on to the theorem 2, we have $\boldsymbol{f\{f^{-1}(y)\} = y}$.

Then again first, we have the definition $\boldsymbol{y = f(x) \Leftrightarrow x = f^{-1}(y)}$.

So we get $\boldsymbol{f(x) = f\{f^{-1}(y)\}}$ if putting $\boldsymbol{f^{-1}(y)}$ into $\boldsymbol{x}$ in $\boldsymbol{f(x)}$, since $\boldsymbol{x = f^{-1}(y)}$.

Thus, we get $\boldsymbol{y = f(x) = f\{f^{-1}(y)\}}$. Therefore, we get: $\boldsymbol{y = f\{f^{-1}(y)\}}$.

• Let's next, move on to the theorem 3, which is saying $\boldsymbol{y = f(x) = x \Rightarrow f^{-1}(f) = f(f^{-1})}$.

First, we have the definition $\boldsymbol{y = f(x) \Leftrightarrow x = f^{-1}(y)}$.
Thus, we can see that
$\boldsymbol{x = f^{-1}(y) = f^{-1}\{f(x)\}}$ if putting $\boldsymbol{f(x)}$ into $\boldsymbol{y}$ in $\boldsymbol{f^{-1}(y)}$ since $\boldsymbol{y = f(x)}$, so we get $\boldsymbol{x = f^{-1}\{f(x)\}}$.

Also, we can see that $\boldsymbol{y = f(x) = f\{f^{-1}(y)\}}$ since $\boldsymbol{x = f^{-1}(y)}$, so we get $\boldsymbol{y = f\{f^{-1}(y)\}}$.

Besides, we have $\boldsymbol{y = x}$ since $\boldsymbol{y = f(x) = x}$. So, we get $\boldsymbol{f\{f^{-1}(y)\} = f^{-1}\{f(x)\}}$.

Thus, we get this, too: $f^{-1}\{f(x)\} = f^{-1}\{f(y)\}$ since $y = x$.

Thus, we now have $f^{-1}\{f(x)\} = f^{-1}\{f(y)\}$, and $f\{f^{-1}(y)\} = f^{-1}\{f(x)\}$.

So we get $f^{-1}\{f(y)\} = f\{f^{-1}(y)\}$.

Also, since $y = x$, we get $f^{-1}\{f(y)\} = f\{f^{-1}(y)\} \Rightarrow f^{-1}\{f(x)\} = f\{f^{-1}(y)\}$.

Therefore, we can now say that $f^{-1}(f) = f(f^{-1})$ if f is an identity function as $y = f(x) = x$.

• Now, let's next, move on to the theorem 4, which is saying this:

> Assuming I_x is an identity function of x, and I_y is an identity function of y, we get $I_x = f^{-1}\bullet f$, $I_y = f\bullet f^{-1}$, and $(g\bullet f = I_x$ and $f\bullet g = I_y) \Leftrightarrow g = f^{-1}$.

The theorem above is in two parts, and one of the two is as follows.

> Assuming I_x is an identity function of x, and I_y is an identity function of y, we get $I_x = f^{-1}\bullet f$, and $I_y = f\bullet f^{-1}$.

Beginning with $I_x = f^{-1}\bullet f$, we can set $I_x = I(x) = x$, since I_x is an identity function of x.

So we get $x = f^{-1}\bullet f$, which means therefore, the composite function $f^{-1}\bullet f$ is an identity function of x, too.

Now, we have $y = f(x) \Leftrightarrow x = f^{-1}(y)$, and we can put it this way, too: $y = f \Leftrightarrow x = f^{-1}$.

So to begin with, we can set $f^{-1}\bullet f = (f^{-1}\bullet f)(x) = f^{-1}(f)$. And we know $y = f$.

Thus, we get $f^{-1}\bullet f = f^{-1}(f) = f^{-1}(y)$, which is x since we have $x = f^{-1}(y)$.

So we get $f^{-1}\bullet f = f^{-1}(y) = x \Rightarrow f^{-1}\bullet f = x$, and thus, we get $I_x = f^{-1}\bullet f$.

• And next, moving on to $I_y = f•f^{-1}$, we can set $I_y = I(y) = y$, since I_y is an identity function of y.

So we get $y = f•f^{-1}$, which means therefore, the composite function $f•f^{-1}$ is an identity function of y, too.

Now, we have $y = f(x) \Leftrightarrow x = f^{-1}(y)$, and we can put it this way, too: $y = f \Leftrightarrow x = f^{-1}$.

So first, we can set $f•f^{-1} = (f•f^{-1})(y) = f(f^{-1})$. And we have $x = f^{-1}$.

Thus, we get $f•f^{-1} = f(f^{-1}) = f(x)$, which is y since we have $y = f(x)$.

So we get $f•f^{-1} = f(x) = y \Rightarrow f•f^{-1} = y$, and thus, we get $I_y = f•f^{-1}$.

• Now, let's next, move on to the second part in the theorem 4, and the second part is

Assuming I_x is an identity function of x, and I_y is an identity function of y, we get $g•f = I_x$ and $f•g = I_y \Leftrightarrow g = f^{-1}$.

Proving a statement that has this sign: $\Leftrightarrow$, which is read as if and only if, we need to show that $\Rightarrow$ is true, and also, that $\Leftarrow$ is true. So we want to prove the two statements as follows.

$g•f = I_x$ and $f•g = I_y \Rightarrow g = f^{-1}$.

$g = f^{-1} \Rightarrow g•f = I_x$ and $f•g = I_y$.

So let's now begin with $g•f = I_x$ and $f•g = I_y \Rightarrow g = f^{-1}$.

First, we have $g•f = I_x$ and $f•g = I_y$.

Also, we have the fact that $I_x = f^{-1}•f$, and $I_y = f•f^{-1}$.

So we get $g•f = I_x = f^{-1}•f$, and $f•g = I_y = f•f^{-1}$.

Thus, we get $g \bullet f = f^{-1} \bullet f$, and $f \bullet g = f \bullet f^{-1}$. So we can see that $g = f^{-1}$.

Thus, we can conclude that $g \bullet f = I_x$ and $f \bullet g = I_y \Rightarrow g = f^{-1}$.

And we know $I_x = I(x) = x$, and $I_y = I(y) = y$, so we have $g \bullet f = x$, and $f \bullet g = y$.

Thus, we may want to put the conclusion the way below:

$g \bullet f = x$, and $f \bullet g = y \Rightarrow g = f^{-1}$, which means g is the inverse of f.

Also, of course, f is the inverse of g.

• Now, let's next, move on to the other way: $g = f^{-1} \Rightarrow g \bullet f = I_x$ and $f \bullet g = I_y$.

First, since $g = f^{-1}$, we can get $g \bullet f = f^{-1} \bullet f$ and $f \bullet g = f \bullet f^{-1}$.

And we have this, too: $I_x = f^{-1} \bullet f$, and $I_y = f \bullet f^{-1}$.

So we get $g \bullet f = I_x$ and $f \bullet g = I_y$.

Therefore, $g = f^{-1} \Rightarrow g \bullet f = I_x$ and $f \bullet g = I_y$.

In other words, if $y = f(x)$ and $g = f^{-1}$, we get $g \bullet f = x$, and $f \bullet g = y$.

• Now, let's move on to another theorem, and the theorem is saying this:

If $z = h(y)$ is one-to-one, we get $(h \bullet f)^{-1} = f^{-1} \bullet h^{-1}$.

First, we know that the theorem above is saying that if $g \bullet f$ and $f \bullet g$ are identity functions, g is the inverse of f.

That is to say that $g \bullet f = x$, and $f \bullet g = y \Rightarrow g = f^{-1}$, which means g is the inverse of f.

• What then, can we mean by $(h \bullet f)^{-1} = f^{-1} \bullet h^{-1}$?

Of course, $f^{-1} \bullet h^{-1}$ is a composite function. More importantly, it is the inverse of $h \bullet f$.

So showing that $(f^{-1} \bullet h^{-1}) \bullet (h \bullet f)$ and $(h \bullet f) \bullet (f^{-1} \bullet h^{-1})$ are identity functions, we have shown that $(h \bullet f)^{-1} = f^{-1} \bullet h^{-1}$.

And looking at $(f^{-1} \bullet h^{-1}) \bullet (h \bullet f)$ and $(h \bullet f) \bullet (f^{-1} \bullet h^{-1})$, we can think of another theorem, covered in the section, **Composite Functions**. And the theorem is as follows.

$r \bullet (q \bullet p) = (r \bullet q) \bullet p = r \bullet q \bullet p$.

That is to say that composite-function-operations are associative.
And the theorem above can be called the composite identity.

• Now, let's first show that $(f^{-1} \bullet h^{-1}) \bullet (h \bullet f)$ is an identity function.

To begin with, we can set $(f^{-1} \bullet h^{-1}) \bullet (h \bullet f) = f^{-1} \bullet (h^{-1} \bullet h) \bullet f$ by the identity above.

Also by the theorem 4 above, we can set $h^{-1} \bullet h = I$, where I is an identity function.

Besides, we have $k \bullet I = k$, where k is a function, of course.
Thus, we get $(f^{-1} \bullet h^{-1}) \bullet (h \bullet f) = f^{-1} \bullet I \bullet f = f^{-1} \bullet f = I$ by the theorem 4 above.
Why not I_y or I_x but just I, though?

