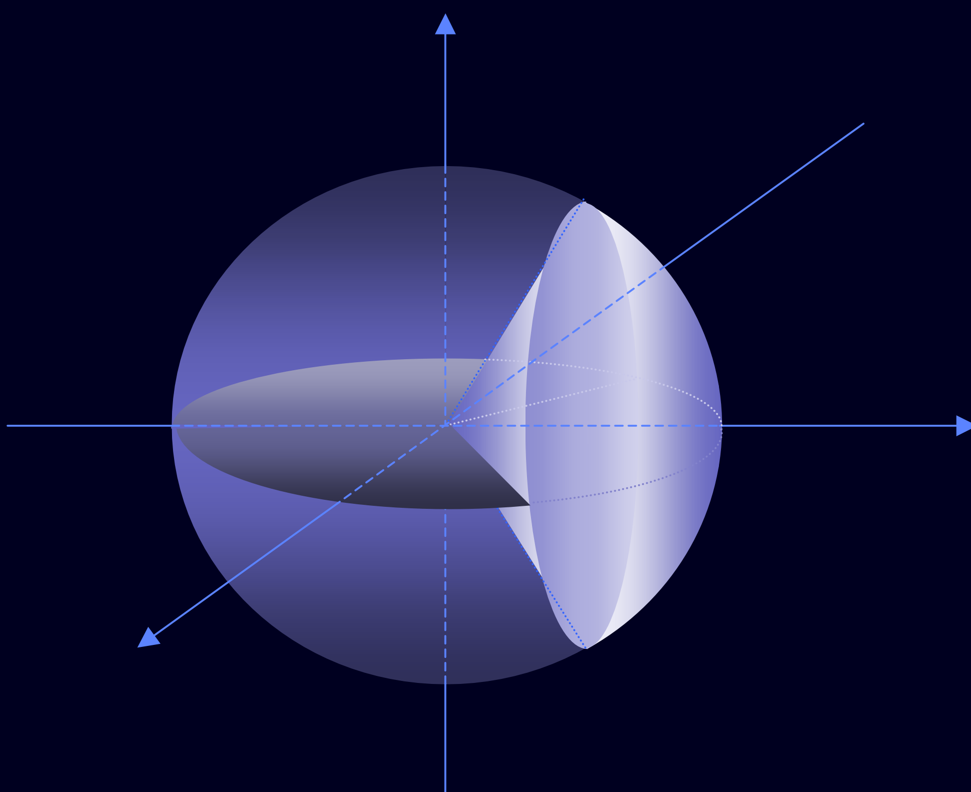
함수의 변환
Function Transformations



김성렬
Seong R. Kim

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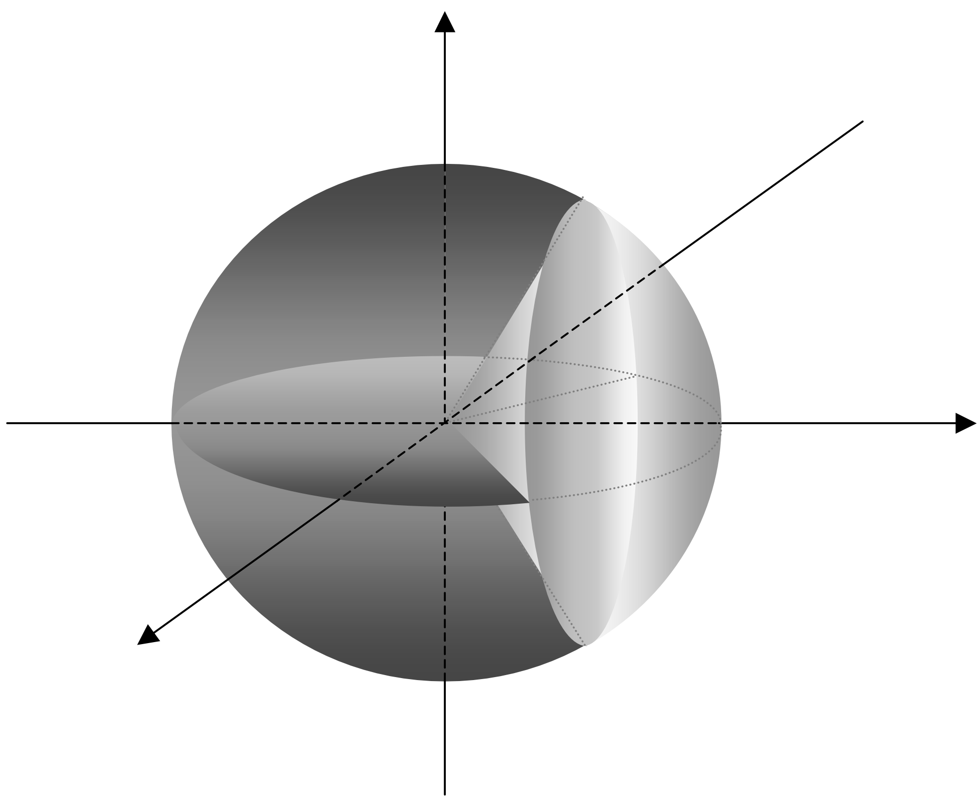

Seong R. KIM

Dear students,

Students need the best teacher, so you need examples, because examples are the best teacher. This is thus, a book of examples. All the examples here are fully worked, and explain **how** the basic but essential tools in math are made, together with **what** they are, **how** they work, and **how** to work with them. Such tools include numbers, formulas, identities, equations, laws, etc.

Doing math, we work with ideas and run ideas, because every thing in math is an idea. What idea though?

It's a concept. So we need to understand it if we want to use it. A number is an idea, for instance, and the same is true for a line or circle, too. And putting ideas together, we can build another, which becomes the base or an element of another, and each is connected. And that's the way your math grows.

So you get to build a circuit, and sometimes, need to fill the gap or repair the circuit so that you can get the sense of it. Your calculation thus, runs properly, and gets to the solution.

Getting the concept doing good examples, and gaining the fluency doing right examples, you can build the circuit strong.

This book is however, nothing but a bunch of examples until you get it powered. How then to get it powered, and make it run and work for you?

Read it closely at your pace, and then, do each example yourself. It is quite important to note that you do it in **your** own writing. Just watching someone doing it, you just only feel that you can do it, too, but can hardly do it when you need it. You do it, you get it. And you've got to do it over with different examples. It's a cliché, of course, but is always true that knowing is one thing and doing is another.

I've been helping students get conceptual understanding and fluency so that they can grow, take care of, and run their own math. The area covers the basics in algebra and geometry, and includes equations for unknowns and those for curves, functions and their graphs, and other math tools, which are the basic elements in calculus, which's been the core of my interest from my early age in high school.

Of my students, some are quite poor in math, and thus, are afraid of or even hate math, some require special education because of exceptional intelligence, some are smart enough, some are naïve and diligent, and many are clever but lazy. All the students are badly after though, one thing in common.

A Strong Foundation in Algebra

It is of course, the prime objective of my work, and I'm always happy and eager to help them achieve it. I can only help though. Your math grows as much as you grow and take care of it yourself. So does mine. What math then do students most often do or use in high schools or colleges?

Algebra and geometry. What algebra then?

Elementary algebra, of course. Doing the algebra, you should be able to not only work with, but manipulate, that is, change, alter, and convert math expressions the way you want, so for instance, you can break apart and factor polynomials in many kinds. Also, you can handle properly ratios and rates, trigonometry, factors or divisors, constants and variables, powers and logarithms or exponents, equations and functions, along with their graphs, equalities and inequalities, identities and formulas, laws and rules, etc.

And if you want to handle them properly, you want to know their natures and how they mingle with each other.

So studying those stated above, you want to know **what** they are, **how** they work, and **how** to work with them or **what** to do with them. What then about the geometry?

Basically, the geometry is about shapes, and has much to do with positions and angles in algebra. Why algebra?

You often need to compute lengths, areas, or volumes in many situations. The shapes begin with points, lines, triangles, and circles, and move on to rectangles, squares, parallelograms or rhombuses, trapezoids, tetragons, other polygons as pentagons, polyhedrons as hexahedrons, etc.

Doing the geometry, too, we need to do the algebra stated above such as trigonometry. So it is analytic geometry, often called coordinate geometry, too. And doing it, we can specify positions using coordinates. So in that geometry, basically, we work with graphs. Putting a math idea in a graph, we can not only effectively think about it, but actually see it, too, and therefore, can not just easily, but efficiently work with it, too. What idea then is it?

The idea begins with a point, a line, a parabola, a circle, an ellipse, and a hyperbola, called a conic section or basic curve, and then, moves on to other curves, planes, surfaces, volumes, and other objects in various dimensional spaces, together with vectors. And using an angle, we can specify and work with an amount of turning or change in direction.

So learning, using, or applying those ideas or tools in math, we get to solve problems. And this book can help. It can help you build and run your own math engine in your brain, and thus, can help achieve your solid math skill.

It is however, not a magic book giving you a math skill of high caliber overnight. And it's just a book of examples, and can have many mistakes. I'm a human, too. There's no mojo making a math genius, and math is science, and is full of facts and ideas. And it is after all, not me, not your teacher, and not any book but you who actually make you put together some of those facts and ideas, and understand it.

Putting facts and ideas together, understanding it, and taking care of what you have learned doing examples, you grow your math. And this book is ready to help and can help you help yourself.

Solving a problem, we put it many different ways. For instance, while expanding or reducing it, or modifying or converting it, we keep searching for the solution, keep approaching the solution, and eventually, can get there.

It is often the case a problem itself is the solution. We can put a problem in many different ways, and eventually, can end up with the solution. How come then is the solution no other than the problem?

For instance, the solution to 3232 ÷ 101 is 32. And we can put it this way:

3232 ÷ 101$=\frac{3232}{101}=\frac{32 \text{ x } 101}{101}=\frac{32}{1}=32$ $\Rightarrow$ 3232 ÷ 101 = 32.

And we can get this, too: 32 = 3232 ÷ 101.   How?

$32=\frac{32}{1}=\frac{32 \text{ x } 101}{101}=\frac{3232}{101}=$ 3232/101 = 3232 ÷ 101.

Too easy?

For another instance, the solution to $\boldsymbol{ax^2} + \boldsymbol{bx} + \boldsymbol{c} = \mathbf{0}$ is

$\boldsymbol{x} = \frac{-\boldsymbol{b} \pm \sqrt{\boldsymbol{b}^2 - 4\boldsymbol{ac}}}{2\boldsymbol{a}}$, which is called the quadratic formula.

How come then is the solution no other than the problem?

We can put it this way:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow 2ax = -b \pm \sqrt{b^2 - 4ac}$$

$$\Rightarrow 2ax + b = \pm\sqrt{b^2 - 4ac} \Rightarrow (2ax + b)^2 = b^2 - 4ac$$

$$\Rightarrow 4a^2x^2 + 4abx + b^2 = b^2 - 4ac \Rightarrow 4a^2x^2 + 4abx = -4ac$$

$$\Rightarrow ax^2 + bx = -c \Rightarrow ax^2 + bx + c = 0.$$

And we can get this, too:

$$ax^2 + bx + c = 0 \quad \Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{How?}$$

$$ax^2 + bx + c = a(x^2 + \frac{b}{a}x) + c = a(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}) + c$$

$$= a(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}) - \frac{b^2}{4a} + c = a(x + \frac{b}{2a})^2 - \frac{b^2 - 4ac}{4a} = 0$$

$$\Rightarrow a(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a} \Rightarrow (x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\Rightarrow x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \Rightarrow x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

So when solving a problem, we expand or reduce it, modify or convert it, and keep searching for and approaching the solution. Then eventually, we can get there. What then do we do when doing the set of processes above?

We do algebra. You need a strong foundation in algebra. So if a problem is well defined, that is, if it makes sense, we should be able to get it solved the way as follows.

A problem $\Rightarrow$ **…** $\Rightarrow$ **…** $\Rightarrow$ **the solution**,

and thus, **the problem** $\Rightarrow$ **the solution**.

So solving a problem, we put it many different ways so that we can get to the solution. And that's the way math runs.

Many more examples are available for free, and you can get them downloaded at **www.runmath.com**. So visit the website, get them downloaded, and do the examples. The website is getting updated weekly or monthly.

May your math run very well.

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Contents

0. What is a transformation?

A transformation is a function. What function then is it?

It is a function taking a point as an input and producing a point as an output.
What point?

It is an arbitrary point in a curve.
An arbitrary point in a curve is the representative of all the points in the curve, so it represents each and every point in the curve, and is random.
So it randomly represents each and every point in the curve. And every point in the curve works exactly the way the arbitrary point works.

And the point taken as an input is an arbitrary point in a curve to be transformed, so the curve is called the original curve.

And the point produced as an output is an arbitrary point in a curve to be made, so the curve is called the new curve.

So we can call the input the original arbitrary point, and call the output the new arbitrary point.

And using the new arbitrary point and the original curve, we can come up with the new curve.

So if using a transformation, we come up with the new curve using the new arbitrary point and the original curve.

And using a transformation, we can say we apply a transformation.
And applying a transformation, we apply it to a point, a curve, a function, or an equation.
Then, we get a new point, a new curve, a new function, or a new equation.

Why functions and equations though?

That's because a function or an equation has the curve. So transforming a curve, we transform the function or the equation that has the curve.