Precisely, we have to use I_y or I_x, and yet, we can just use I for simplicity.

We have the definition $y = f(x) \Leftrightarrow x = f^{-1}(y)$, so f^{-1} is a function of y.

So we can set $h^{-1} \bullet h = I_y$, where $z = I(y) = y$, which is an identity function of y.

Also, we can get the same by the theorem 1, which is $f^{-1}(f) = x$.

Note that in the function $z = I_y$, z is the output variable, and y is the input variable.

• Now next, let's show that $(h \bullet f) \bullet (f^{-1} \bullet h^{-1})$ is an identity function.

To begin with, we can set $(h \bullet f) \bullet (f^{-1} \bullet h^{-1}) = h \bullet (f \bullet f^{-1}) \bullet h^{-1}$ by the composite identity above.
Also by the theorem 4 above, we can set $f \bullet f^{-1} = I$.

Besides, we have $k \bullet I = k$, where k is a function, of course.
Thus, we get $(h \bullet f) \bullet (f^{-1} \bullet h^{-1}) = h \bullet I \bullet h^{-1} = h \bullet h^{-1} = I$ by the theorem 4 above.

Now, putting threads together, we have $(f^{-1} \bullet h^{-1}) \bullet (h \bullet f) = I$, and $(h \bullet f) \bullet (f^{-1} \bullet h^{-1})\ I$.
And therefore, we get $(h \bullet f)^{-1} = f^{-1} \bullet h^{-1}$.

Working with functions, we often put them in a graph.
And quite readily, we can do so if the function is an inverse.
It is the case though, only if the curve of the original function is put in a graph already.
That's because the curve of the inverse is symmetric to the original's about the line $y = x$.

And a curve is a set of points in a coordinate plane as the x-y plane.
So we can put the idea the way as follows.

> Suppose f: $X \longrightarrow Y$, $y = f(x)$, f is one-to-one, and its curve is the set of points below.
> $\{(x, f(x)) \mid x \in X\}$, which equals to $\{(x, y) \mid x \in X$ and $y \in Y\}$.
>
> Then, the curve of f^{-1} is $\{(f(x), x) \mid x \in X\}$, that is, $\{(y, x) \mid x \in X$ and $y \in Y\}$.

Now, an inverse function is a function, too, so we can name it differently if we want to.
And we can use any letters as the variables as long as their relation is maintained.

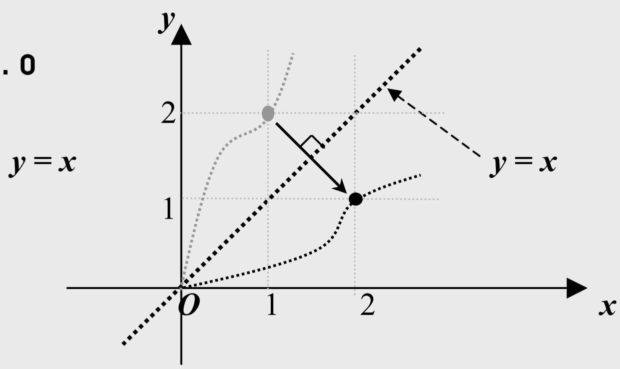
So suppose now, $y = g(x)$ is the inverse of f defined above.
Then, the input variable in g is x, and y is the output variable.

So in the inverse function g, the input variable x gets each output of the original f.
Suppose for instance, in $f(x)$, x get 1 as an input, and 2 is the output for 1.
Then, a point (1, 2) gets produced by f.
Next, moving on to $g(x)$, we can see that x gets 2 as an input, which is the output of f.

Then, $g(2) = 1$, which is the input of f, and is assigned to y, which is the output variable in the inverse function g. So a point (2, 1) gets produced by g.

That is, the point (1, 2) in the curve of $\boldsymbol{f}$ corresponds to the point (2, 1) in the curve of $\boldsymbol{g}$.

Fig. 0



So the two points are symmetric to each other about the line $\boldsymbol{y = x}$.

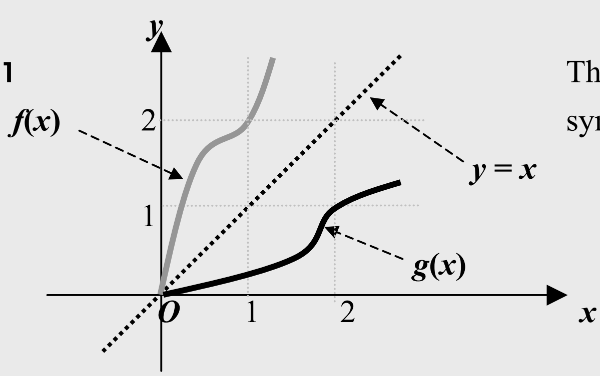
And the same is true, also, for all the points in $\boldsymbol{f}$ and $\boldsymbol{g}$.
So if the curve of $\boldsymbol{f}$ is a set of points as follows $\{(x, f(x)) | x \in X\}$, then $\{(f(x), x) | x \in X\}$ is the curve of $\boldsymbol{g}$.
Therefore, what happens between $\boldsymbol{f}$ and $\boldsymbol{g}$ is as follows.

$$y = f(x) \Leftrightarrow x = f^{-1}(y) \Rightarrow y = g(x).$$

So $g = f^{-1}$, y in $y = g(x)$ gets the x-value in $f(x)$, and x in $g(x)$ gets the y-value in $y = f(x)$.

Thus, we can put the two curves in the same graph the way below:

Fig. 1

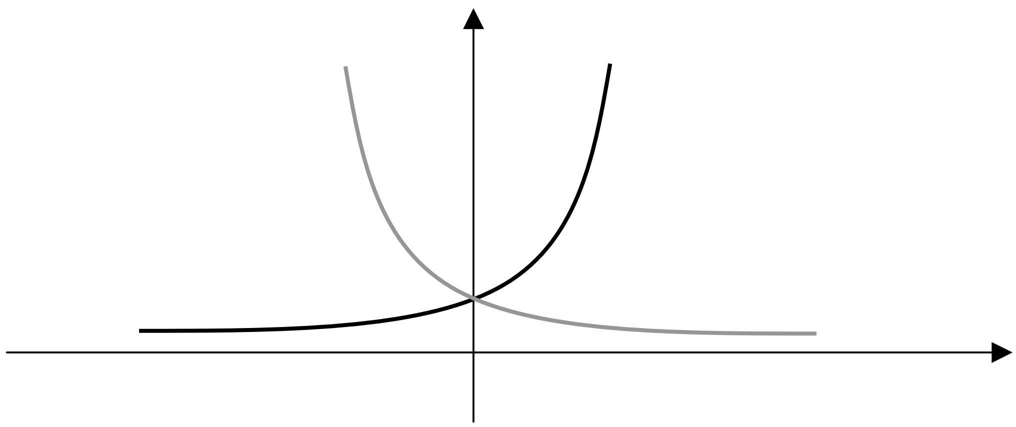


The two curves are symmetric about the line $\boldsymbol{y = x}$.

So two corresponding points from the two curves make a pair, and the two points in each pair are symmetric about the line $\boldsymbol{y = x}$.

The analytic approach (algebraic approach) to how all the pairs of points are symmetric about the line $y = x$ is covered in the book, **Function Transformations**, which covers how to manipulate curves of functions and equations.

6. Exponential & Log Functions



Normally, indicating a value, we use a number as 5, 12, 1.4, 1/3, etc. We cannot indicate however, some values using such numbers. Indicating for instance, a diagonal in a square, we cannot use just a number. Indicating it, we have to put numbers in a special form, or need to use a special sign. What sign then is it?

It is called a radical sign as a square root sign, called a second radical sign or a second root sign, too. So for instance, if a square is 1 by 1, its diagonal is $\sqrt{2}$.

Without using such a sign though, we can still indicate such a value using just numbers only. That is, we use a special expression to indicate a value using numbers. We put together numbers in a special form or structure. So what expression is it?

It is called a power.
Putting a value in a power, we express the value in terms of a base and an exponent.
For instance, we can put the diagonal above this way, too: $2^{0.5}$.
And of course, we can put it this way, too: $2^{\frac{1}{2}}$. So we can set $\sqrt{2} = 2^{\frac{1}{2}} = 2^{0.5}$.

Using, for another instance, the power expression, we can put 9 in 3^2, where the base is 3, and the exponent is 2. And of course, we can put it this way, too: 9^1, where the base is 9, and the exponent is 1.

And in fact, every number can be said to have a base and an exponent. For instance, we can say that a number 7 has 7 as the base, and has 1 as the exponent, since $7 = 7^1$, which is a power, a power of 7.

So technically, a number can be said to be in two parts, one is a base, and the other is an exponent. Thus, every number can be taken as a power.

Of course, it is probably the case you've already known all the fact above. It's just a warming-up.

In this section in fact, it is assumed that you have some knowledge on expressions with powers, that is, exponential expressions. Also, you are expected to be familiar with irrational numbers, radicals, and logarithms, too. For instance, you should be able to understand such numbers and expressions as follows.

$\mathbf{2^{-3} = \frac{1}{2^3}}$, $\sqrt[3]{x}$ or $\sqrt{x+2}$, and $\mathbf{2^3 2^5 = 2^{3+5} = (2^2)^4 = 2^{10-2} = \frac{2^{10}}{2^2} = 2^{\frac{16}{2}} = (2^{16})^{0.5} = (2^{1.6})^5}$ = etc.

$\log_2 8 = 3$, $\log 2 + \log 5 = \log 10 = 1$, $\log 1 = \log_2 1 = \log_{0.3} 1 = \log_7 1 = \log_b 1 = 0$, etc.

For details on exponents, powers, logarithms, radicals, and how they work, refer to **POWERS AND LOGARITHMS**.

Now, what if the exponent changes in a power?

Then, the value of the power changes, too, of course. How though?

Using a function, we can see how values change.
More specifically, how a value changes as other value changes.

So using a function, we can see how the value of a power changes as the value of its exponent changes. What function then, is it?

It is called an exponential function. What function is it though?

If an expression is a power, or has a power, it is called an exponential expression.

So an exponential function is an exponential expression, which can be a power, or can be made of powers.
For instance, an exponential function can be any of the expressions as follows.

$\mathbf{2^x}$, $\mathbf{-3^x}$, $\mathbf{3^x + 2^{x+1}}$, $\mathbf{3^x - 2^{x+1} + 9}$, $\mathbf{3^x - 5 \cdot 2^{x+1} + 1}$, $\mathbf{2a^x - b^{x+1} + c}$, etc.

Thus, of an exponential function, the expression has a power, where the base is usually a number or a constant, and the exponent is the input variable.

In short, an exponential function has a variable exponent.

So a function exponential is a function of an exponent.
And for instance, it can be any of the functions as follows.

$\boldsymbol{y = f(x) = a^x}$ for $\boldsymbol{x}$ real, $\boldsymbol{y = g(x) = 2^x + b}$ for $\boldsymbol{x \geq 3}$, and $\boldsymbol{y = h(x) = 3^x + x + 1}$ for $\boldsymbol{x \geq 0}$.

In all the functions above, $\boldsymbol{a^x}$, $\mathbf{2^x}$, and $\mathbf{3^x}$ are powers, and in each, the exponent is the input variable.

In some exponential functions however, we can use an input variable as a base, too.
And such a function can be $\boldsymbol{y = f(x) = x^x}$. In high school math though, we don't use such functions.

- Note however, no matter what exponential function it may be, the base is positive, but is unequal to 1. The base cannot be $\leq \mathbf{0}$, and cannot be 1, either.

Why not though, negative, 0, or 1?

We have $0^2 = 0^3 = ... = 0$, and $1^2 = 1^{0.3} = ... = 1$.
So if the base is 0 or 1, there is not much of any significance in making such a function.

And if the base is negative, we cannot use many numbers as the exponent. Why not?

If the exponent is a fraction where the numerator is odd, but the denominator is even, and the base is negative, the function cannot be defined. That's because such a power cannot exist in the real number space. That is, such a power is not a real number.

For instance, $\sqrt{-2} = (-2)^{\frac{1}{2}}$ is not a real number, and we have infinitely many of such numbers even in such a tiny interval as $\mathbf{1 < x < 1.000000000000000000001}$.
So we can hardly set up a domain if the base is negative.

> Thus, using a constant as the base, we want to consider two cases, one is that the base is between 0 and 1, and the other is that the base is greater than 1.

So assuming $\boldsymbol{y = f(x) = a^x}$, we want to consider the case where $\mathbf{0 < a < 1}$, and the other case where $\boldsymbol{a > 1}$. And the same is true, too, for a function as $\boldsymbol{y = g(x) = 2a^x + 3x^2 - x + 1}$.

> And in fact, the function behaves very differently depending on the case the base belongs to. The behavior in one case is directly opposite of the behavior in the other.

How then, do we make an exponential function?

Making a function, we define it, so making a function exponential, we define it, too. And defining it, we produce a description of a function. Then, the description is a function definition, called a definition of a function, too.

So a definition of an exponential function can be as follows: $\boldsymbol{y = g(x) = 2^x}$ for $\boldsymbol{x > 3}$.

And the function above is called $\boldsymbol{g}$, which is therefore, the name of the function, and is defined only if $\boldsymbol{x > 3}$, which is thus, the domain, so $\boldsymbol{x}$ cannot get any value bigger than 3.

And defining in general, such a function as above, we can define it the way as follows.

$$\boldsymbol{y = f(x) = a^x} \text{ where } \boldsymbol{a > 0}, \text{ but } \boldsymbol{a \neq 1}$$

That's because the base has to be positive, but is not equal to 1.
So just given $y = f(x) = a^x$ only, we need to assume that $a > 0$, but $a \neq 1$.

What then about the domain?

If not specified with a function, the domain is assumed to be the largest set of numbers the function can hold for. The function f can hold for all real numbers, that is, x can be all real numbers, so the domain is a set of all real numbers. What then is the range?

We know the base a is positive, and not equal to 1, so a is positive anyway, and thus, every number a^x produces is positive, that is, every output is positive. And we know the range is the set of all the outputs, and x takes all real numbers. What then is the range?

The range is a set of all *positive* numbers.

So assuming R is a set of all real numbers, and r is a set of all *positive* numbers, we can set f: $R \longrightarrow r$. Specifically, we can set f: $x \longrightarrow a^x$ where $x \in R$, $a > 0$, $a \neq 1$.

So the exponential function f is a function from R to r.

And of course, that's not the only exponential function we can have.

For instance, another function exponential can be $y = g(x) = -2c^x + 3x - 1$ for $x > 0$.

And in that case, too, it is assume that the base $c > 0$, and $\neq 1$.
And the same is true for these, too: $y = p(x) = m^{3x+1} + x + 1$, and $y = q(x) = m^x n^{2x}$.

So in those cases, too, it is assumed that $m > 0$, and $\neq 1$, and that $n > 0$, and $\neq 1$.

And we know some functions can have their inverses.

So what about the inverse of an exponential function?

If a function has its inverse, the function is invertible, and thus, is one-to-one.
So if we want to see if a function is invertible, that is, if it can have its inverse, we want to check to see if the function is one-to-one.
In the case of a function one-to-one, the same output cannot be produced more than once.

So showing a function f is one-to-one, we want to show: $f(m) \neq f(n)$, assuming of course, $m \neq n$ where m and n are constant, and belong to the domain.

So let's now check to see if the function f below is one-to-one.

$$y = f(x) = a^x \text{ where } a > 0 \text{ but } a \neq 1$$

To begin with, we can get: $\dfrac{f(m)}{f(n)} = \dfrac{a^m}{a^n} = a^{m-n}$. The domain is a set of all real numbers.

Assuming thus, $m \neq n$ where m and n are constant and real, we get $a^{m-n} \neq a^0 = 1$.

So we get $\dfrac{f(m)}{f(n)} \neq 1$, and thus, we get $f(m) \neq f(n)$. So f is one-to-one.

Thus, f is invertible, so it has its inverse. What then is the inverse?

We know that of the exponential function f, each output is a value of a power, for instance, 9 is the value of 3^2, and each input is an exponent.
For instance, assuming $a = 3$ in a^x, we get $y = f(x) = 3^x$, so we get $f(2) = 3^2 = 9$.

So in short, of the function f, each output is a power, and each input is an exponent.

What then is each output of the inverse?

It is an exponent. And each input of the inverse is (the value of) a power.
That's because outputs of the inverse are inputs of the original function f, and outputs of the inverse are inputs of the original function f.

And we have a function, of which each input is a number that is the value of a power, and each output is an exponent.
In short, of such a function, each input is a power, and each output is an exponent.

And such a function is called a logarithmic function, called a log function for short.

So the inverse of the exponential function f is a log function.
What then, is the domain of the log function above?

We know that the log function is the inverse of the exponential function f, and also that the domain of an inverse is the range of the original function, that is, the function we take the inverse of.

So the domain of the log function is the range of the exponential function f.

And we know that the range of the exponential function f is a set of all positive numbers.

So the domain of the log function is a set of all positive numbers.
What then, about the range of the log function?

We know that the domain of the exponential function f is a set of all real numbers.

So the range of the log function is a set of all real numbers.

Thus in short, the domain of the log function is a set of all positive numbers, and the range is a set of all real numbers.

And a log function is a function, too. So making a log function, we define it, too.
How then do we define a log function?
In other words, how do we make a log function definition?

The log function stated above is the inverse of the exponential function f.

So beginning with the definition for inverse functions, we have $y = f(x) \Leftrightarrow x = f^{-1}(y)$.

Next, we have $y = f(x) = a^x$ where $a > 0$, but $a \neq 1$.   And the range of f is $y > 0$.

Next, the definition for logs is as follows $A = b^c \Leftrightarrow c = \log_b A$ where $b > 0$, but $\neq 1$.

So using both the two definitions above, we can put together the exponential function f and the inverse of f, that is, the log function the way as follows.

$y = f(x) = a^x \Leftrightarrow x = f^{-1}(y) = \log_a y$ where x is real, $y > 0$, and $a > 0$, but $a \neq 1$.

What is f^{-1}, though?

It is the name of the log function, which is $\boldsymbol{x = f^{-1}(y) = \log_a y}$.

The name of the log function is however, a bit bulky to carry around with.

And we can name a function differently if consistency is kept. So we can rename the log function above, and for instance, we can call it $\boldsymbol{g}$, and can set: $\boldsymbol{x = g(y) = \log_a y}$ for $\boldsymbol{y > 0}$.