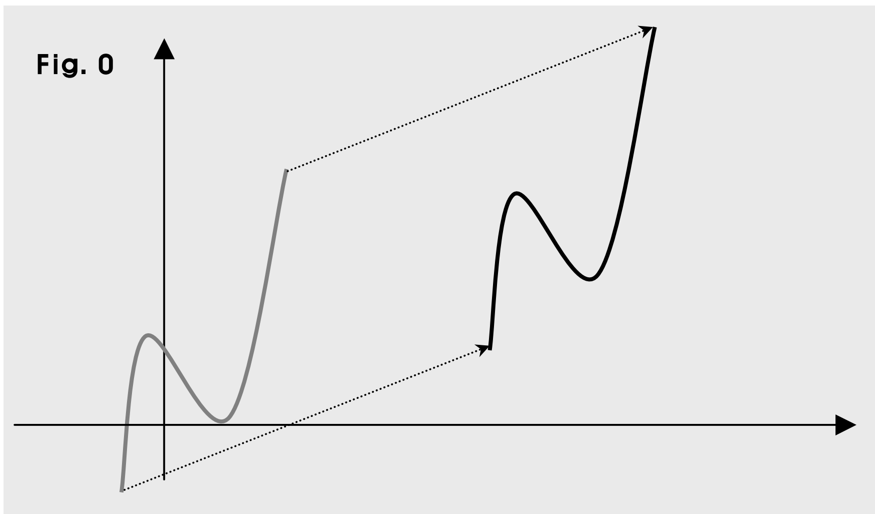
So a transformation changes a function or equation to a new function or equation.

And thus, if we want to make a change to a function, an equation, or a point, we can apply a transformation to it, and get the new one, which is the function, the equation, or the point changed.

So in sum, applying a transformation to a curve, we transform the curve. And transforming it, we change it. And changing it, we change its form or its location, or do both. So we get to have a new curve, that is, we get a new equation or a new function, because a new curve gets a new equation or a new function.

And we are going to begin with a simple transformation called a parallel transformation, and get to the bottom of it, and cover all the details so that you can get the concept of it. It is called a translation, too, and is in the next section, **Parallel Transformations**.

1.1. Parallel Transformations 1



Applying a transformation to a curve, we transform the curve. And transforming it, we change its form or its location, or do both. So we get a new curve, that is, we get a new equation, because a new curve gets a new equation.

What then is a parallel transformation?

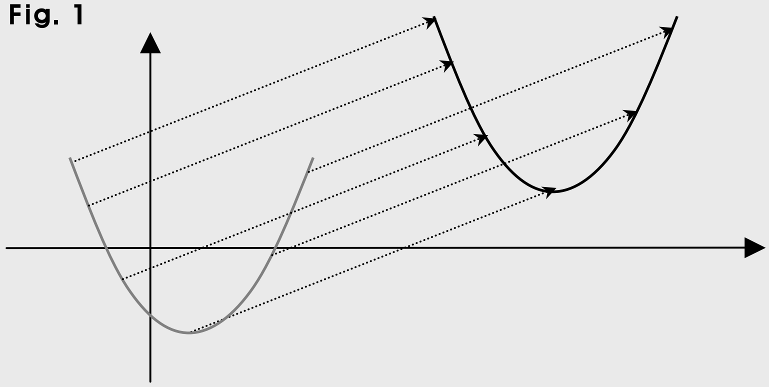
Applying a parallel transformation to a curve, we transform the curve the way as follows.

We move the curve in a particular amount along a particular line. So the curve gets a new location. Thus, we get a new curve, which is the curve that gets the new location.

Suppose now that the ***curve we transform*** is called an ***original curve***.
Then, applying a parallel transformation, we translate an original curve in a particular direction and amount, and then, get a ***new curve***, which is the ***original curve translated***.
Why parallel though?

All the points in the original curve move not only in the same amount but in the same direction, too. So all the paths all the points make are lines parallel to each other. So the transformation is said to be parallel. And we call such a transformation a translation, too.

Fig. 1

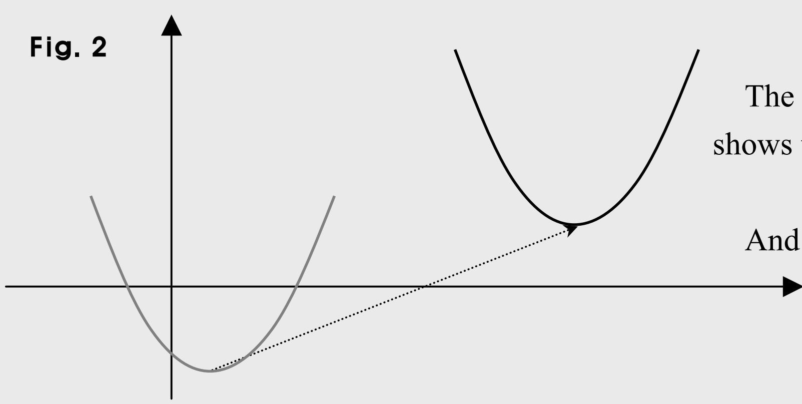


Applying thus, a parallel transformation to a curve, we translate the curve.

And translating a curve, we apply a parallel transformation to the curve.

So we can say that in the case of a parallel transformation, the original curve moves in a particular amount along a particular line without rotation and deformation. So we can simply put the graph above the way below, too.

Fig. 2



The line segment dotted shows the amount in which the curve translates. And its arrowhead shows the direction.

How then, can a curve make such a move?

Suppose now, that P is a parallel transformation, and that if we apply P to the curve of a function $y = f(x)$, we get the curve of another function $y = g(x)$.

Then, we can say that the curve of the function f is the original curve, and the curve of the function g is the new curve.

And of course, we can say that the function f is the original function, and the other function g is the new function.
So we can say that applying the transformation P to the function f, we get the function g.

In other words, transforming the function f by the transformation P, we get the function g. In short, g is the transformation of f by P. And more briefly, using a transformation operator, we can put it this way: $g = P \bullet f$, in which therefore, $\bullet$ is the operator.

In sum, applying P to f, we get $P \bullet f$, which is g. That is, we get $g = P \bullet f$.

How then can we get the new function g?

Transforming in fact, a function or an equation, we don't really change its curve directly. We change the points the curve has rather than the curve itself. We don't change though, each and every point the curve has. We can't change infinitely many points.   So?

We change one point only. If changing the one point, we make all the other points change. And they all change exactly the way the one point changes. So a new curve is made. Then, we get the new function or the new equation that has the new curve.
And the same is true for all the other transformations, too.

What then is the one point?

It is called the arbitrary point in the curve.

An arbitrary point in a curve is the representative of all the points in the entire curve, that is, it represents each and every point in the curve, and can be called a variable-point or a point-variable, too. So <u>transforming a curve</u> in fact, we <u>transform the arbitrary point</u> in the curve.

How then can we get the arbitrary point in the original curve?

It is a point, too. So we can specify it specifying the coordinates of it.
And we can specify the coordinates using the variables used in the function or equation that has (indicates) the original curve. Since the arbitrary point is in the original curve, we can just call it the original arbitrary point without mentioning the original curve.

So in the case of transforming the function $\boldsymbol{f}$, when specifying the original arbitrary point, we specify its coordinates by the variables used in the original function $\boldsymbol{y = f(x)}$, where $\boldsymbol{x}$ and $\boldsymbol{y}$ are the variables. Usually in fact, we put a function or equation in the $\boldsymbol{x}$-$\boldsymbol{y}$ system. So we normally use $\boldsymbol{x}$ and $\boldsymbol{y}$ as the variables.

Thus, the original arbitrary point is usually specified by $\boldsymbol{(x, y)}$.
How then can we get the new function $\boldsymbol{g}$?

Let's first, get back to a point. Changing a point, we get a new point. So changing the arbitrary point in the original curve, we get a new arbitrary point, which is of course, in a new curve, which is the very new curve we get transforming the original curve.

Changing thus, the original arbitrary point, we get the new arbitrary point.
Then, using the new arbitrary point, together with the original curve, we get the new curve. Specifically, using the coordinates of the new arbitrary point, along with the original function or equation, we can get the new function or the new equation.

How then do we get the new arbitrary point?

We set it up first. That is, we assume it to be the new arbitrary point. And setting it up, we specify the coordinates of it. And specifying the coordinates, we use the variables to be used in the new function or equation. Naming the variables though, we use names different from the ones used in the original. For instance, we can use $\boldsymbol{(s, t)}$ as the new arbitrary point, since $\boldsymbol{(x, y)}$ is used as the original. And after getting the new function or equation, we replace $\boldsymbol{s}$ and $\boldsymbol{t}$ with $\boldsymbol{x}$ and $\boldsymbol{y}$, because the new is in the $\boldsymbol{x}$-$\boldsymbol{y}$ system, too.

Usually, transforming, we use the definition of the transformation. And the definition has the new arbitrary point. If the definition is not given, we need to come up with the one we need, that is, we need to find it.   What then is the definition about?

The definition specifies how the new arbitrary point gets made.

So we are going to see now, how the definition can be made. There are two kinds. One defines a specific parallel transformation, and the other is the definition for parallel transformations, and thus, shows what it is in general.

So let's now begin with getting the definition for parallel transformations.

First, we can apply a transformation to a point, too.
That is, we can move a point by a transformation.
So let's begin with moving a point in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane by the parallel transformation $\boldsymbol{P}$.

Suppose now, we want to move a particular point $\boldsymbol{p_1}$ at a position $\boldsymbol{(x_1, y_1)}$ in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane to another position $\boldsymbol{(x_2, y_2)}$ by the parallel transformation $\boldsymbol{P}$.

Then, since a point in math means a position, we can set $\boldsymbol{p_1 = (x_1, y_1)}$ and $\boldsymbol{p_2 = (x_2, y_2)}$.

Suppose next, $\boldsymbol{a}$ is the amount of change in the $\boldsymbol{x}$-coordinate, and $\boldsymbol{b}$ is the amount of change in the $\boldsymbol{y}$-coordinate. That is, we have $\boldsymbol{x_2 - x_1 = a}$, and $\boldsymbol{y_2 - y_1 = b}$.

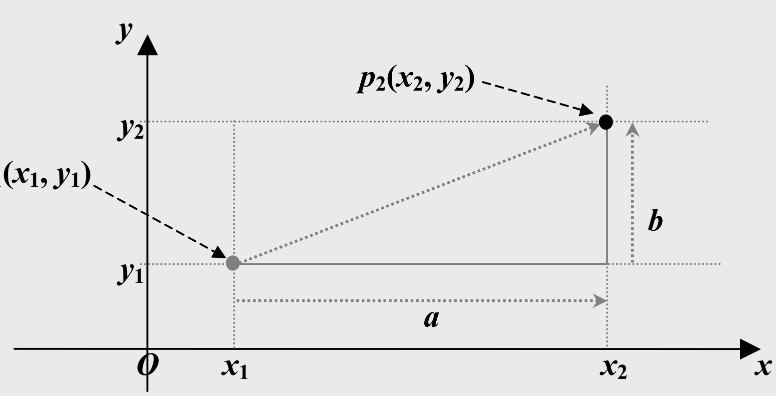
Then, if we apply the parallel transformation $\boldsymbol{P}$ to the point $\boldsymbol{p_1}$, the point $\boldsymbol{p_1}$ moves in the amount of $\boldsymbol{x_2 - x_1 = a}$ along the $\boldsymbol{x}$-axis, and in the amount of $\boldsymbol{y_2 - y_1 = b}$ along the $\boldsymbol{y}$-axis.

That is to say that we get $\boldsymbol{x_2 = x_1 + a}$, and $\boldsymbol{y_2 = y_1 + b}$.

So we can put all the ideas above this way: $\boldsymbol{(x_1, y_1) \longrightarrow (x_2, y_2) = (x_1 + a, y_1 + b)}$.

And putting in a graph, the expression above, we get

Fig. 3



Then, we can say that the point $\boldsymbol{p_1}$ has been transformed to the point $\boldsymbol{p_2}$ by $\boldsymbol{P}$.

Suppose now, $\boldsymbol{F}$ is the original curve, that is, the curve of the original function $\boldsymbol{y = f(x)}$, and $\boldsymbol{G}$ is the new curve, that is, the curve of the new function $\boldsymbol{g}$.
That is, we are now going to apply the transformation $\boldsymbol{P}$ to the entire curve of $\boldsymbol{f}$.

Then, we want to make all at once every point in the curve $\boldsymbol{F}$ move exactly the way the point $\boldsymbol{p_1}$ has been moved by $\boldsymbol{P}$. What then do we want to work with?

It is the arbitrary point in the curve $\boldsymbol{F}$. Since the arbitrary point represents every point in $\boldsymbol{F}$, transforming the arbitrary point by $\boldsymbol{P}$, we transform all the points in $\boldsymbol{F}$ in the same manner all at once. Then, the entire curve $\boldsymbol{F}$ gets transformed by $\boldsymbol{P}$, that is, it becomes $\boldsymbol{G}$.

So now, moving the arbitrary point in the curve $\boldsymbol{F}$, what do we get?

The arbitrary point in the curve $\boldsymbol{F}$, that is, the original arbitrary point is $\boldsymbol{(x, y)}$.

So moving $\boldsymbol{(x, y)}$ in the amount of $\boldsymbol{a}$ in the direction of the $\boldsymbol{x}$-axis, and in the amount of $\boldsymbol{b}$ in the direction of the $\boldsymbol{y}$-axis, we get $\boldsymbol{(x + a, y + b)}$, which is the new arbitrary point.

Thus, we can put the parallel transformation $\boldsymbol{P}$ the way as follows.

$\boldsymbol{P: (x, y) \longrightarrow (x + a, y + b)}$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are constant.

And the expression above is called the <u>definition for parallel transformations</u>.