We don't usually though, take the equality above as a log function definition.
It is quite conventional that we use $\boldsymbol{x}$ as the input variable, and use $\boldsymbol{y}$ as the output variable. That is, we usually put a function in the $\boldsymbol{x}$-$\boldsymbol{y}$ system, and not $\boldsymbol{y}$-$\boldsymbol{x}$ system.

So assuming the log function is called $\boldsymbol{g}$, and putting it in the $\boldsymbol{x}$-$\boldsymbol{y}$ system, we can put the log function $\boldsymbol{g}$ the way as follows:

$$\boldsymbol{y = g(x) = \log_a x} \text{ for } \boldsymbol{x > 0}.$$

Then, of the log function $\boldsymbol{g}$ above, each <u>x-value</u> is an input, and is the value of <u>a power</u> $\boldsymbol{a^y}$ for each <u>y-value</u>, which is an output, and is <u>an exponent</u>.

For instance, if $\boldsymbol{a = 2}$ and $\boldsymbol{x = 8}$, we get $\boldsymbol{y = 3}$, because $\boldsymbol{8 = 2^3}$. So $\boldsymbol{3 = g(8) = \log_2 8}$.

And we can put all the ideas above formally (that is, symbolically) the way as follows.

First, assuming $\boldsymbol{R}$ is a set of all real numbers, and $\boldsymbol{r}$ is a set of all positive numbers, we can set: $\boldsymbol{f: R \longrightarrow r}$. So the exponential function $\boldsymbol{f}$ is a function from $\boldsymbol{R}$ to $\boldsymbol{r}$.

And specifically, we can set $\boldsymbol{f: x \longrightarrow a^x}$ where $\boldsymbol{x \in R}$, $\boldsymbol{a > 0}$, and $\boldsymbol{\neq 1}$.

And usually, we just put it this way: $\boldsymbol{y = f(x) = a^x}$.

And next, $\boldsymbol{g}$ is the inverse of $\boldsymbol{f}$, and we know the domain of an inverse is the range of the original function, and the range of the inverse is the domain of the original.

So we can set $\boldsymbol{g: r \longrightarrow R}$. Thus, the log function $\boldsymbol{g}$ is a function from $\boldsymbol{r}$ to $\boldsymbol{R}$.

And specifically, we can set $\boldsymbol{g: x \longrightarrow \log_a x}$ where $\boldsymbol{x \in r}$, $\boldsymbol{a > 0}$, and $\boldsymbol{\neq 1}$.

Usually, we just put it this way: $\boldsymbol{y = g(x) = \log_a x}$.
And just given $\boldsymbol{y = f(x) = a^x}$, we need to assume that $\boldsymbol{x}$ is real, $\boldsymbol{a > 0}$, and $\boldsymbol{a \neq 1}$.

Also, just given $\boldsymbol{y = g(x) = \log_a x}$, we want to assume that $\boldsymbol{x > 0}$, $\boldsymbol{a > 0}$, and $\boldsymbol{a \neq 1}$.

And we want to keep in mind that a log function is an inverse of an exponential function, and vice versa. That is, both are the inverse of each other. And in short,

> Each input of an exponential function is an exponent, and the output is a power.
>
> Each input of a log function is a power, and the output is an exponent.

How do we call then $\boldsymbol{x}$ in $\boldsymbol{\log_a x}$?

It is called an antilogarithm, often just called an antilog, for short.
What then is $\boldsymbol{\log_a x}$?

It is in fact, an exponent, since if we have $\boldsymbol{y = g(x) = \log_a x}$, we get $\boldsymbol{x = a^y}$, where $\boldsymbol{y}$ is an exponent. So ($\boldsymbol{\log_a x}$) itself is an exponent.

Specifically thus, we can put an exponent in a structured form. An exponent in such a form above can let us see specifics on itself.

For instance, specifying 81, we can use 2 as the exponent using 9 as the base, and also, we can use 4 as the exponent using 3 as the base. So we can have $\mathbf{81 = 9^2 = 3^4}$.

Specifying a number therefore, we can use a different exponent using a different base. And thus, a log shows what an exponent is about. We have $2 = \log_9 81$, and $4 = \log_3 81$.

And the same is true, too, for specifying an exponent.

So specifying an exponent, we can use a different number using a different base.
For instance, we can have $8 = 2^3$, $27 = 3^3$, $64 = 4^3$, etc.
Thus, using logarithms, we can put the exponent 3 in such many ways as follows.

$\mathbf{3 = \log_2 8 = \log_3 27 = \log_4 64 = \ldots}$

So each of $\log_2 8$, $\log_3 27$, and $\log_4 64$ is a number, is an exponent, and is 3.

Thus, $\log_a x$ is in fact, an exponent.

So for another instance, we can put 16 this way, too: $16 = 2^{\log_2 16}$ since $4 = \log_2 16$, which is thus, an exponent. We can therefore, indicate x in $\log_a x$ this way, too: $x = a^{\log_a x}$.

Now, let's take a look at what's happening in the exponential function $y = f(x) = a^x$ as the input changes. That is, we are going to keep track of the y-value as the x-value changes. Then, we can see how the exponential function f behaves.

We have two cases though. In one case, the base a is positive but is less than 1, and in the other, the base a is greater than 1.

So beginning with the case where $a > 1$, we can see y gets increased as x gets increased. That is, as the exponent increases, the power increases also if the base is bigger than 1.

So for instance, we have $\ldots\ 2^{-3} < 2^{-2} < 2^{-1} < 2^0 = 1 < 2^1 < 2^2 < 2^3 < 2^4$, and so forth. How large then, the value of y can be?

It is the value of a power, and will be infinitely large if the exponent is infinitely large, that is, if the exponent x goes infinity, because the base is bigger than 1.

And thus, if the base a is bigger than 1, the power a^x keeps increasing as the exponent x increases, and the power goes infinity when the exponent goes infinity.

If however, we have $0 < a < 1$, we can see that y gets decreased as x gets increased. That is, as the exponent increases, the power decreases if the base is between 0 and 1. So for instance, we have

$\ldots\ 0.1^{-2} = 10^2 > 0.1^{-1} = 10 > 0.1^0 = 1 > 0.1^1 > 0.1^2 > 0.1^3 > 0.1^4$, and so forth.

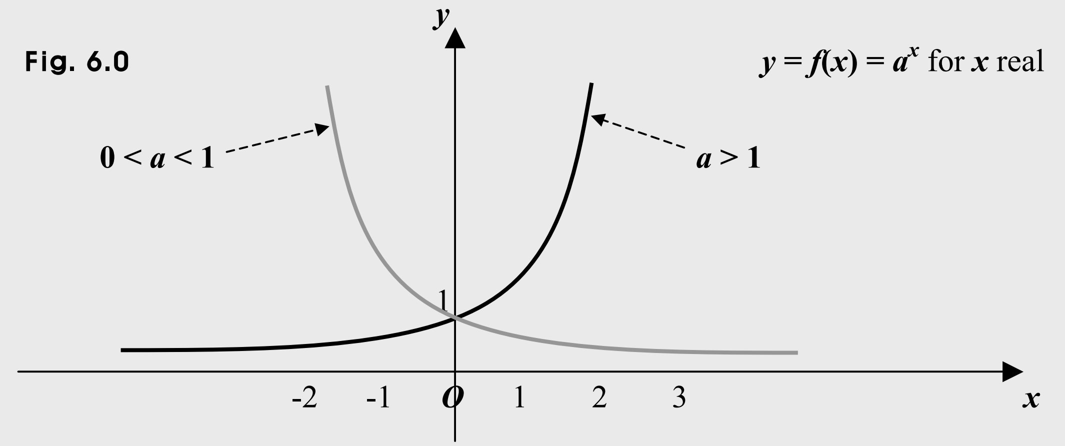
How small then the value of y can be? Can it be 0 or negative?

We know $y > 0$, so y cannot be negative, and it cannot be 0, either, but can be infinitely small being positive, though. How small then can it be anyway?

It will be infinitesimal if the exponent is infinitely large, that is, if in a^x, the exponent x goes infinity. An infinitesimal is as good as 0, but is not 0 itself.

Thus, if the base a is between 0 and 1, as the exponent increases, the power a^x keeps decreasing, and as the exponent x grows bigger, the power a^x approaches 0, then eventually, will be infinitesimal when the exponent x goes infinity.
So we can put in a graph, the curve of the exponential function f the way as follows.

Fig. 6.0



Notice that the two curves meet at the point **(0, 1)**.
Now, let's next, take a look at what's happening in the log function $y = g(x) = \log_a x$ as the input changes. That is, we are going to keep track of the value of y as x changes its value. Then, we can see what the curve of the log function g looks like.

As in the case of the exponential function, we have two cases. In one case, the base a is positive but is less than 1, and in the other, the base a is greater than 1.
So beginning with, the case where $a > 1$, we can see y gets increased as x gets increased. That is, as the antilog increases, the exponent increases, too, if the base is bigger than 1. So for instance, we have

$$\ldots\ \log_2 \tfrac{1}{8} = -3 < \log_2 \tfrac{1}{4} = -2 < \log_2 \tfrac{1}{2} = -1 < \log_2 1 = 0 < \log_2 2 = 1 < \log_2 4 = 2 \ldots$$

That's because we have $\ldots\ 2^{-3} < 2^{-2} < 2^{-1} < 2^0 = 1 < 2^1 < 2^2 < 2^3 < 2^4$, and so forth.
How large then the exponent y, that is, the value of ($\log_a x$) can be?