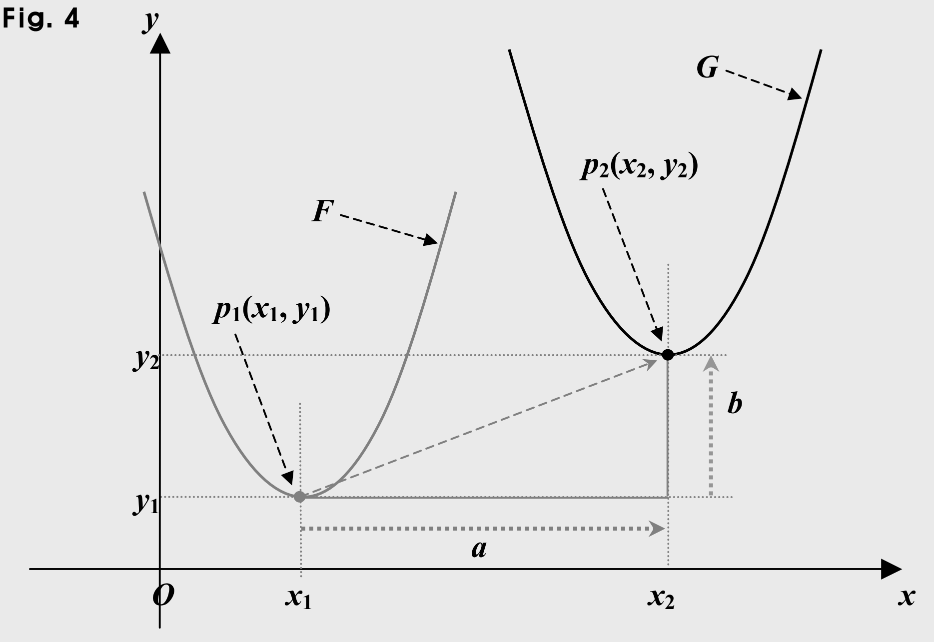
Using the new point, along with the original curve $\boldsymbol{F}$, we get the transformation of the curve $\boldsymbol{F}$ by $\boldsymbol{P}$, that is, we get the new curve $\boldsymbol{G}$.

And using the definition above, we can define (or make) a particular transformation.
If for instance, $\boldsymbol{Q}$ is a parallel transformation where $\boldsymbol{a = 5}$, and $\boldsymbol{b = 2}$, we can put $\boldsymbol{Q}$ this way: $\boldsymbol{Q: (x, y) \longrightarrow (x + 5, y + 2)}$.

Let's now, take a close look at the way parallel transformations work in general. That is, we are now going to see closely how the transformation $\boldsymbol{P}$ works.

First, assuming for instance, the original curve ***F*** stated above is a parabola, the point $\boldsymbol{p_1}$ is in the curve ***F***, and ***G*** is the new curve we get applying ***P*** to the curve ***F***, we can put in a graph, the effect of ***P*** acted on the curve ***F***, the way as follows.

Fig. 4



Then, we get $(x_2, y_2) = (x_1 + a, y_1 + b)$, and thus, applying the transformation ***P*** to the curve ***F***, we get another curve called ***G*** in a different location but in the same graph.

Note however, the two curves themselves are identical to each other.

The two ***F*** and ***G*** are said to be though, different from each other, since they don't share the same equation. That is, the equation of ***F*** is different from the equation of ***G***.

And we can notice that the new curve ***G*** can be a curve of a function, too, because no vertical line can meet the curve ***G*** at more than one point. Applying in fact, a parallel transformation to a function, we can get a function. That's because the original curve gets translated, that is, moves along a line with no rotation and deformation.

Now, if the function that has the curve ***G*** is $y = g(x)$, the function ***g*** is the new function, which is therefore, the transformation of the original function ***f*** by ***P***. In short, $g = P \bullet f$.

How then can we get the new curve ***G***, that is, the equation of ***G***, and get the function ***g***?

We can get it using the arbitrary point in $\boldsymbol{G}$, together with the original curve $\boldsymbol{F}$. Using it in fact, so to speak, we mix it into the original curve. And getting the mixture, we get the new curve, that is, the new equation. How then do we get the mixture?

Let's first, get back to $\boldsymbol{P}$. Then, we have $\boldsymbol{P}: (\boldsymbol{x}, \boldsymbol{y}) \longrightarrow (\boldsymbol{x} + \boldsymbol{a}, \boldsymbol{y} + \boldsymbol{b})$.

Then, $(\boldsymbol{x}, \boldsymbol{y})$ is the arbitrary point in the original curve.
So $(\boldsymbol{x}, \boldsymbol{y})$ can be just quickly called the original arbitrary point.
What then do we mean by $(\boldsymbol{x} + \boldsymbol{a}, \boldsymbol{y} + \boldsymbol{b})$? What does it say?

It shows how each and every point in the new curve gets made. So it explains ***how*** the arbitrary point in the new curve is made, that is, ***how*** the new arbitrary point gets made.

Given thus, the transformation we need to apply, we are given ***how*** the new arbitrary point gets made from the original arbitrary point.

So assuming now, $(\boldsymbol{s}, \boldsymbol{t})$ is the new arbitrary point, we can say that the transformation $\boldsymbol{P}$ is saying that the point $(\boldsymbol{x}, \boldsymbol{y})$ becomes or changes to $(\boldsymbol{s}, \boldsymbol{t})$ by means of $(\boldsymbol{x} + \boldsymbol{a}, \boldsymbol{y} + \boldsymbol{b})$.

So given the transformation definition, we are given specifically ***how*** each and every point in the new curve gets made. In short, the definition shows the way the new arbitrary point gets made. And the way is $(\boldsymbol{x} + \boldsymbol{a}, \boldsymbol{y} + \boldsymbol{b})$, which is in a form of a point.

So $\boldsymbol{x} + \boldsymbol{a}$ shows how the $\boldsymbol{x}$-coordinate at the new arbitrary point gets made, and $\boldsymbol{y} + \boldsymbol{b}$ shows how the $\boldsymbol{y}$-coordinate at the new point gets made.

In short, the two expressions show how the new arbitrary point gets made.

Adding $\boldsymbol{a}$ to the $\boldsymbol{x}$-coordinate at a point in the original curve, we get the $\boldsymbol{x}$-coordinate at the corresponding point in the new curve. And we can put it this way: $\boldsymbol{x} \longrightarrow \boldsymbol{x} + \boldsymbol{a}$.

And adding $\boldsymbol{b}$ to the $\boldsymbol{y}$-coordinate at a point in the original curve, we get the $\boldsymbol{y}$-coordinate at the corresponding point in the new curve. So we can put it this way: $\boldsymbol{y} \longrightarrow \boldsymbol{y} + \boldsymbol{b}$.

Thus, $(\boldsymbol{x} + \boldsymbol{a}, \boldsymbol{y} + \boldsymbol{b})$ is no other than a point, and shows how all the points in the new curve get made, and is random. So $(\boldsymbol{x} + \boldsymbol{a}, \boldsymbol{y} + \boldsymbol{b})$ is no other than the arbitrary point in the new curve, and can be set equal to the new arbitrary point set up already above.

Assuming thus, (s, t) is the new arbitrary point, we can define P the way below, too.

$$P: (x, y) \longrightarrow (s, t) = (x + a, y + b).$$

In the definition of P, $(x + a, y + b)$ is the new arbitrary point, and shows how (x, y) changes to (s, t), that is, how the original curve F changes to the new curve G.
How then can we get the equation of the new curve G?

We can get it getting the connective equation between the coordinates of (s, t), the new arbitrary point. How then do we get the connective equation?

We can get it, so to speak, mixing the new arbitrary point (s, t) into the original curve F.
In other words, we can get it, blending the new arbitrary point (s, t) into the original equation $y = f(x)$. And getting the mixture, we use this: $(x + a, y + b)$.
Then, the mixture is the connective equation.

In short, mixing or blending (s, t) into $y = f(x)$ using $(x + a, y + b)$, we get the equation connecting s and t.
So when mixing happens, $(x + a, y + b)$ acts as an agent, a transformation agent. And the mixture is the new curve, that is, the new equation. How then do we get the mixture?

Suppose now that the original function f is $y = f(x) = x^2 - 4x + 5$.
Then, $y = f(x)$ means $y = x^2 - 4x + 5$, which is the equation of the original curve F. So either of $y = f(x)$ and $y = x^2 - 4x + 5$ indicates the equation of F, and can be taken as the original equation. How then do we mix (s, t) into $y = f(x)$, and get s and t connected?
That is, how do we use this: $(x + a, y + b)$?

We can use it the way as follows.

First, we have set $(s, t) = (x + a, y + b)$. So we can get $s = x + a$, and $t = y + b$.
That is, we get a simple system of two equations, and the system is for x and y.
Thus next, solving the system for x and y, we get $x = s - a$, and $y = t - b$.
So we can now mix (s, t) into $y = f(x)$. How?

We know $x = s - a$, and $y = t - b$, so mixing (s, t) into $y = f(x)$, we put $(s - a)$ into x in $f(x)$, and put $(t - b)$ into y in $y = f(x)$.

Then, we get $\boldsymbol{y = f(x) \Rightarrow t - b = f(s - a)}$, which is the equation connecting $\boldsymbol{s}$ and $\boldsymbol{t}$.
So using the agent $\boldsymbol{(x + a, y + b)}$ means that we get the system of equations stated above, and solve the system for $\boldsymbol{x}$ and $\boldsymbol{y}$, and then, do the substitutions as explained above.

What do we mean by $\boldsymbol{f(s - a)}$ though?

We have $\boldsymbol{y = f(x) = x^2 - 4x + 5}$. So we get $\boldsymbol{f(s - a) = (s - a)^2 - 4(s - a) + 5}$.
So we can express $\boldsymbol{t - b = f(s - a)}$ this way, too: $\boldsymbol{t - b = (s - a)^2 - 4(s - a) + 5}$.
What equation then is it?

We know $\boldsymbol{(s, t)}$ is the arbitrary point in the new curve $\boldsymbol{G}$. And the equation connects $\boldsymbol{s}$ and $\boldsymbol{t}$, so it is the equation connecting $\boldsymbol{s}$ and $\boldsymbol{t}$, and thus, indicates the new curve $\boldsymbol{G}$.

That is, $\boldsymbol{t - b = f(s - a)}$ is the very equation indicating the new curve $\boldsymbol{G}$.
So it is the new equation, and can be put this way, too: $\boldsymbol{t - b = (s - a)^2 - 4(s - a) + 5}$.
It's because $\boldsymbol{f(s - a) = (s - a)^2 - 4(s - a) + 5}$.

And as mentioned earlier, $\boldsymbol{s}$ is the $\boldsymbol{x}$-coordinate at the new arbitrary point, so $\boldsymbol{s}$ is just another name for the variable $\boldsymbol{x}$ in the new equation, and $\boldsymbol{t}$ is the $\boldsymbol{y}$-coordinate at the new arbitrary point, so $\boldsymbol{t}$ is just also, another name for the variable $\boldsymbol{y}$ in the new equation.

So replacing $\boldsymbol{s}$ with $\boldsymbol{x}$, and $\boldsymbol{t}$ with $\boldsymbol{y}$, we put the new equation in the $\boldsymbol{x}$-$\boldsymbol{y}$ system. So doing the replacement, we get $\boldsymbol{y - b = f(x - a)}$, which can be put this way, too: $\boldsymbol{y = f(x - a) + b}$.

So the equation of the new curve $\boldsymbol{G}$ is $\boldsymbol{y - b = f(x - a)}$. And more specifically, we can put it this way too: $\boldsymbol{y - b = (x - a)^2 - 4(x - a) + 5}$, since $\boldsymbol{f(x - a) = (x - a)^2 - 4(x - a) + 5}$.
And of course, we can put it this way, too: $\boldsymbol{y = (x - a)^2 - 4(x - a) + 5 + b}$.

How then do we get the new function $\boldsymbol{g}$?

We have $\boldsymbol{y - b = f(x - a)}$. So we get $\boldsymbol{y = f(x - a) + b}$.
And we have $\boldsymbol{y = g(x)}$. So we get $\boldsymbol{g(x) = f(x - a) + b}$.
So the new function $\boldsymbol{g}$ is $\boldsymbol{y = g(x) = f(x - a) + b}$. And we can put it the way below, too.
$\boldsymbol{y = g(x) = (x - a)^2 - 4(x - a) + 5 + b}$, since $\boldsymbol{f(x - a) = (x - a)^2 - 4(x - a) + 5}$.

Thus, summing up, we can put the whole idea above the way below.

Assuming first, (s, t) is the arbitrary point in the new curve, we can set

P: $(x, y) \longrightarrow (s, t) = (x + a, y + b)$.

So we can set $s = x + a$, and $t = y + b$, so next, we can get $x = s - a$, and $y = t - b$.
Thus next, mixing the new arbitrary point (s, t) into the original curve, that is, getting the connective equation between s and t, we get $y = f(x) \Rightarrow t - b = f(s - a)$, which is the equation connecting s and t, that is, the new equation.

And assuming for instance, the original equation is $y = f(x) = x^2 - 4x + 5$, we get
$f(s - a) = (s - a)^2 - 4(s - a) + 5$.

Thus, we get $t - b = f(s - a) \Rightarrow t - b = (s - a)^2 - 4(s - a) + 5$.

And we know s is just another name for the variable x in the new equation, and t is just also, another name for the variable y in the new equation. So replacing s with x, and t with y, we get the new equation. Doing thus, the replacements, we get $y - b = f(x - a)$, which is the new equation. And we can put it this way, too: $y = f(x - a) + b$.
And assuming the new function is $y = g(x)$, we get $y = g(x) = f(x - a) + b$, where
$f(x - a) = (x - a)^2 - 4(x - a) + 5$.

So we can put the new function g the way below, too.
$y = g(x) = (x - a)^2 - 4(x - a) + 5 + b$.

And assuming for instance, Q is a parallel transformation where $a = 5$, and $b = 2$, we can put Q this way: Q: $(x, y) \longrightarrow (s, t) = (x + 5, y + 2)$.

Then, taking $y = f(x) = x^2 - 4x + 5$ as the original function, and taking $y = g(x)$ as the new function, we get $y = g(x) = (x - 5)^2 - 4(x - 5) + 5 + 2$.

In other words, assuming $g = Q \bullet f$, where $f(x) = x^2 - 4x + 5$, and finding the new function $g(x)$, we get $y = g(x) = (x - 5)^2 - 4(x - 5) + 7$.

And also, we can put the idea the way below, too.

Moving the curve of an equation $y = f(x) = x^2 - 4x + 5$ in the amount of 5 in the direction of the x-axis, and in the amount of 2 in the direction of the y-axis, we get the curve of a new equation, and the new equation is as follows.
$y - 2 = f(x - 5)$, where $f(x - 5) = (x - 5)^2 - 4(x - 5) + 5$.

So we can put it this way: $\boldsymbol{y - 2 = f(x - 5) \Rightarrow y = f(x - 5) + 2 = (x - 5)^2 - 4(x - 5) + 5 + 2}$.

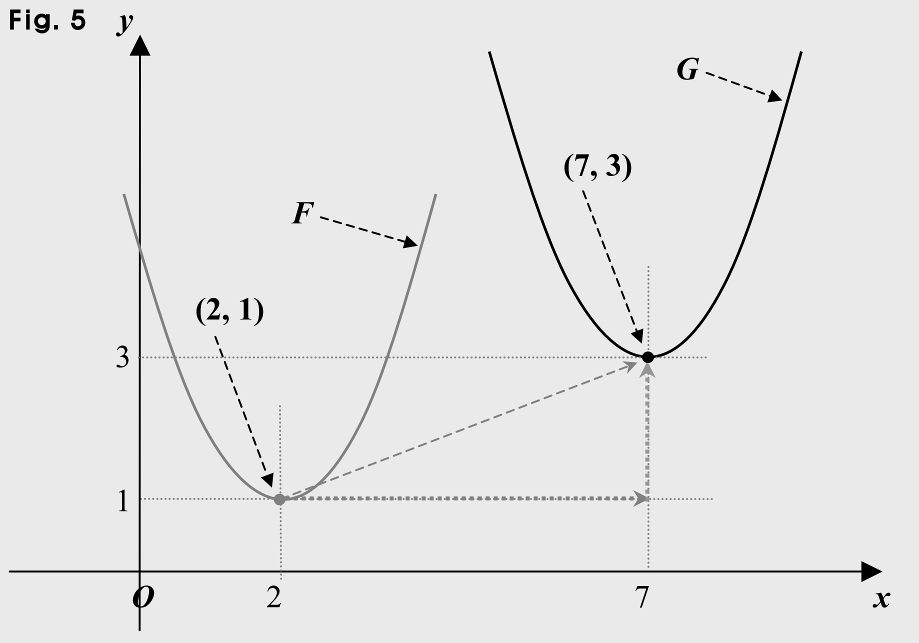
So we can put the new equation this way, too: $\boldsymbol{y = (x - 5)^2 - 4(x - 5) + 7}$.

And also, we can put the idea the way below, too.

Moving the curve of a function $\boldsymbol{y = f(x) = x^2 - 4x + 5}$ in the amount of 5 along the $\boldsymbol{x}$-axis, and in the amount of 2 along the $\boldsymbol{y}$-axis, we get the curve of a new function $\boldsymbol{g}$, and the new function $\boldsymbol{g}$ is $\boldsymbol{y = g(x) = (x - 5)^2 - 4(x - 5) + 7}$.

And putting the whole idea above in a graph, we get

Fig. 5



In the graph above, we can see that not only the point **(2, 1)** but all the other points in the curve $\boldsymbol{F}$, too, have moved in the amount of 5 along the $\boldsymbol{x}$-axis, and in the amount of 2 along the $\boldsymbol{y}$-axis. In fact, they all have moved diagonally.

And we can put the original equation $\boldsymbol{y = x^2 - 4x + 5}$ this way:

$\boldsymbol{y = x^2 - 4x + 5 = (x^2 - 4x) + 5 = (x^2 - 4x + 4 - 4) + 5 = (x^2 - 4x + 4) - 4 + 5}$
$\boldsymbol{= (x - 2)^2 + 1 \Rightarrow y = (x - 2)^2 + 1}$, which is now in the vertex form. And the vertex is at **(2, 1)**.

And we know in the parallel transformation $\boldsymbol{Q}$, every point in the original curve moves in the amount of 5 along the $\boldsymbol{x}$-axis, and in the amount of 2 along the $\boldsymbol{y}$-axis.

So the vertex **(2, 1)** gets transformed to **(2 + 5, 1 + 2)**, which is **(7, 3)**.
Thus, moving a parabola, we can easily move it moving its vertex first.
That is, we can just move the parabola exactly the way the vertex moves.

And of course, the transformation can be horizontal only or vertical only, too.

If it's horizontal, we get $\boldsymbol{b = 0}$ in $\boldsymbol{(x + a, y + b)}$ as in the case of $\boldsymbol{(x, y) \longrightarrow (x + 3, y)}$. Then, the curve moves in the amount of 3 along the $\boldsymbol{x}$-axis only, that is, horizontally only.

And if it's vertical, we get $\boldsymbol{a = 0}$ as in the case of $\boldsymbol{(x, y) \longrightarrow (x, y - 2)}$. Then, the curve moves in the amount of -2 along the $\boldsymbol{y}$-axis only, that is, it only goes downward.

And if necessary, the diagonal distance and direction can be easily found by means of the distance formula (called Pythagorean theorem, too) and a trigonometric ratio called the tangent. So the direction is indicated by an angle.

Assuming the distance is $\boldsymbol{d}$, and using the distance formula, we get $\boldsymbol{d^2 = (\Delta x)^2 + (\Delta y)^2}$.
And assuming the angle is $\boldsymbol{\theta}$, we get $\boldsymbol{\tan\ \theta = \frac{\Delta y}{\Delta x}}$.
So in the case of the transformation $\boldsymbol{Q}$, we have $\boldsymbol{a = \Delta x = 5}$, and $\boldsymbol{b = \Delta y = 2}$.
Thus, we get $\boldsymbol{d = \sqrt{29}}$, and $\boldsymbol{\tan\ \theta = \frac{2}{5}}$.

And finding the angle $\boldsymbol{\theta}$ using a calculator, we get $\boldsymbol{\theta = \tan^{-1} \frac{2}{5} \approx 21.8^{o}}$.
Then, we can say that by the parallel transformation $\boldsymbol{Q}$, the curve $\boldsymbol{F}$ itself has moved to a new location in the amount of $\boldsymbol{d}$ at the angle of $\boldsymbol{\theta}$ against the $\boldsymbol{x}$-axis, and we can say that the curve at the new location is the new curve $\boldsymbol{G}$.

And notice that applying $\boldsymbol{Q}$ to $\boldsymbol{y = f(x)}$, we get $\boldsymbol{y - 2 = f(x - 5)}$, which is the new one.

So applying $\boldsymbol{Q}$ to $\boldsymbol{y = f(x)}$ to get the new equation, we can get the new just replacing $\boldsymbol{x}$ with $\boldsymbol{x - 5}$, and $\boldsymbol{y}$ with $\boldsymbol{y - 2}$ in the original $\boldsymbol{y = f(x)}$. That is, the new is $\boldsymbol{y - 2 = f(x - 5)}$.

Thus in general, if a parallel transformation is $\boldsymbol{(x, y) \longrightarrow (x + p, y + q)}$, and the original is $\boldsymbol{y = f(x)}$, then the new is $\boldsymbol{y - q = f(x - p)}$.

And in the next section, we will go over the idea of parallel transformation, and will do it with another example. If sure of the transformations however, skip the next two sections.

1.2. Parallel Transformations 2

In this section, we are going to go over the idea of parallel transformation covered in the previous section, and will do so with another example.

So to begin with, suppose for instance, we want to move a curve $\boldsymbol{F}$ in the amount of -4 in the direction of the $\boldsymbol{x}$-axis, and in the amount of -2 in the direction of the $\boldsymbol{y}$-axis, and the curve $\boldsymbol{F}$ is the curve of a function $\boldsymbol{y = f(x) = x^3 - 14x^2 + 63x - 88}$.

Then, we want to make at once all the points in the curve $\boldsymbol{F}$ move in the amount of -4 in the direction of the $\boldsymbol{x}$-axis, and in the amount of -2 in the direction of the $\boldsymbol{y}$-axis.

Or briefly, we can put it this way, too: all the points in the curve $\boldsymbol{F}$ move at once by -4 along the $\boldsymbol{x}$-axis, and by -2 along the $\boldsymbol{y}$-axis.

Then, the curve $\boldsymbol{F}$ translates, so it moves along a line with no turning and deformation. And making a curve translate, we apply a parallel transformation to the curve. So we want to make a parallel transformation and apply it to the function $\boldsymbol{f}$ or to the curve $\boldsymbol{F}$. How then, do we make such a transformation?

Beginning with the definition for parallel transformations, we can put it the way below.

Assuming $\boldsymbol{P}$ is a transformation where a curve gets moved in the amount of $\boldsymbol{a}$ in the direction of the $\boldsymbol{x}$-axis, and in the amount of $\boldsymbol{b}$ in the direction of the $\boldsymbol{y}$-axis, we call $\boldsymbol{P}$ a parallel transformation, and can put $\boldsymbol{P}$ the way below.

$\boldsymbol{P: (x, y) \longrightarrow (u, v) = (x + a, y + b)}$. And just briefly, $\boldsymbol{P: (x, y) \longrightarrow (x + a, y + b)}$.

Then, we call each set of expressions above the definition for parallel transformations, and can call $\boldsymbol{(x, y)}$ the original (arbitrary) point, and call $\boldsymbol{(u, v)}$ the new (arbitrary) point. So $\boldsymbol{(x + a, y + b)}$ is the new (arbitrary) point, and shows the way $\boldsymbol{(x, y)}$ changes to $\boldsymbol{(u, v)}$, that is, how each and every point in the new curve gets made, that is, how the new curve gets made. If not sure of the original and new points above, refer to the previous section.

And in (u, v), u is the x-coordinate at the new (arbitrary) point, so u is just another name for the variable x in the new equation, and v is the y-coordinate at the new point, so v is just also, another name for the variable y in the new equation. And the new equation is the equation of the new curve, of course.

Next, if defining a particular transformation, we make the particular transformation.

And in the parallel transformation we want to apply to the curve F, a is -4, and b is -2.

So assuming Q is the transformation we want, and defining Q, we can define it in such a way as follows. Q: $(x, y) \longrightarrow (u, v) = (x - 4, y - 2)$.

How then, do we apply the transformation Q to the function f (or to the curve F)?

Applying a transformation to a curve, we can use an operator, which is a dot, •.
And applying the transformation Q to the function f, we put it this way: $Q \bullet f$, which is the new function. So assuming g is the new function, we can set $g = Q \bullet f$.

And applying the transformation Q to the function f, we make at once every point in the curve F move by -4 along the x-axis, and by -2 along the y-axis.

In other words, we want to move at once all the points of F in the amount of 4 to the left, and in the amount of 2 downward. What then do we want to work with?

It is the arbitrary point in the curve F. Since the arbitrary point represents every point in F, transforming the arbitrary point by Q, we transform all the points in F all at once exactly the way we transform the arbitrary point. Thus, if we transform by Q the arbitrary point in the curve F, the entire curve F gets transformed by Q.

And in this book, the arbitrary point above is called the original arbitrary point or just the original point, for short. And the same is true for the arbitrary point in the new curve, too. So we call it the new arbitrary point or just the new point, for short. Note however, in other books, they may not be called that way.

How then do we get the new curve?

Let's now begin with the original curve ***F***.

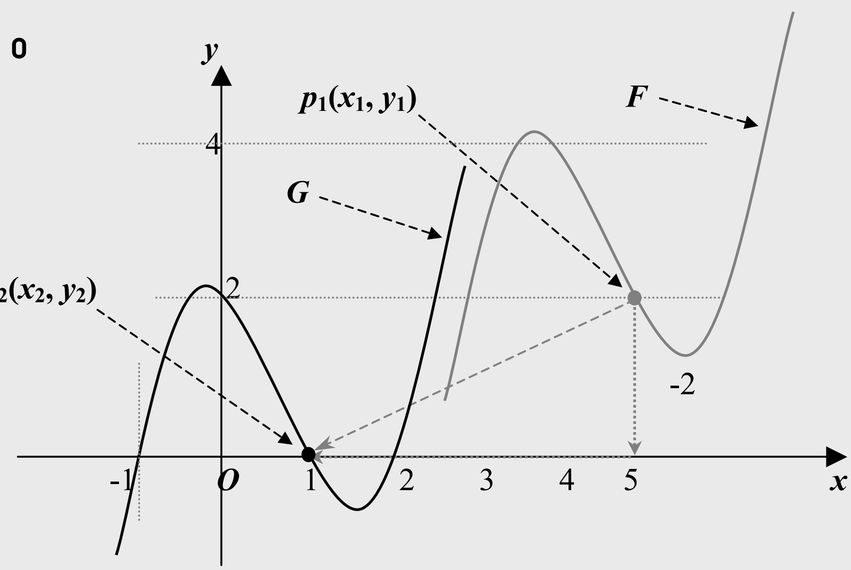
The original curve ***F*** is the curve of the function $\boldsymbol{y = f(x) = x^3 - 14x^2 + 63x - 88}$.
So we can call the function ***f*** the original function.

Assuming thus, the new function is ***g***, and applying the parallel transformation ***Q*** to the original function ***f***, we get the new function ***g***.

That is to say that if we apply the transformation ***Q*** to the function ***f***, the curve of ***f*** moves in the amount of –4 along the ***x***-axis, and in the amount of –2 along the ***y***-axis, and gets a new position, and then, the curve located at the new position is the curve of ***g***.

Assuming thus, ***G*** is the new curve, we can put all the ideas in a graph the way below.

Fig. 0



So we get another curve called ***G*** in a different location but in the same graph.
The two curves themselves are identical to each other.
The two ***F*** and ***G*** are however, said to be different from each other, since they don't share the same equation. That is, the equation of ***F*** is different from the equation of ***G***.
Also, we can notice that the curve ***G*** can be a curve of a function, too, since every vertical line meets the curve ***G*** at one point only. Applying to a function in fact, a parallel transformation, we can get a function. That's because the original curve is translated, that is, moves along a line with no rotation and deformation.

So if the function that has the curve $\boldsymbol{G}$ is $\boldsymbol{y = g(x)}$, the function $\boldsymbol{g}$ is the new function, which is therefore, the transformation of the original function $\boldsymbol{f}$ by $\boldsymbol{Q}$. In short, $\boldsymbol{g = Q \bullet f}$.