It will be infinitely large if the antilog x is infinitely large, that is, goes infinity.
Thus, if the base a is bigger than 1, the exponent y keeps increasing as the antilog x increases, and the exponent ($y = \log_a x$) goes infinity when the antilog x goes infinity.

If however, $\mathbf{0 < a < 1}$, we can see y decreases as x increases.
That is, as the antilog increases, the exponent decreases if the base is between 0 and 1.
So for instance, we have:

$$\ldots\ \log_{0.1}\tfrac{1}{100} = 2 > \log_{0.1}\tfrac{1}{10} = 1 > \log_{0.1} 1 = 0 > \log_{0.1} 10 = -1 > \log_{0.1} 100 = -2\ \ldots$$

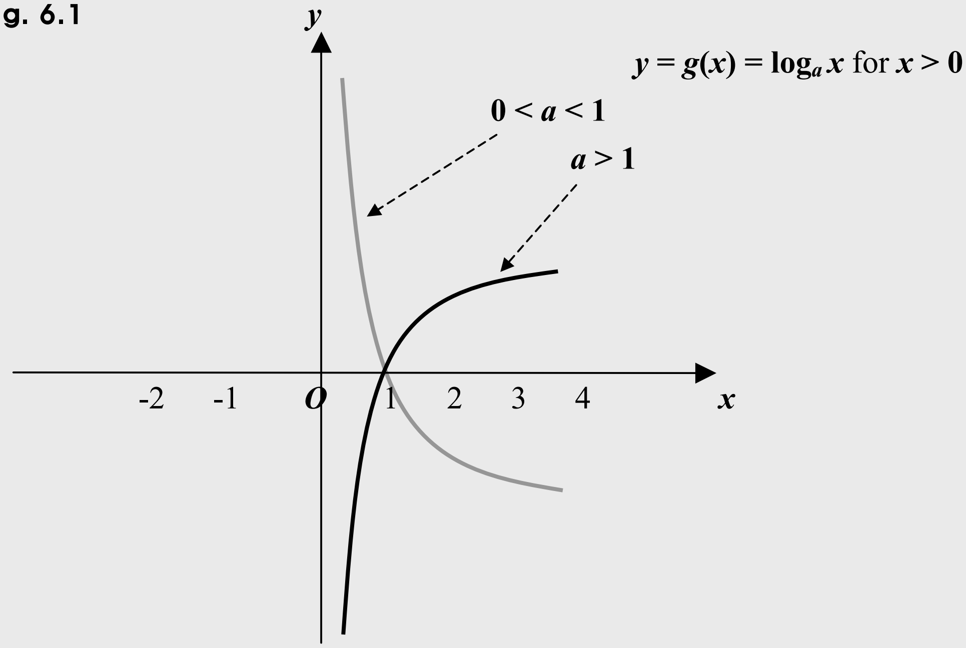
That's because we have

$\ldots\ \mathbf{0.1^{-2} = 10^2 > 0.1^{-1} = 10 > 0.1^0 = 1 > 0.1^1 > 0.1^2 > 0.1^3 > 0.1^4}$, and so on.

How small then the exponent y can be? Can it go negative infinity?

Yes, it will go negative infinity if the antilog is infinitely large, that is, if x goes infinity.
Thus, if the base is between 0 and 1, the exponent (the y-value) keeps decreasing as the antilog (the x-value) increases, and the exponent goes negative infinity when the antilog goes infinity.
So putting in a graph, the curve of the log function g, we can put it the way below.

Fig. 6.1



We can notice that the two curves meet at the point **(1, 0)**.

And note that if $a > 1$, the y-value keeps increasing, then goes infinity as the x-value keeps increasing, then goes infinity, and that if $0 < a < 1$, the y-value keeps decreasing, then goes negative infinity as the x-value keeps increasing, then goes infinity.

Now, we have $y = f(x) = a^x$ for <u>x real</u>, and $y = g(x) = \log_a x$ for <u>$x > 0$</u>.

And we know that the log function g is the inverse of the exponential function f.

So the curves of both functions are symmetric about the line $y = x$.

Thus, putting in a graph both curves of the two functions f and g, we get

Fig. 6.2

<u>$0 < a < 1$</u>

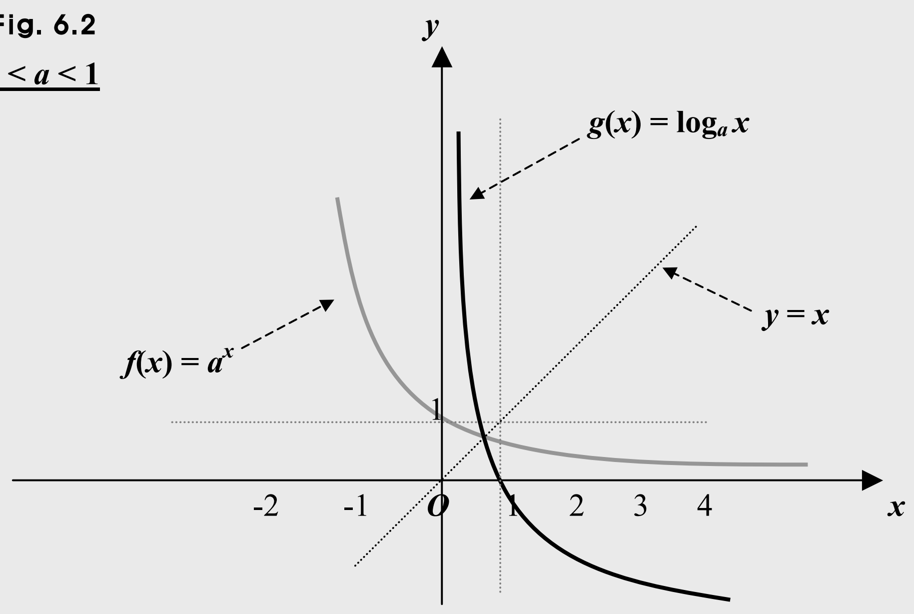
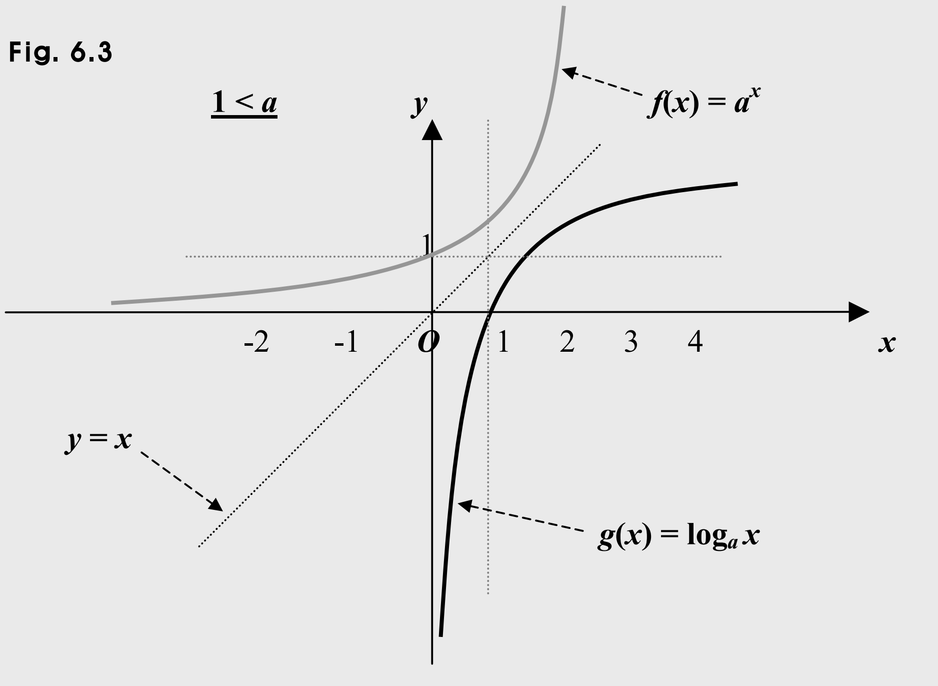


Fig. 6.3



And in this case, the line $\boldsymbol{y = 0}$ (i.e., the $\boldsymbol{x}$-axis) is called the asymptote for $\boldsymbol{y = a^x}$. Specifically, it is called a horizontal asymptote.

Also, the line $\boldsymbol{x = 0}$ (i.e., the $\boldsymbol{y}$-axis) is called the asymptote for $\boldsymbol{y = \log_a x}$, and specifically, it is called a vertical asymptote.

7. Periodic Functions

Generally, a function produces many different outputs.

In particular, a function one-to-one produces a different output for each input. That is, of such a function, all outputs are different. For any two inputs therefore, the outputs are different. So all outputs are different,.

From a function many-to-one however, same outputs can be produced. That is, from such a function, a particular output can be produced more than once, so some outputs can be the same. And thus, some outputs are the same.

From some functions many-to-one though, many different outputs get produced repeatedly. That is, from such a function, an output gets produced over and over, and the same is true for all the other outputs. And thus, each output repeats. How?

> The same outputs get produced for a group of particular inputs, and the difference between any two inputs in the group is an integer multiple of a particular value.