How then can we get the equation of the new curve $\boldsymbol{G}$, and also, the new function $\boldsymbol{g}$?

We can get it getting the connective equation between the coordinates of $\boldsymbol{(u, v)}$, the new arbitrary point.
And we can get it mixing the new arbitrary point into the original curve, if you will.
And getting the mixture, we get the connective equation.
How then do we get the mixture?

We have set $\boldsymbol{(u, v) = (x - 4, y - 2)}$. So we can get $\boldsymbol{u = x - 4}$, and $\boldsymbol{v = y - 2}$.
Thus, we get $\boldsymbol{x = u + 4}$, and $\boldsymbol{y = v + 2}$. So we can now mix $\boldsymbol{(u, v)}$ into $\boldsymbol{y = f(x)}$. How?

Getting the mixture, since $\boldsymbol{x = u + 4}$, and $\boldsymbol{y = v + 2}$, we put $(\boldsymbol{s + 4})$ into $\boldsymbol{x}$ in $\boldsymbol{f(x)}$, and put $(\boldsymbol{v + 2})$ into $\boldsymbol{y}$ in $\boldsymbol{y = f(x)}$.

Then, we get $\boldsymbol{y = f(x) \Rightarrow v + 2 = f(u + 4)}$, which is the equation connecting $\boldsymbol{u}$ and $\boldsymbol{v}$.

And we have $\boldsymbol{y = f(x) = x^3 - 14x^2 + 63x - 88}$.
So we get $\boldsymbol{f(u + 4) = (u + 4)^3 - 14(u + 4)^2 + 63(u + 4) - 88}$.

So we can put $\boldsymbol{v + 2 = f(u + 4)}$ the way below, too.
$\boldsymbol{v + 2 = (u + 4)^3 - 14(u + 4)^2 + 63(u + 4) - 88}$. What equation then is it?

We know $\boldsymbol{(u, v)}$ is the arbitrary point in the curve $\boldsymbol{G}$. And the equation connects $\boldsymbol{u}$ and $\boldsymbol{v}$, so it is the connective equation between $\boldsymbol{u}$ and $\boldsymbol{v}$, and thus, indicates the new curve.

That is, $\boldsymbol{v + 2 = f(u + 4)}$ is the very equation indicating the new curve $\boldsymbol{G}$.

And we know $\boldsymbol{f(u + 4) = (u + 4)^3 - 14(u + 4)^2 + 63(u + 4) - 88}$. So we can put the equation above this way, too: $\boldsymbol{v + 2 = (u + 4)^3 - 14(u + 4)^2 + 63(u + 4) - 88}$.

And as mentioned above, $\boldsymbol{u}$ is the $\boldsymbol{x}$-coordinate at the new arbitrary point, so $\boldsymbol{u}$ is just another name for the variable $\boldsymbol{x}$ in the new equation, and $\boldsymbol{v}$ is the $\boldsymbol{y}$-coordinate at the new arbitrary point, so $\boldsymbol{v}$ is just also, another name for the variable $\boldsymbol{y}$ in the new equation.

So replacing $\boldsymbol{u}$ with $\boldsymbol{x}$, and $\boldsymbol{v}$ with $\boldsymbol{y}$, we put the new equation in the $\boldsymbol{x}$-$\boldsymbol{y}$ system. So doing the replacement, we get $\boldsymbol{v + 2 = f(x + 4)}$, which can be put this way, too: $\boldsymbol{y = f(x + 4) - 2}$.

So the equation of the new curve $\boldsymbol{G}$ is $\boldsymbol{y + 2 = f(x + 4)}$.

And of course, we can put it this way, too: $\boldsymbol{y + 2 = (x + 4)^3 - 14(x + 4)^2 + 63(x + 4) - 88}$, since $\boldsymbol{f(x + 4) = (x + 4)^3 - 14(x + 4)^2 + 63(x + 4) - 88}$.

How then do we get the new function $\boldsymbol{g}$?

We have $\boldsymbol{y = f(x + 4) - 2}$. And we have $\boldsymbol{y = g(x)}$. So we get $\boldsymbol{g(x) = f(x + 4) - 2}$.

So the new function $\boldsymbol{g}$ is $\boldsymbol{y = g(x) = f(x + 4) - 2}$.

And we can put it this way, too: $\boldsymbol{y = g(x) = (x + 4)^3 - 14(x + 4)^2 + 63(x + 4) - 88 - 2}$, since $\boldsymbol{f(x + 4) = (x + 4)^3 - 14(x + 4)^2 + 63(x + 4) - 88}$.

Thus, summing up, we can put the whole idea above the way below.

Assuming first, $\boldsymbol{(u, v)}$ is the arbitrary point in the new curve, we can set

$\boldsymbol{Q: (x, y) \longrightarrow (u, v) = (x - 4, y - 2)}$.

So we can set $\boldsymbol{u = x - 4}$, and $\boldsymbol{v = y - 2}$, so next, we can get $\boldsymbol{x = u + 4}$, and $\boldsymbol{y = v + 2}$.

Thus next, mixing the new point $\boldsymbol{(u, v)}$ into the original curve, that is, getting the connective equation between $\boldsymbol{u}$ and $\boldsymbol{v}$, we get

$\boldsymbol{y = f(x) \Rightarrow v + 2 = f(u + 4)}$, which is the connective equation between $\boldsymbol{u}$ and $\boldsymbol{v}$.

And we have $\boldsymbol{y = f(x) = x^3 - 14x^2 + 63x - 88}$.

So we get $\boldsymbol{f(u + 4) = (u + 4)^3 - 14(u + 4)^2 + 63(u + 4) - 88}$.

Thus, we get $\boldsymbol{v + 2 = f(u + 4) \Rightarrow v + 2 = (u + 4)^3 - 14(u + 4)^2 + 63(u + 4) - 88}$.

And we know $\boldsymbol{u}$ is just another name for the variable $\boldsymbol{x}$ in the new equation, and $\boldsymbol{v}$ is just also, another name for the variable $\boldsymbol{y}$ in the new equation. So replacing $\boldsymbol{u}$ with $\boldsymbol{x}$, and $\boldsymbol{v}$ with $\boldsymbol{y}$, we get the new equation. Doing thus, the replacement, we get $\boldsymbol{y + 2 = f(x + 4)}$, which is the new equation. And we can put it this way, too: $\boldsymbol{y = f(x + 4) - 2}$.

And assuming the new function is $\boldsymbol{y = g(x)}$, we get $\boldsymbol{y = g(x) = f(x + 4) - 2}$, where $\boldsymbol{f(x + 4) = (x + 4)^3 - 14(x + 4)^2 + 63(x + 4) - 88}$.

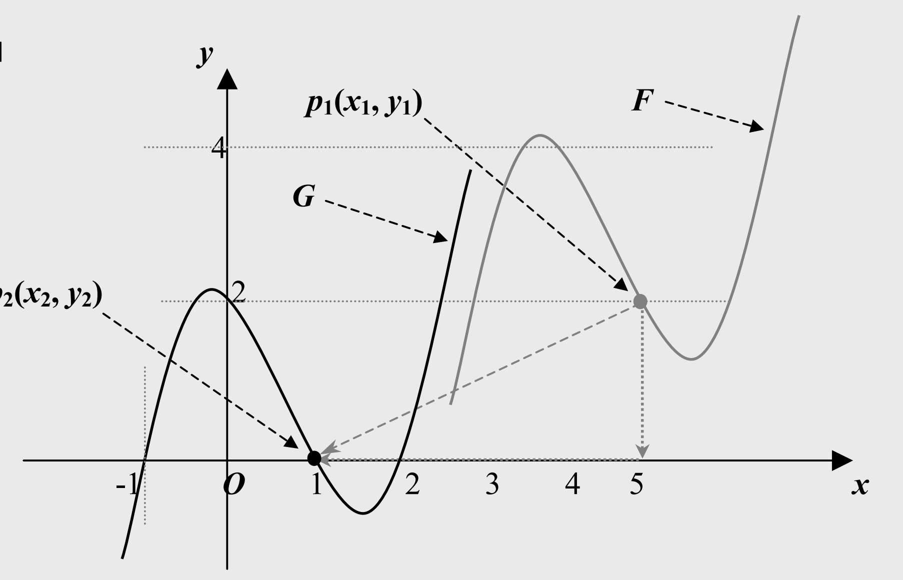
So we can put the new function $\boldsymbol{g}$ the way below, too.
$\boldsymbol{y = g(x) = (x + 4)^3 - 14(x + 4)^2 + 63(x + 4) - 90}$.

And simplifying the right hand side, we get $\boldsymbol{y = g(x) = x^3 - 2x^2 - x + 2}$.

Also, factorizing it, we get $\boldsymbol{y = g(x) = (x + 1)(x - 1)(x - 2)}$.

And putting the whole idea in a graph, we get

Fig. 1



1.3. Parallel Transformations 3

Translating a curve, we apply a parallel transformation to the curve. So moving a curve along a line with no rotation and deformation, we apply to the curve a parallel transformation.

And we can put the definition for parallel transformations the way below.

$\boldsymbol{P}$: $(\boldsymbol{x}, \boldsymbol{y}) \longrightarrow (\boldsymbol{s}, \boldsymbol{t}) = (\boldsymbol{x} + \boldsymbol{a}, \boldsymbol{y} + \boldsymbol{b})$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are constant.

Usually though, it is put briefly the way as follows. $\boldsymbol{P}$: $(\boldsymbol{x}, \boldsymbol{y}) \longrightarrow (\boldsymbol{x} + \boldsymbol{a}, \boldsymbol{y} + \boldsymbol{b})$.

In the definition above, $(\boldsymbol{x}, \boldsymbol{y})$ is the arbitrary point in the original curve, and is called briefly the original point in this book, and $(\boldsymbol{s}, \boldsymbol{t})$ is the arbitrary point in the new curve, and is thus, quickly called the new point. What then about $(\boldsymbol{x} + \boldsymbol{a}, \boldsymbol{y} + \boldsymbol{b})$?

It shows the ***way*** the original point becomes the new point, and is in a form of a point. So it is the new point, and we set it equal to the new point $(\boldsymbol{s}, \boldsymbol{t})$.

So given a transformation definition, we are given the new point.

And in the definition $\boldsymbol{P}$ above, $\boldsymbol{a}$ indicates the change in the $\boldsymbol{x}$-coordinate, and $\boldsymbol{b}$ indicates the change in the $\boldsymbol{y}$-coordinate. How then, do we get the new curve or the new equation?

Suppose first, applying the transformation $\boldsymbol{P}$ to a function $\boldsymbol{f}$, we get a function $\boldsymbol{g}$. Then, the original curve is the curve of the function $\boldsymbol{f}$, and the new curve is the curve of the function $\boldsymbol{g}$, and we can say that the function $\boldsymbol{g}$ is the transformation of the function $\boldsymbol{f}$ by $\boldsymbol{P}$.

And using the transformation operator •, we can put it the way as follows. $\boldsymbol{g} = \boldsymbol{P} \bullet \boldsymbol{f}$.

Assuming next, (s, t) is the new point, we can set $(s, t) = (x + a, y + b)$.

So we get $s = x + a$, and $t = y + b$.   Thus, we get $x = s - a$, and $y = t - b$.

Next, putting $x = s - a$ and $y = t - b$ into the original equation, we can connect s and t.

And getting the equation connecting s and t, and then, replacing s and t with x and y, we get the new equation.

So for instance, finding the new function $g = P \bullet f$, where $f(x) = x^3 - 2x^2 - x + 2$, we can find the new function g the way below.

Assuming first, (s, t) is the new arbitrary point, we can set $(s, t) = (x + a, y + b)$.

So we get $s = x + a \Rightarrow x = s - a$, and $t = y + b \Rightarrow y = t - b$.

So next, putting $x = s - a$ and $y = t - b$ into the original equation, we can connect s and t. Putting thus, $x = s - a$ and $y = t - b$ into $y = f(x) = x^3 - 2x^2 - x + 2$, we get

$t - b = f(s - a) = (s - a)^3 - 2(s - a)^2 - (s - a) + 2$, which connects s and t.

So next, replacing s and t with x and y to get the new equation, we get

$y - b = f(x - a) = (x - a)^3 - 2(x - a)^2 - (x - a) + 2$, which is the new equation.

If for instance, $a = 4$, and $b = 2$, the new equation is $y - 2 = f(x - 4)$.

More specifically, it is put this way: $y - 2 = f(x - 4) = (x - 4)^3 - 2(x - 4)^2 - (x - 4) + 2$.

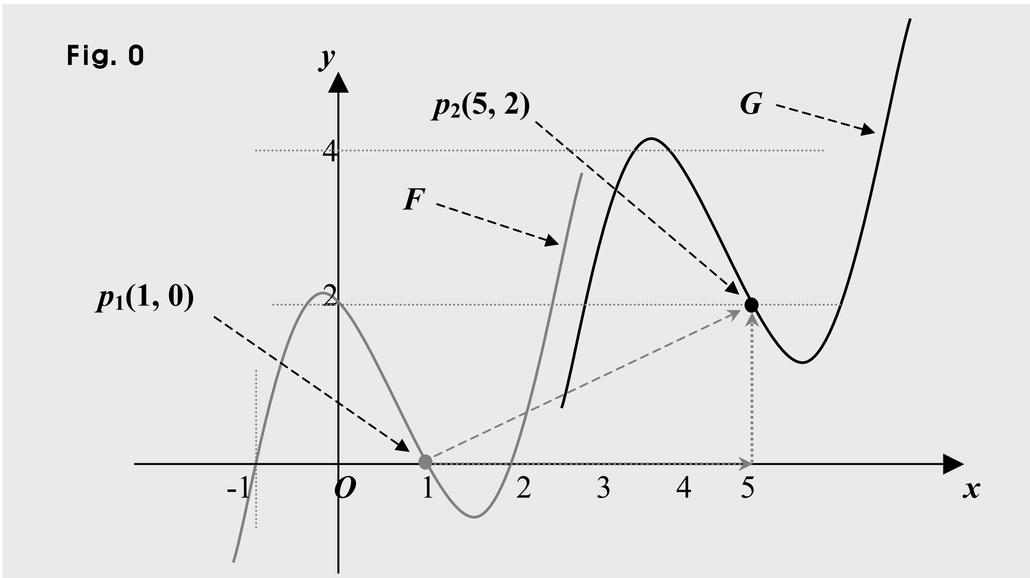
And of course, we can put it this way, too: $y - 2 = (x - 4)^3 - 2(x - 4)^2 - (x - 4) + 2$.

Or this way: $y = (x - 4)^3 - 2(x - 4)^2 - (x - 4) + 4$.

And simplifying the right hand side, we get $y = x^3 - 14x^2 + 63x - 88$.

And assuming the new function is g, we get $y = g(x) = x^3 - 14x^2 + 63x - 88$.

And assuming $\boldsymbol{F}$ is the original curve, $\boldsymbol{G}$ is the new curve, we can put them in a graph the way below.



All the points in $\boldsymbol{F}$ move exactly the way the point $\boldsymbol{p_1}$ moves to $\boldsymbol{p_2}$.

And for another instance, finding the new function $\boldsymbol{g = P \bullet f}$, where $\boldsymbol{f(x) = (x - 1)^2 + 1}$ for $\boldsymbol{0 < x < 2}$, we can find the new function $\boldsymbol{g}$ the way below.

Assuming first, $\boldsymbol{(s, t)}$ is the new point, we can get
$\boldsymbol{s = x + a \Rightarrow x = s - a}$, and $\boldsymbol{t = y + b \Rightarrow y = t - b}$.

Next, we want to get $\boldsymbol{s}$ and $\boldsymbol{t}$ connected, so putting $\boldsymbol{x = s - a}$ and $\boldsymbol{y = t - b}$ into the original equation, that is, putting $\boldsymbol{x = s - a}$ and $\boldsymbol{y = t - b}$ into $\boldsymbol{y = f(x) = (x - 1)^2 + 1}$, we get

$\boldsymbol{t - b = f(s - a) = \{(s - a) - 1\}^2 + 1}$, which connects $\boldsymbol{s}$ and $\boldsymbol{t}$.

So next, replacing $\boldsymbol{s}$ and $\boldsymbol{t}$ with $\boldsymbol{x}$ and $\boldsymbol{y}$ to get the new equation, we get
$\boldsymbol{y - b = f(x - a) = \{(x - a) - 1\}^2 + 1}$, which is the new equation.

And if for instance, $\boldsymbol{a = 4}$, and $\boldsymbol{b = 2}$, the new equation is $\boldsymbol{y - 2 = f(x - 4)}$.
More specifically, it is put this way $\boldsymbol{y - 2 = f(x - 4) = \{(x - 4) - 1\}^2 + 1}$.
And of course, we can put it this way, too: $\boldsymbol{y - 2 = (x - 5)^2 + 1}$.

Or this way: $y = (x - 5)^2 + 3$.

And simplifying the right hand side, we get $y = x^2 - 10x + 28$.

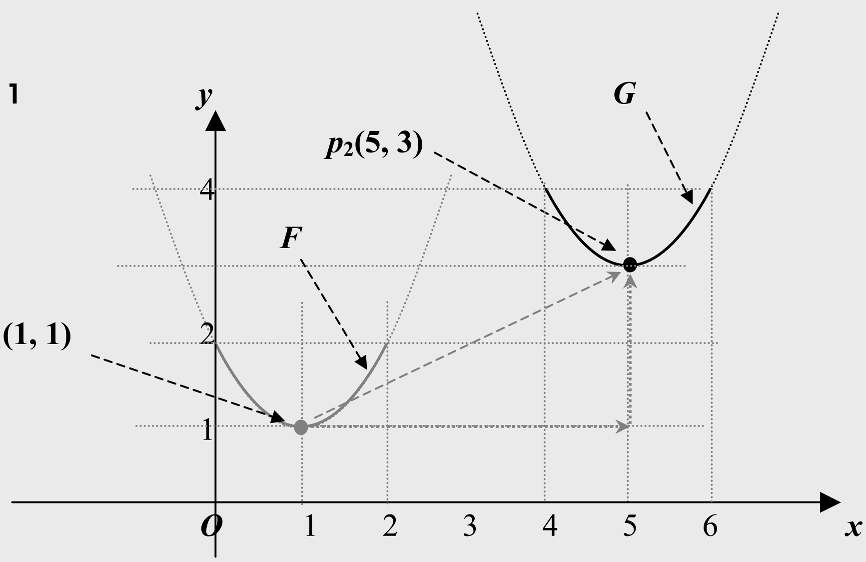
And assuming the new function is g, we get $y = g(x) = x^2 - 10x + 28$.

It is not the case however, the original function is defined for all real numbers.
The domain, that is, the original domain is $0 < x < 2$. What then about the new domain?

If the curve translates, the domain does, too. The domain is the set of all the x-values.
So finding the new domain, we count the translation in the direction of x-axis only.
And the amount is 4 to the right. So the new domain is $0 + 4 < x < 2 + 4 \Rightarrow 4 < x < 6$.

And assuming F is the original curve, G is the new curve, we can put them in a graph the way below.

Fig. 1



And notice that the range gets translated, too. The range is the set of all the y-values.
So since the original is $1 < y < 2$, the new is $1 + 2 < y < 2 + 2 \Rightarrow 3 < y < 4$.

And for another instance, finding a new equation $g = P \bullet f$, where $f(x, y) = x^2 + y^2 - 1 = 0$, we can find the new equation g the way below.

Assuming first, (s, t) is the new arbitrary point, we can get
$s = x + a \Rightarrow x = s - a$, and $t = y + b \Rightarrow y = t - b$.

Next, we want to get s and t connected, so putting $x = s - a$ and $y = t - b$ into the original equation, that is, putting $x = s - a$ and $y = t - b$ into $f(x, y) = x^2 + y^2 - 1 = 0$, we get

$f(s - a, t - b) = (s - a)^2 + (t - b)^2 - 1 = 0$, which connects s and t.

So next, replacing s and t with x and y to get the new equation, we get
$f(x - a, y - b) = (x - a)^2 + (y - b)^2 - 1 = 0$, which is the new equation.

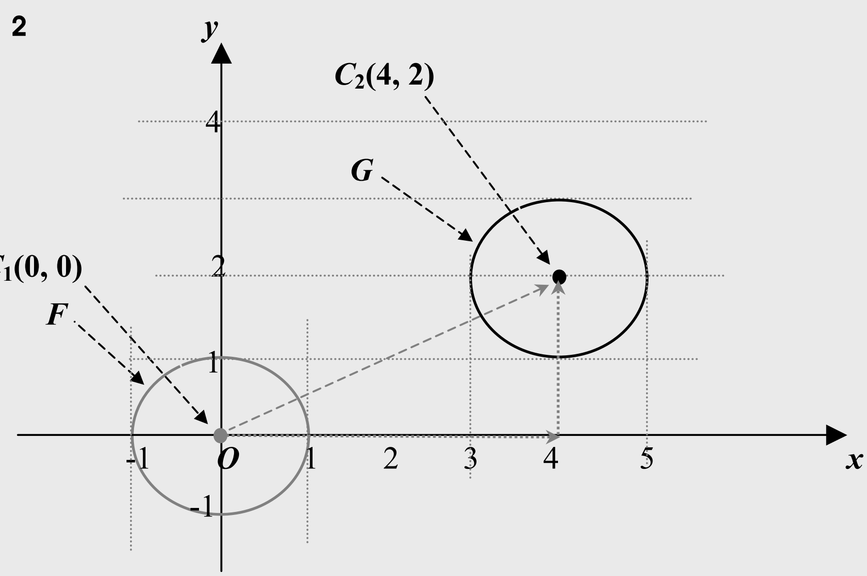
And we have assumed that g is the new equation, that is, $g(x, y) = 0$ is the new equation.
So we get $g(x, y) = f(x - a, y - b) = (x - a)^2 + (y - b)^2 - 1 = 0$.

In short, $g(x, y) = (x - a)^2 + (y - b)^2 - 1 = 0$, which is an equation of a circle of radius 1 centered at (a, b), in the x-y plane, of course.

In fact, the original equation is the equation of a circle of radius 1 centered at the origin.

And if for instance, $a = 4$, and $b = 2$, the new equation is $g(x, y) = f(x - 4, y - 2) = 0$.
More specifically, it is put this way: $g(x, y) = f(x - 4, y - 2) = (x - 4)^2 + (y - 2)^2 - 1 = 0$.
And of course, we can put it this way, too: $(x - 4)^2 + (y - 2)^2 - 1 = 0$.
Or this way: $(x - 4)^2 + (y - 2)^2 = 1$.
And simplifying the equation above, we get $x^2 - 8x + y^2 - 4y + 19 = 0$.

Fig. 2



Let's find this time, a new equation $\boldsymbol{g = P \bullet f}$, where $\boldsymbol{f(x, y) = (x - 1)^2 + (y - 3)^2 - 1 = 0}$, which is the equation of a circle of radius 1 centered at (1, 3).

Take first, $\boldsymbol{(s, t)}$ as the new point, we get $\boldsymbol{s = x + a \Rightarrow x = s - a}$, and $\boldsymbol{t = y + b \Rightarrow y = t - b}$.

Next, put $\boldsymbol{x = s - a}$ and $\boldsymbol{y = t - b}$ into the original equation to get $\boldsymbol{s}$ and $\boldsymbol{t}$ connected.
So putting $\boldsymbol{x = s - a}$ and $\boldsymbol{y = t - b}$ into $\boldsymbol{f(x, y) = (x - 1)^2 + (y - 3)^2 - 1 = 0}$, we get

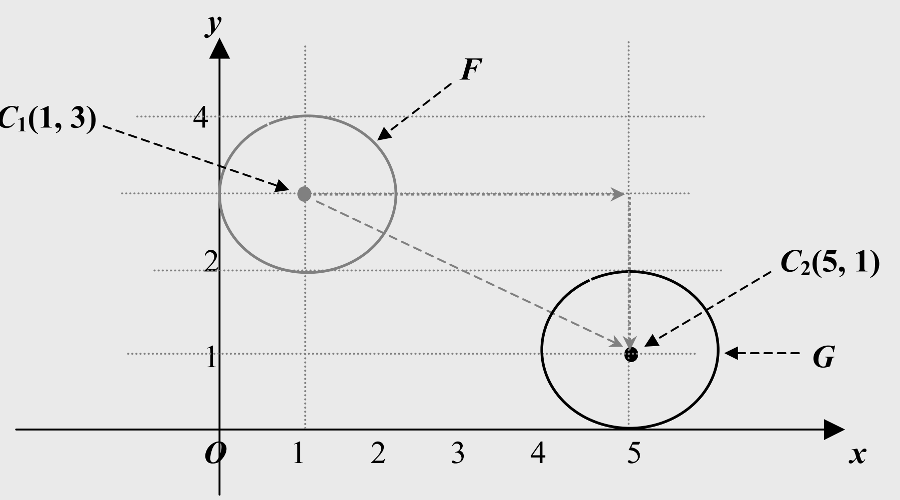
$$\boldsymbol{f(s - a, t - b) = \{(s - a) - 1\}^2 + \{(t - b) - 3\}^2 - 1 = (s - a - 1)^2 + (t - b - 3)^2 - 1 = 0}.$$

So next, replacing $\boldsymbol{s}$ and $\boldsymbol{t}$ with $\boldsymbol{x}$ and $\boldsymbol{y}$ to get the new equation, we get
$\boldsymbol{f(x - a, y - b) = (x - a - 1)^2 + (y - b - 3)^2 - 1 = 0}$, which is the new equation.

And we have assumed that $\boldsymbol{g}$ is the new equation, that is, $\boldsymbol{g(x, y) = 0}$ is the new equation.
So we get $\boldsymbol{g(x, y) = f(x - a, y - b) = (x - a - 1)^2 + (y - b - 3)^2 - 1 = 0}$, which is the equation of a circle of radius 1 centered at $\boldsymbol{(a + 1, b + 3)}$.

And if for instance, $\boldsymbol{a = 4}$, and $\boldsymbol{b = -2}$, the new equation is $\boldsymbol{g(x, y) = f(x - 4, y + 2) = 0}$.
More specifically, it is put this way: $\boldsymbol{g(x, y) = f(x - 4, y + 2) = (x - 5)^2 + (y - 1)^2 - 1 = 0}$.
And of course, we can put it this way, too: $\boldsymbol{(x - 5)^2 + (y - 1)^2 - 1 = 0}$.
Or this way: $\boldsymbol{(x - 5)^2 + (y - 1)^2 = 1}$, which is a circle of radius 1 centered at (5, 1).
And simplifying the equation above, we get $\boldsymbol{x^2 - 10x + y^2 - 2y + 25 = 0}$.