So for instance, if the particular value is $\boldsymbol{p}$, and $\boldsymbol{n}$ is an integer, the difference is $\boldsymbol{np}$. Thus, if $\boldsymbol{p} = \mathbf{0.3}$, the differences can be:

$\mathbf{-2 \cdot \boldsymbol{p} = -0.6}$, $\mathbf{-1 \cdot \boldsymbol{p} = -0.3}$, $\mathbf{1 \cdot \boldsymbol{p} = 0.3}$, $\mathbf{2 \cdot \boldsymbol{p} = 0.6}$, $\mathbf{3 \cdot \boldsymbol{p} = 0.9}$, $\mathbf{4 \cdot \boldsymbol{p} = 1.2}$, etc.

What then can be the particular inputs?

The particular inputs can be

-1,	**-0.7 = -1 + 0.3**,	**-0.4 = -0.7 + 0.3**,	**-0.1 = -0.4 + 0.3**,	**0.2 = -0.1 + 0.3**,
0.5 = 0.2 + 0.3,	**0.8 = 0.5 + 0.3**,	**1.1 = 0.8 + 0.3**,	**1.4 = 1.1 + 0.3**, and so forth.	

So for instance, taking the difference between the two inputs **-1** and **-0.1**, we get

$\mathbf{-1 - (-0.1) = -1 + 0.1 = -0.9 = -3 \cdot 0.3 = -3}\boldsymbol{p}$, or $\mathbf{-0.1 - (-1) = -0.1 + 1 = 0.9 = 3 \cdot 0.3 = 3}\boldsymbol{p}$.

So the differences among all the particular inputs are integer multiples of a particular value. In other words, the set of all the particular inputs forms an *arithmetic sequence*.

Assuming thus, $\boldsymbol{f}$ is the function described above, and $\boldsymbol{f}\mathbf{(-1) = 2}$, we can get

$f(-1.6) = 2$,	$f(-1.3) = 2$,	$f(-1) = 2$,	$f(-0.7) = 2$,	$f(-0.4) = 2$,
$f(-0.1) = 2$,	$f(0.2) = 2$,	$f(0.5) = 2$,	$f(0.8) = 2$,	etc.

That is, we can get

$$f(-1.6) = f(-1.3) = f(-1) = f(-0.7) = f(-0.4) = f(-0.1) = f(0.2) = f(0.5) = f(0.8) = f(1.1)$$
$$= f(1.4) = f(2) = f(2.6) = f(5.6) = f(6.2) = f(9.5) = f(15.5) = f(16.7) = f(46.7) = \ldots = 2$$

And the same is true, also, for all the other outputs from the function $\boldsymbol{f}$. So for instance,

$$f(-1.5) = f(-1.2) = f(-0.9) = f(-0.6) = f(-0.3) = f(0) = f(0.3) = f(0.6) = f(0.9) = f(1.2)$$
$$= f(1.5) = f(1.8) = \ldots = 3$$

$$f(-1.4) = f(-1.1) = f(-0.8) = f(-0.5) = f(-0.2) = f(0.1) = f(0.4) = f(0.7) = f(1) = f(4.3)$$
$$= f(4.6) = f(7.9) = \ldots = 1$$

So every time the input gets increased by the particular value $\boldsymbol{p}$, one same output gets produced *periodically*. What then, should we call such a function?

We can call it a *periodic function*. What then, can we call such a particular value?

We call it a *period*, and thus, the particular value $\boldsymbol{p}$ is the period for the function $\boldsymbol{f}$.
So for instance, the particular value **0.3** is the period for the function $\boldsymbol{f}$, which is therefore, a periodic function.

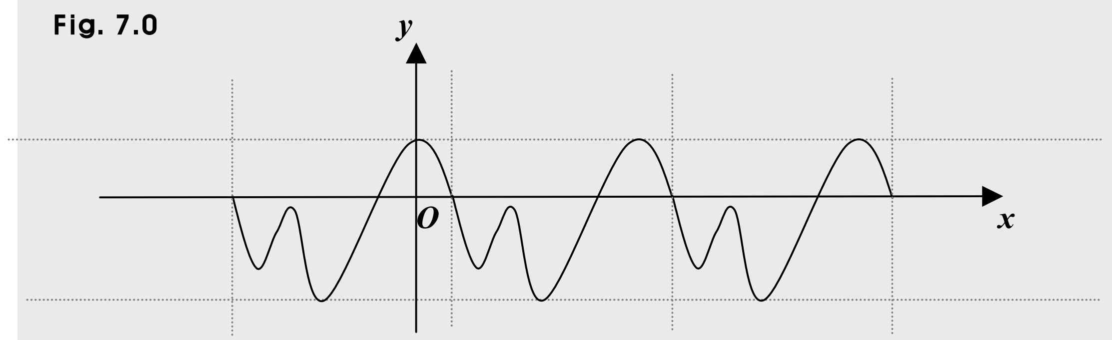
Such a period can be taken as an interval in a number line as the $\boldsymbol{x}$-axis in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane.
So in a function periodic, all the outputs get produced in such an interval.
What then, can we say about the range?

The range repeats for every period.
Does a periodic function then, have to be one-to-one within a period?

Not necessarily. So it doesn't have to be one-to-one.

The graph of a periodic function can be as follows.

Fig. 7.0



A function periodic can be therefore, many-to-one within a period, too.

That is, some numbers in the range, that is, some outputs can show up more than once within one period. And in particular, if a periodic function is continuous, the same output gets produced at the beginning and at the end of each period, that is, at both ends.

Otherwise, the function cannot be periodic.

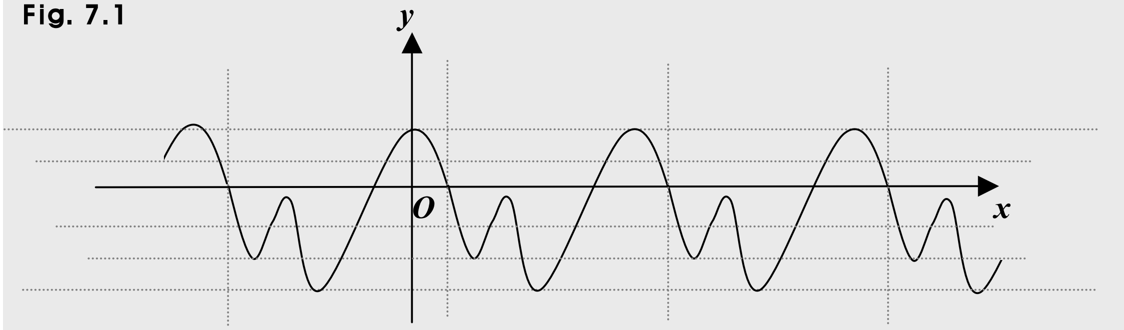
If a periodic function is not continuous at the two points each period begins and ends, the two outputs at both ends are different.

So from a periodic function, all the outputs repeat periodically.
More specifically, each and every output equally repeats with the same period.

And thus, in a function periodic, is the period the shortest length between the two inputs for the same output?

No, it's not the case. It's because the function can be many-to-one within the period, too.

Fig. 7.1



And also, we can see in the graph above, the curve of a periodic function is composed of the same curve segments, each of which is for the period.

What then, is the period?

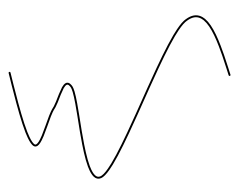
If we construct the graph of a periodic function, we can see that a particular section of the curve of the function regularly repeats for every constant interval in the x-axis.

Suppose now in the curve, we have found the *smallest section* that repeats continuously for every constant interval in the $\boldsymbol{x}$-axis.
Then, the magnitude of the interval is the period. So we can take the entire curve as a linear collection of duplicates of the smallest section that are put along a line.

We want to note however, that the line stated above is *horizontal*. What do we mean by though, the line horizontal?

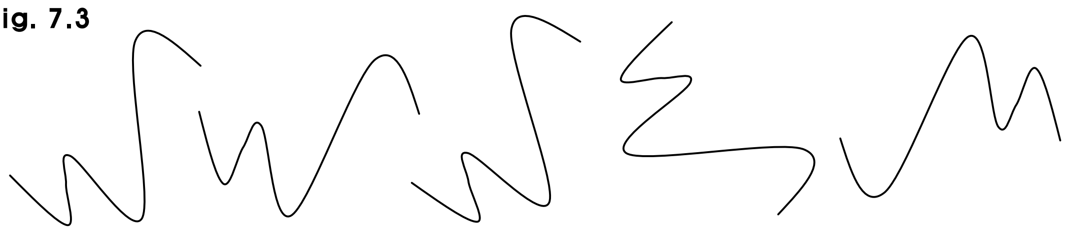
The horizontal line is parallel to the axis for the input variable as the $\boldsymbol{x}$-axis in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane. Suppose for instance, the smallest section repeating in a curve is as below.

Fig. 7.2



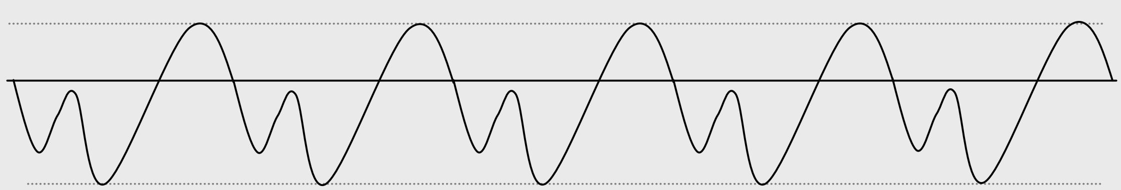
Next, making duplicates of the section above, we can get these:

Fig. 7.3



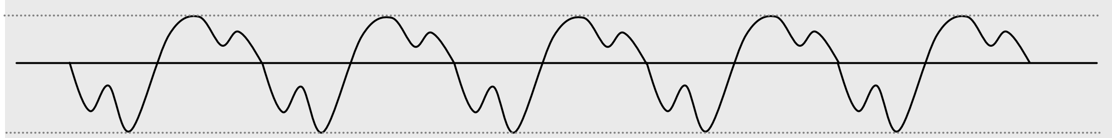
Now, making a linear collection of the sections putting them along a line horizontal, we can get a periodic curve as below.

Fig. 7.4



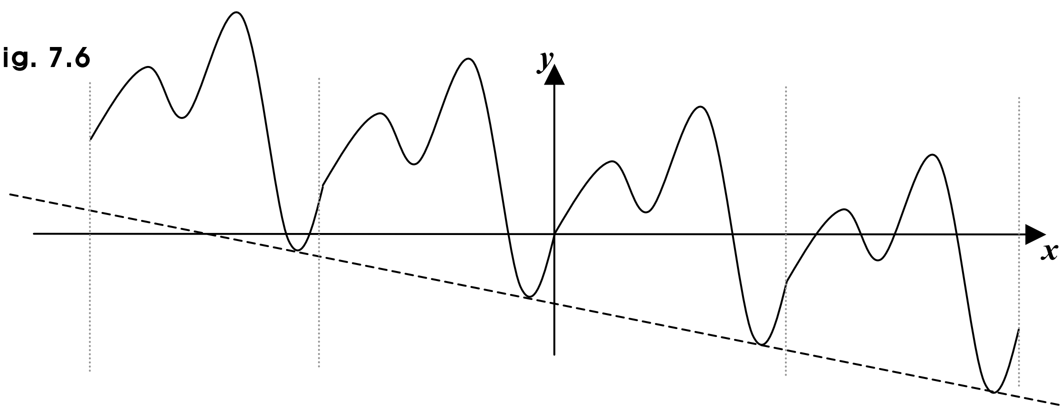
Taking another example, we can get a periodic curve as follows.

Fig. 7.5



Some curves can look periodic, but they are not from periodic functions. For instance, in the curve below, the same section repeats for every particular interval, but the function is not periodic, because the entire curve is not such a linear collection as described above. The sections are lined up on a line, which is however, not horizontal but slanted. In other words, the sections are connected along a line, not parallel to the ***x***-axis.

Fig. 7.6



That's because the function is continuous, and the output at the beginning of the interval is not the same as the output at the end of the interval. Assuming for instance, ***g*** is the function above, and the interval is ***p***, we can see in the graph below that $g(p) \neq g(2p)$.

Fig. 7.7

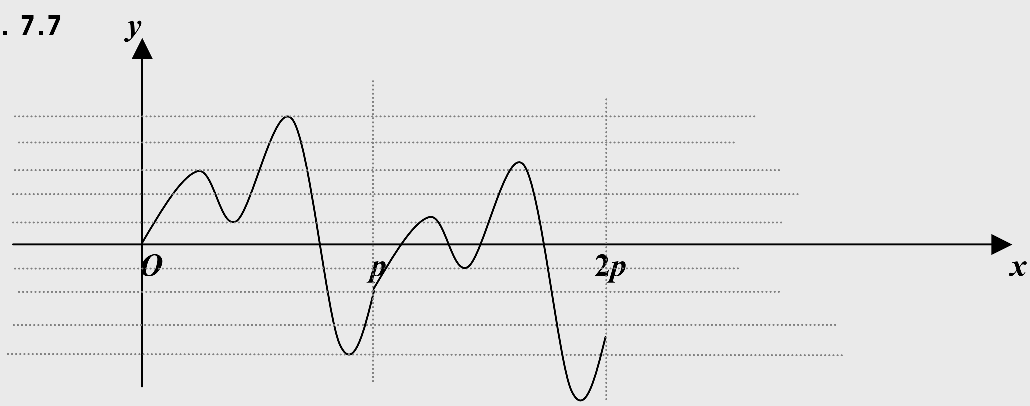
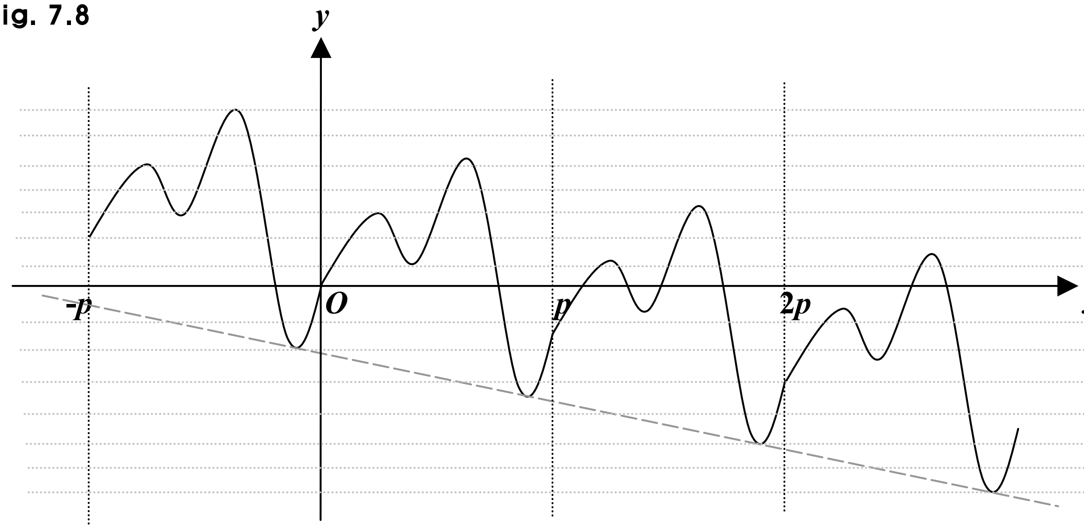
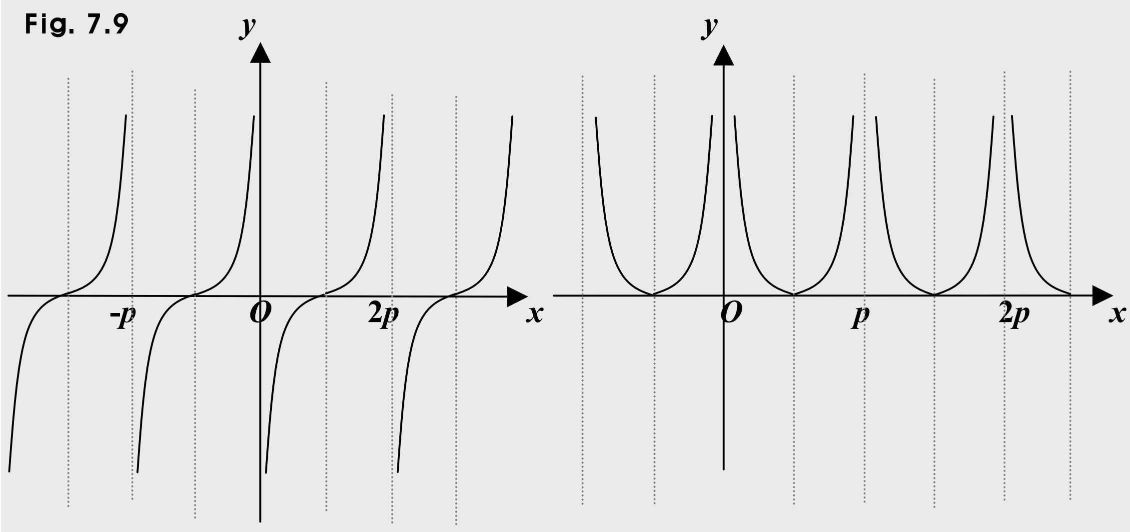


Fig. 7.8



Functions periodic are however, not necessarily continuous. So we can have periodic functions not continuous. Normally though, such functions are continuous within their periods. In other words, such a function is discontinuous at the time only when it moves on to the next period. And examples of such functions periodic and not continuous are:

Fig. 7.9



The first of the two graphs above is the graph of a function $\boldsymbol{y = t(x) = \tan x}$ where the period $\boldsymbol{p = 2\pi}$, and the second is the graph of a function $\boldsymbol{y = T(x) = |\tan x|}$ where the period $\boldsymbol{p = 2\pi}$. Such a function as above is called a trigonometric function, a typical function periodic.

How then, can we define a periodic function?
Since the function is periodic, we need to specify the period in the definition.
How then, can we specify the period?

From a periodic function, the same output gets produced for particular inputs.
And we know the particular inputs form an *arithmetic sequence*.

So let's first, make such a sequence, where the terms are the particular inputs.

Forming an arithmetic sequence, we add a particular value to each term to make the next term.
So for instance, assuming a_0 is the first term, and d is the particular value, we can get an arithmetic sequence called A as follows: $A = \{a_0, a_1, a_2, a_3, \ldots\}$, where

$a_1 = a_0 + d$, $a_2 = a_1 + d$, $a_3 = a_2 + d$, $a_4 = a_3 + d$, and so forth.