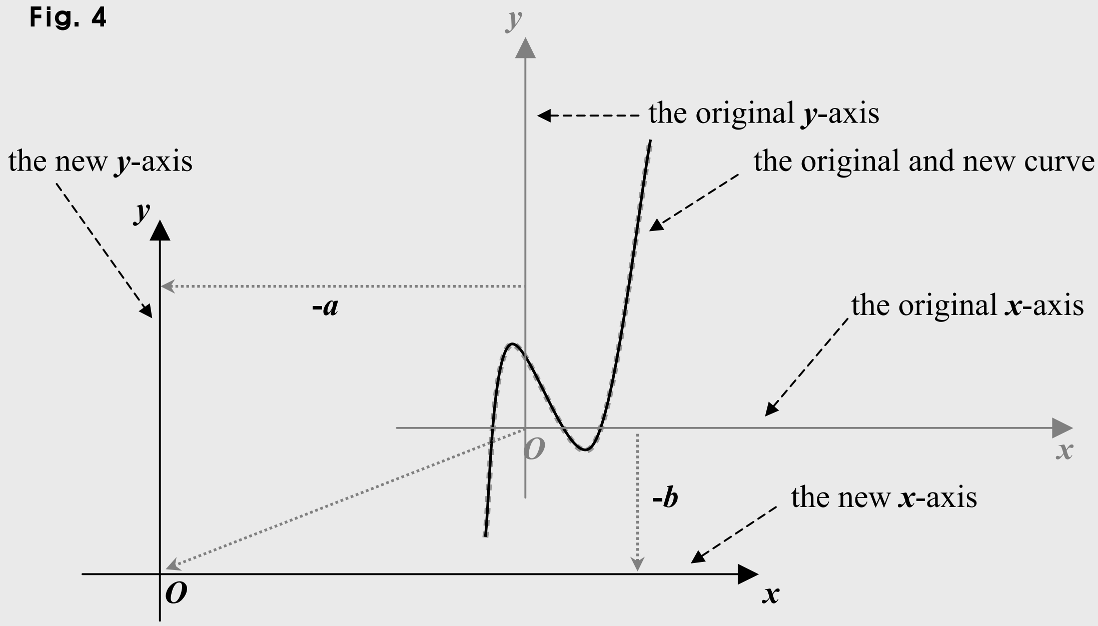
Fig. 3



And we can put a parallel transformation this way, too: applying a parallel transformation to a curve, we copy the curve, and then, paste it with no rotation.

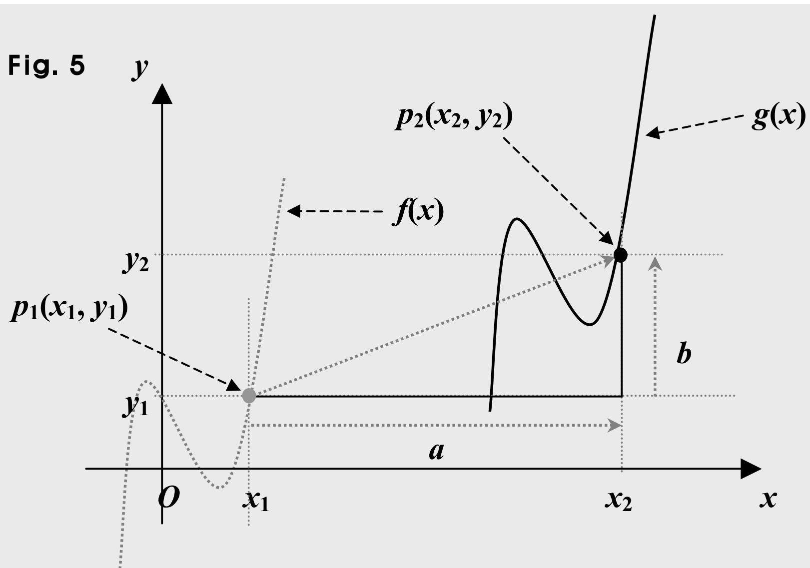
Also, there are two ways we can interpret a parallel transformation.
In one, moving the axes, that is, moving the origin keeping the curve intact, we get

Fig. 4



And in the other, moving the curve, we get

Fig. 5



So transforming a curve by a parallel transformation, we translate the curve in a particular direction and amount, and then, we get a new curve.

That is, applying a parallel transformation to a curve, we shift the curve up or down, sideways, or diagonally.

Shifting the curve up or down, we set $\boldsymbol{a} = \mathbf{0}$, shifting the curve sideways, we set $\boldsymbol{b} = \mathbf{0}$, and shifting the curve diagonally, we set for instance, $\boldsymbol{a} = \mathbf{-1}$, and $\boldsymbol{b} = \mathbf{2}$.

• And note that in this book, some important terminologies are used the way as follows.

The original curve: the curve of the function or equation a transformation is applied to
The new curve: the curve of the function or equation made by a transformation

The original function or equation: the function or equation a transformation is applied to
The new function or equation: the function or equation made by a transformation

The original arbitrary point or the original point: the arbitrary point in the original curve.
The new arbitrary point or the new point: the arbitrary point in the new curve.

2.0. Symmetric Transformations

In the case of a symmetric transformation, the new curve is symmetric to the original curve. So the two curves, the original and the new are symmetric to each other.
And thus, the two are the same in shape. That is, both have the same look.

It is often the case however, they are not identical, that is, their equations are different.
What do we mean by a symmetry though, between two curves?

Suppose for instance, the original and the new are parabolas symmetric to each other in the ***x-y*** plane. Then, first, the two curves share the same shape.

It is often the case though, the two don't share the same equation.
In other words, the original and the new are usually taken as two different curves.

That is because it is normally the case the new curve is at a different position.

It doesn't mean however, the new curve can be anywhere in the ***x-y*** plane.
The new one has to be positioned in a particular manner. What particular manner?

In the case of a symmetric transformation, what matters is <u>how</u> the new and the original curves are symmetric to each other.
More specifically, besides the fact that both share the same shape, what really matters is <u>about what</u> the two are symmetric. So what can the two be symmetric about?

If the two curves are in a plane, the two can be symmetric <u>about a point or a line</u>.
So for instance, the two can be symmetric about a point **(1, 2)** or a line $\boldsymbol{y = 2x + 1}$.

And thus, applying a transformation symmetric about a point **(1, 2)**, we get a new curve, and the new curve is symmetric to the original curve about the point **(1, 2)**.

And also, if a transformation is symmetric about a line $y = 2x + 1$, the new curve is symmetric to the original curve about the line $y = 2x + 1$.
So if we fold the plane along the line $y = 2x + 1$, the two curves match exactly.

That is, the two coincide. It is usually the case though, the two are different, because they have different equations. Is there any case then, the two curves are the same? That is, is there any case, both share the same equation?

If we apply to a parabola a transformation symmetric about the axis of symmetry of the parabola, the new curve is the very parabola to which we apply the transformation.
In that case, we say that both are the same or identical, because their equations are the same. So depending on the symmetry we apply, the two can share the same equation.

What do we mean by though, curves symmetric about a point?

Transforming a curve, we don't actually move or change the curve as a whole.
What then do we actually move or change?

It is each and every point of the curve. And in a curve, its arbitrary point represents all the points in the curve. So transforming a curve, we actually apply a transformation to the arbitrary point of the curve.

In the case of a transformation symmetric about a point therefore, each and every point of the original curve gets moved symmetrically about the point.

And the same is true for a transformation symmetric about a line, too.
So in the case of a transformation symmetric about a line, each and every point of the curve gets moved symmetrically about the line.

Thus, applying a symmetric transformation, we actually apply the transformation to the arbitrary point of the original curve.

So for instance, applying a transformation symmetric about a point **(1, 2)**, we get each and every point to be moved symmetrically about a point **(1, 2)**. And then, all the points moved form a new curve, which is said to be symmetric about the point **(1, 2)**.

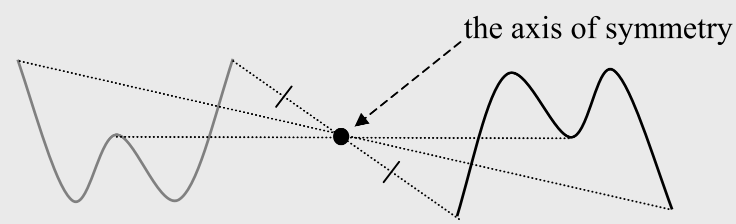
That is, the new curve is said to be symmetric to the original curve about the point **(1, 2)**.

Thus, saying two curves are symmetric to each other, we mean in both curves, each and every pair of corresponding points are symmetric to each other.
What do we mean by then, two points symmetric about a point **(1, 2)**?

If two points are symmetric about a point **(1, 2)**, the point **(1, 2)** is the midpoint between the two points. And the two points and the point **(1, 2)** are in a line.

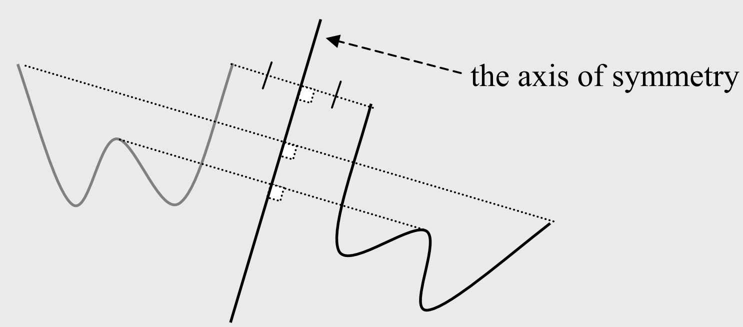
And the point **(1, 2)** can be called the axis of symmetry.
So if a point is the axis of symmetry, the point is the midpoint between every pair of two corresponding points in the two curves, the new and the original.

Fig. 0



And the same is true, also, for two points symmetric about a line $y = 2x + 1$.
So in that case, we call the line the axis of symmetry.

Fig. 1



So if a line is the axis of symmetry, the line is perpendicular to the line that connects the two points, and passes through the midpoint between the two points.

And thus, saying two curves are symmetric to each other, we mean in both curves, each and every pair of corresponding points are equal distance away from a particular point or line called the axis of symmetry.

And of course, the axis of symmetry is right in the middle of the two points in each of the pairs.

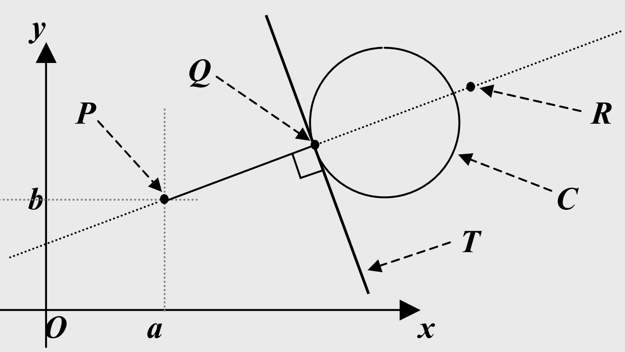
What do we mean by though, a distance in math?

Saying a distance in math, we mean the shortest distance, called the perpendicular distance, too.

What do we mean by the perpendicular distance though?

Suppose ***D*** is the distance from a point ***P*(*a*, *b*)** to a circle ***C*** as shown below.

Fig. 3



Suppose also, ***T*** is a line tangent to the circle ***C*** at a point ***Q*** as shown above.

Then, if ***D*** is the length of the line segment between ***P*** and ***Q***, the line segment is perpendicular to the tangent line ***T***.

So taking the perpendicular distance from ***P*** to the circle, we take ***D*** as the distance. Also, just saying a distance, we take a perpendicular distance as a distance.

Besides, the line containing the line segment passes through the center of the circle. That is, the line passing through the three points ***P***, ***Q***, and ***R*** passes though the center of the circle.

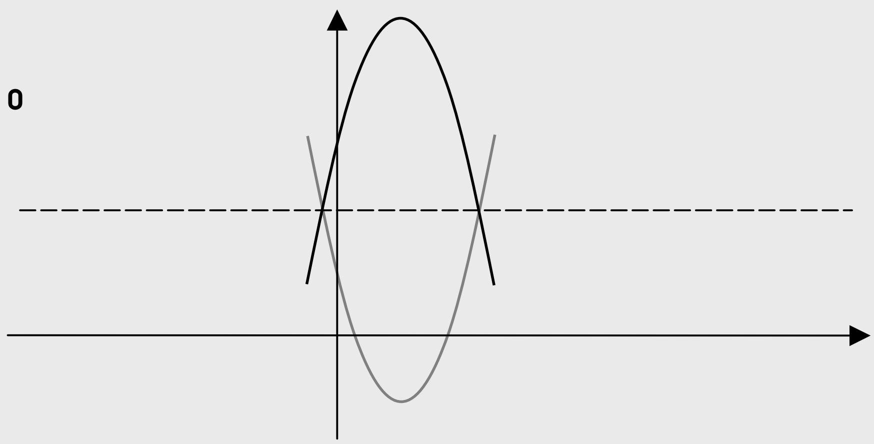
Also, ***D*** is called the distance from the point ***P*** to the line ***T***.

So in the graph above, if the point ***R*** is symmetric to the point ***P*** about the point ***Q*** or the line ***T***, the point ***R*** is in the line containing the line segment connecting ***P*** and ***Q***, and of course, we take ***D*** as the distance from ***R*** to ***Q***, also.

So let's see now how symmetric transformations work. We are going to begin with a simple transformation, which is symmetric about a line ***y* = *b***, which is a horizontal line, which is a line parallel to the ***x***-axis in the ***x-y*** plane.

2.1. Symmetric about a line 1

Fig. 0



• First, note that in this book, some important terminologies are used the way as follows.

The original curve: the curve of the function or equation a transformation is applied to
The new curve: the curve of the function or equation made by a transformation

The original function or equation: the function or equation a transformation is applied to
The new function or equation: the function or equation made by a transformation

The original arbitrary point or the original point: the arbitrary point in the original curve.
The new arbitrary point or the new point: the arbitrary point in the new curve.

Applying a transformation symmetric about a line, we get a new curve, which is symmetric to the original curve about the line, which is called therefore, the axis of symmetry. How then is the new curve made?

In the case of a transformation symmetric about a line, every point in the original curve gets moved symmetrically about the line, and then, all the points moved form the new curve, which is thus, said to be symmetric to the original curve about the line.

What do we mean by then, transforming a function symmetrically about a line?

Transforming a function symmetrically about a line, we get a new curve, which is symmetric about the line to the curve of the function. So do we get a new function?