Thus, the difference between two consecutive terms is the same, and is d in the sequence A. So we call the difference d the *common difference* in the sequence A.

Now, we know all the terms in the sequence A are the particular inputs, for each of which, the same output gets produced.

So what can be the period in the periodic function?

The common difference is the period. How come?

We know all the terms in the sequence A are the particular inputs, for each of which, the same output gets produced. So assuming the periodic function is f, we get

$$f(a_0) = f(a_1) = f(a_2) = f(a_3) = \ldots$$

And we know: $a_1 = a_0 + d$, $a_2 = a_1 + d$, $a_3 = a_2 + d$, $a_4 = a_3 + d$, …
Thus, we get: $f(a_0) = f(a_0 + d) = f(a_1 + d) = f(a_2 + d) = \dots$

And assuming 2 is the output for the input a_0, we get

$$2 = f(a_0) = f(a_0 + d) = f(a_1 + d) = f(a_2 + d) = \dots$$

So if we add the period to the current input and take the sum as the next input, the output will be the same. And the same is true, too, for each of all the other outputs.
In other words, every output gets produced periodically.

So if for instance, $y = f(x) = f(x + c)$, what is c?

We have: $f(a_0) = f(a_0 + d) = f(a_1 + d) = f(a_2 + d) = \dots$, and we can put it this way, too:

$$f(a_0) = f(a_0 + d) = f(a_1) = f(a_1 + d) = f(a_2) = f(a_2 + d) = \dots$$

So when $x = a_0$, we get $f(a_0) = f(a_0 + d)$. When $x = a_1$, we get $f(a_1) = f(a_1 + d)$. And so forth.
Thus, for any value of x, we get: $f(x) = f(x + c)$. What then is c?

It is the common difference d, so we get: $c = d$, which is therefore, the period.
So we can define a periodic function f in such a way as follows:

$y = f(x) = f(x + p)$ where p is constant.

From the definition above, we can see that the function f is periodic, and p is the period.
Also, we can see that

$$f(x) = f(x + p) \Rightarrow f(x + p) = f\{(x + p) + p\} = f(x + 2p) \Rightarrow f(x) = f(x + 2p)$$

$$f(x) = f(x + p) \Rightarrow f(x + 2p) = f\{(x + 2p) + p\} = f(x + 3p) \Rightarrow f(x) = f(x + 3p)$$

$$f(x) = f(x + p) \Rightarrow f(x + 3p) = f\{(x + 3p) + p\} = f(x + 4p) \Rightarrow f(x) = f(x + 4p)$$

. . .

So assuming a function $\boldsymbol{g}$ is periodic, and the period is $\boldsymbol{p}$, we can set $\boldsymbol{g(x) = g(x + np)}$ where $\boldsymbol{n}$ is an integer. And of course, the definition of the function $\boldsymbol{g}$ can be as follows.

$\boldsymbol{y = g(x) = g(x + p)}$ where $\boldsymbol{p}$ is constant.

So for instance, if we get $\boldsymbol{h(x) = h(x + 3p)}$, or $\boldsymbol{h(x) = h(x - 2p)}$, we can see that $\boldsymbol{h}$ is a periodic function where the period is $\boldsymbol{p}$, since $\boldsymbol{3p}$ and $\boldsymbol{-2p}$ are integer multiples of $\boldsymbol{p}$.

And for another instance, assuming the period $\boldsymbol{p}$ is 1 in the periodic function $\boldsymbol{y = f(x)}$, we get: $\boldsymbol{y = f(x) = f(x + 1)}$, so we can get

When $\boldsymbol{x = 1}$, we get $\boldsymbol{f(1) = f(1 + 1) = f(2) = f(1 + 1p)}$

When $\boldsymbol{x = 2}$, we get $\boldsymbol{f(2) = f(2 + 1) = f(3) = f(1 + 2p)}$

When $\boldsymbol{x = 3}$, we get $\boldsymbol{f(2) = f(3 + 1) = f(4) = f(1 + 3p)}$

. . .

When $\boldsymbol{x = 0.2}$, we get $\boldsymbol{f(0.2) = f(0.2 + 1) = f(1.2) = f(0.2 + 1p)}$

When $\boldsymbol{x = 2.2}$, we get $\boldsymbol{f(2.2) = f(2.2 + 1) = f(3.2) = f(0.2 + 3p)}$

When $\boldsymbol{x = 5.2}$, we get $\boldsymbol{f(5.2) = f(5.2 + 1) = f(6.2) = f(0.2 + 6p)}$

$\boldsymbol{\ldots f(5.2) = f(4.2 + 1) = f(4.2) = f(0.2 + 4p) = f(3.2 + 1) = f(3.2) = f(0.2 + 3p) \ldots}$

When $\boldsymbol{x = 17.9}$, we get $\boldsymbol{f(17.9) = f(17.9 + 1) = f(18.9) = f(0.9 + 18p) = f(0.9)}$

When $\boldsymbol{x = 23.9}$, we get $\boldsymbol{f(23.9) = f(23.9 + 1) = f(24.9) = f(0.9 + 24p) = f(0.9)}$

When $\boldsymbol{x = -3}$, we get $\boldsymbol{f(-3) = f(-3 + 1) = f(-2) = f(-3 + 1p)}$

When $\boldsymbol{x = -2}$, we get $\boldsymbol{f(-2) = f(-2 + 1) = f(-2) = f(-3 + 2p)}$

When $\boldsymbol{x = -1}$, we get $\boldsymbol{f(-1) = f(-1 + 1) = f(0) = f(-3 + 2p)}$

…

Thus, in sum, we get $f(x) = f(x +$ every multiple of $p)$, where p is the period, of course.

So for instance, we get $f(0.7) = f(0.7 + np)$ where n is an integer, and p is the period.

Thus, if $p = 1$, we can get $f(0.7) = f(-0.3)$, because $f(0.7) = f(0.7 + (-1)p) = f(0.7 - 1)$.

However, the same output can be produced for other inputs, too.

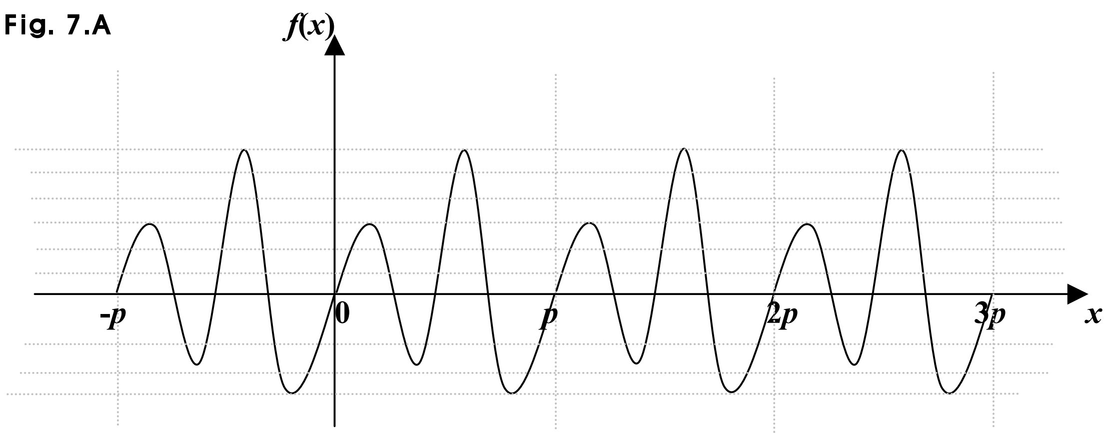
That is, a periodic function can be many-to-one for each period.

It is the case if the function periodic is continuous. In other words, if a periodic function is continuous, two or more same outputs can be produced in one period.

So for instance, assuming f is periodic and continuous, and p is the period, we can get not only $f(x) = f(x + p)$ but also $f(x) = f(x + q)$, where q is constant, but is not a period.

For instance, as shown below, $f(x)$ can be 0, also, for x not an integer multiple of p, too.

Fig. 7.A



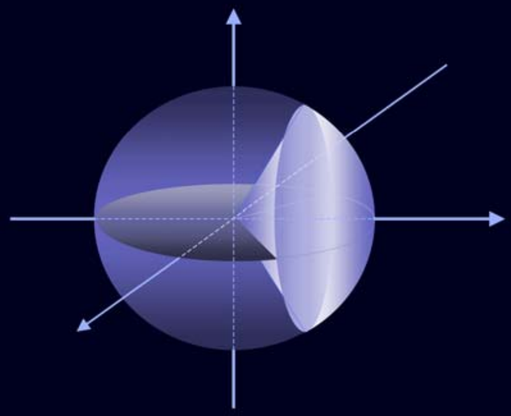
The period is p, and it is *not* the case where $f(x) = f(x + nq)$ where n is an integer.

That is, we get $f(x) \neq f(x + nq)$ where q is *not* the period, and n is an integer.

Frequent examples of periodic functions are trigonometric functions, often called briefly trig-functions. Among those, the typical ones are sine, cosine, and tangent functions.

Trig-functions are from trigonometry, which is right triangle geometry, which covers very important ratios, called trigonometric-ratios, and the details are covered in the book **Trigonometry**.

And in fact, many examples of periodic functions are some combinations of the trig-functions, and are wave functions in general. Among such, for instance, we can see functions called Saw tooth function, Square function, and Triangular function.



기본 함수

Basic Functions

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