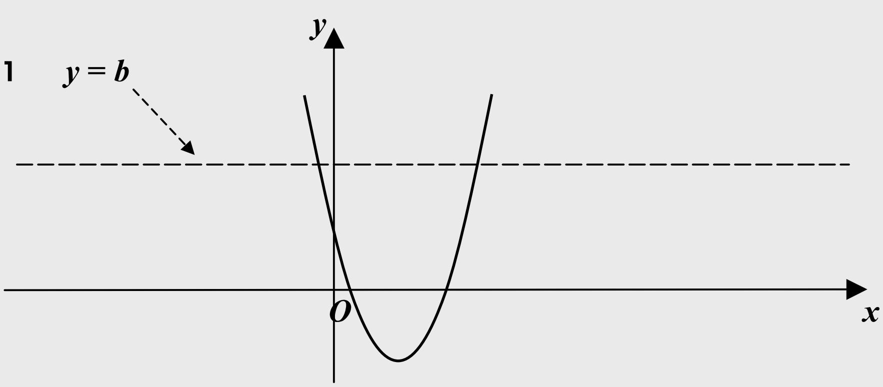
Maybe. It depends on the original function and the line taken as the axis of symmetry. If the line is however, horizontal or vertical, that is, parallel or perpendicular to a coordinate axis as the $\boldsymbol{x}$-axis, we get a new function.

And what's covered in this section is a case where the axis of symmetry is a horizontal line. So covered here is a transformation symmetric about a line parallel to the $\boldsymbol{x}$-axis.

Thus, the two curves, the original and the new in such a transformation are symmetric about a line $\boldsymbol{y = b}$, where $\boldsymbol{b}$ is constant. And if $\boldsymbol{b = 0}$, the axis of symmetry is the $\boldsymbol{x}$-axis.

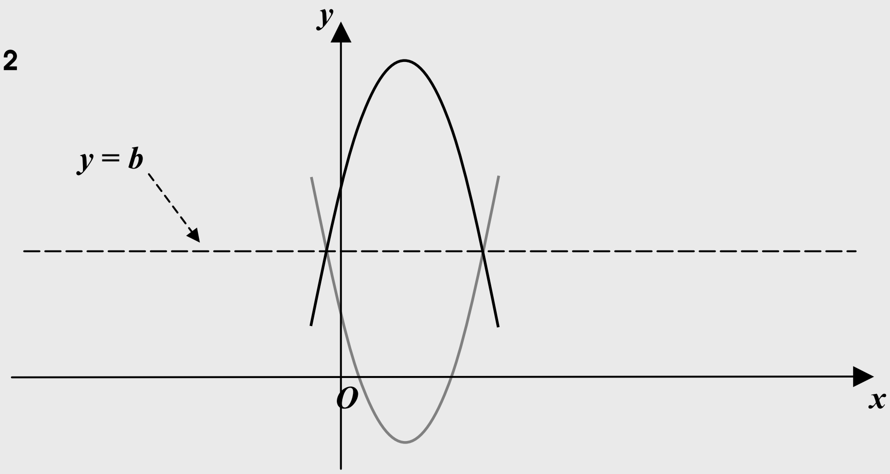
Suppose now that the original curve is the curve of a function $\boldsymbol{y = f(x)}$, and is put in a graph the way as follows.

Fig. 1



Then, applying to the curve of the function $\boldsymbol{f}$ the symmetric transformation stated above, we get a new curve, which is of course, symmetric to the curve of the function $\boldsymbol{f}$ about the line $\boldsymbol{y = b}$. Thus, the new curve is the curve in black in the graph as follows.

Fig. 2



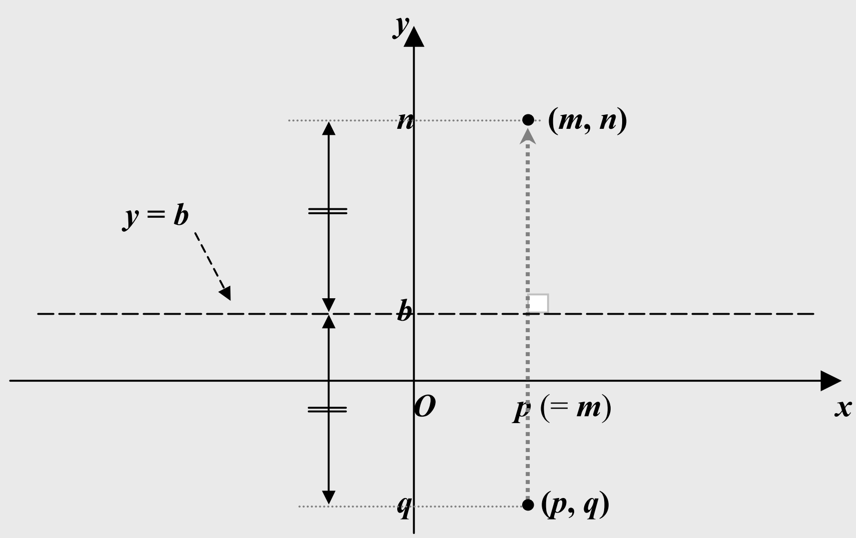
How then can we get the new curve, that is, the new equation?

A transformation definition shows specifically how every point gets changed in the original curve. So it specifies the way the original curve changes, that is, the way the new curve gets made. Thus, we want to get the definition first.

To begin with, if two points are symmetric about a line, the two are equal distance away from the line, and both points are in a line perpendicular to the line.

So for instance, assuming two points (p, q) and (m, n) are symmetric about the line $y = b$, we can put the two points in a graph the way as follows.

Fig. 3



Then first, since the two points (p, q) and (m, n) are symmetric about the line $y = b$, their x-coordinates are the same, that is, we get first, $p = m$.

And next, the distance from (p, q) to the line equals the distance from (m, n) to the line. In other words, b is right in the middle of n and q. That is, b is the average of n and q.

So we get $b = \dfrac{n+q}{2} \Rightarrow 2b = n + q \Rightarrow n = 2b - q$. How though, is it the average?

Since b is at the center between n and q, adding to b the distance from b to q, we get n.

That is, we get $n = b + (b - q) \Rightarrow 2b = n + q \Rightarrow b = \dfrac{n+q}{2}$, which is the average.

So we get $m = p$, and $n = 2b - q$.

Thus, assuming B is the transformation symmetric about the line $y = b$, and applying B to the point (p, q), we get $(m, n) = (p, 2b - q)$, which shows therefore, the correlation between (p, q) and (m, n) in the symmetry about the line $y = b$.

What then is the correlation between the original arbitrary point and the new arbitrary point in the symmetric transformation?

We know that the arbitrary point in a curve represents all the points in the curve, which means that every point in the curve acts the way the arbitrary point does, and vice versa.

So assuming (p, q) is in the original curve, and (x, y) is the original arbitrary point, we can say that (p, q) acts the way (x, y) does, and (x, y) acts the way (p, q) does.

Also, assuming (m, n) is in the new curve, and (s, t) is the new arbitrary point, we can say that (m, n) acts the way (s, t) does, and (s, t) acts the way (m, n) does.

So we get a correlation as follows. $(s, t) = (x, 2b - y)$.

Thus, we can put the transformation B the way as follows.

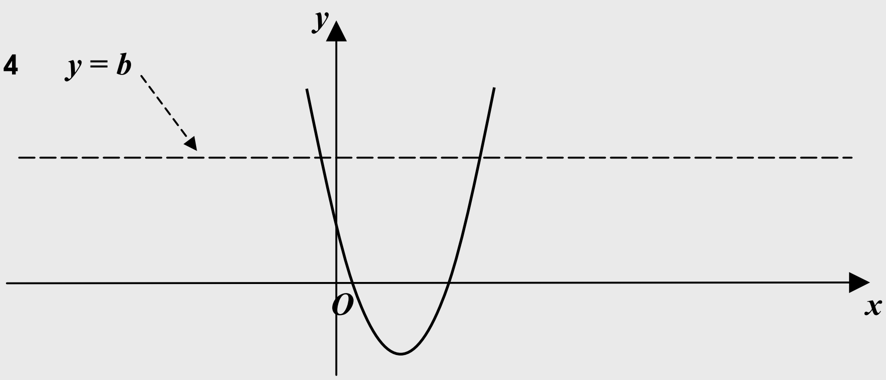
B: $(x, y) \longrightarrow (s, t) = (x, 2b - y)$, where b is constant.

And briefly, we can put it this way, too: B: $(x, y) \longrightarrow (x, 2b - y)$.

So we can call the expression above the definition of the transformation symmetric about a line $y = b$. And the core of the transformation definition is $(x, 2b - y)$, which is the new arbitrary point, so we set $(s, t) = (x, 2b - y)$. And it shows how each and every point in the original curve gets changed, that is, the way every point in the new curve gets made. Is the new curve then a curve of a function?

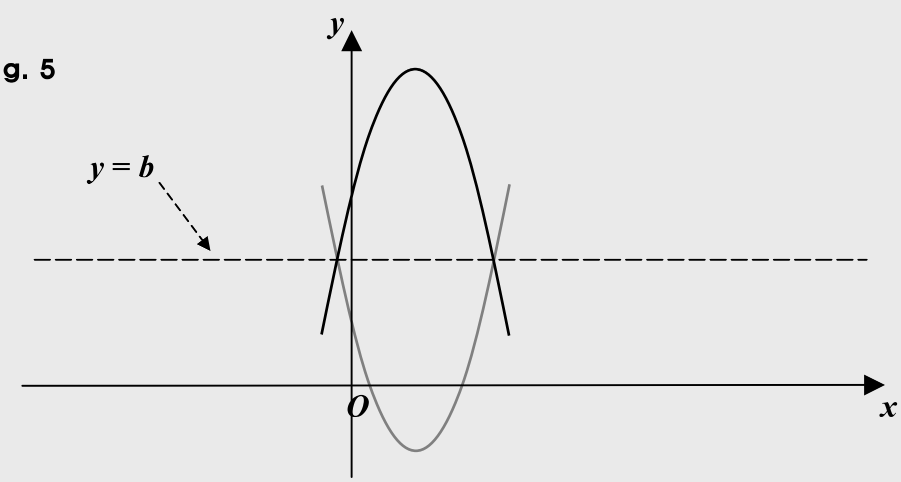
Yes. Applying to a function a transformation symmetric about a horizontal line $y = b$, we get another function. That is, if the original curve is a curve of a function, the new curve is a curve of a function, too. In this example, the original curve is as follows.

Fig. 4



Applying the transformation B to the original curve above, we get a new curve, which is the curve in black below, and can be a curve of a function, which is a new function.

Fig. 5



And the transformation of a function by B is a function, too.
How then do we get the new curve, that is, the equation of the new curve?

We can get it getting the equation connecting the coordinates of the new arbitrary point (s, t), that is, the connective equation between s and t. And we can get the connective equation mixing the new arbitrary point (s, t) into the original curve. And we can get them mixed using $(x, 2b - y)$. And using it, we begin with this: $(s, t) = (x, 2b - y)$.

Then, we can set $s = x$, and $t = 2b - y$, which is the system of equations stated above.
So solving it for x and y, we get $x = s$, and $y = 2b - t$.

Thus, we can now mix the new arbitrary point (s, t) into the original curve. That is, we can mix now, (s, t) into the original equation $y = f(x)$ to get the connective equation.

And getting the mixture, that is, the connective equation, putting $x = s$ and $y = 2b - t$ into the original equation. That is, in $y = f(x)$, we replace x with s, and y with $2b - t$.

So doing the substitutions, we get $y = f(x) \Rightarrow 2b - t = f(s)$, which is the connective equation between s and t, and thus, is the new equation indicating the new curve.

And we know that the new curve is in the x-y plane, and that s is another name for the variable x, and t is another name for the variable y.

So replacing s with x, and t with y, we get the new equation in the x-y system.
That is, we get $2b - t = f(s) \Rightarrow 2b - y = f(x)$, which is the new equation.

> So in sum, transforming an equation $y = f(x)$ symmetrically about a line $y = b$, we get a new equation, and the new equation is $2b - y = f(x)$.

And we can put it the way below, too.

> Transforming an equation $h(x, y) = 0$ symmetrically about a line $y = b$, and assuming k is the new equation, we get $k(x, y) = h(x, 2b - y) = 0$.   Why?

We have $(s, t) = (x, 2b - y)$.
So we can get $s = x$, and $t = 2b - y$, and thus, we get $x = s$, and $y = 2b - t$.

Next, mixing (s, t) into the original equation $h(x, y) = 0$, we get the connective equation.

And getting the mixture, that is, the connective equation, putting $x = s$ and $y = 2b - t$ into the original equation. That is, in $h(x, y) = 0$, we replace x with s, and y with $2b - t$.

So doing the substitutions, we get $h(x, y) = 0 \Rightarrow h(s, 2b - t) = 0$, which is the connective equation between s and t, and thus, is the new equation.

And we know that the new curve is in the x-y plane, and that s is another name for the variable x, and t is another name for the variable y.

So replacing s with x, and t with y, we get the new equation in the x-y system.
That is, we get $h(s, 2b - t) = 0 \Rightarrow h(x, 2b - y) = 0$, which is the new equation.

Thus, assuming k is the new equation, we get $k(x, y) = h(x, 2b - y) = 0$.

And next, we know applying to a function such a symmetric transformation, we get a new function. How then can we get the new function?

Expressing a function, we put the output variable in terms of the input variable.
That is, we set y equal to an expression in terms of x, and the expression is the function.
In this case, we have $2b - y = f(x)$. So we can set $y = 2b - f(x)$.
Assuming thus, g is the new function, we can set $y = g(x) = 2b - f(x)$.

Suppose for a concrete case, the original function is $y = f(x) = 2x^2 - 4x + 1$, and it gets transformed symmetrically about a line $y = 2$, that is, $b = 2$ in the axis of symmetry $y = b$.

Then, we can get the new equation, the way as follows.

First, we have $(x, y) \longrightarrow (x, 2b - y)$. So we have $(x, y) \longrightarrow (x, 4 - y)$.

So next, we can set $(s, t) = (x, 4 - y)$. Thus, we get $s = x$, and $t = 4 - y \Rightarrow y = 4 - t$.

So we get $y = f(x) \Rightarrow 4 - t = f(s) \Rightarrow 4 - t = 2s^2 - 4s + 1 \Rightarrow t = -2s^2 + 4s + 3$.

So we now have the equation connecting s and t. Replacing thus, s with x, and t with y, we get $y = -2x^2 + 4x + 3$, which is the new equation.

Next, assuming g is the new function, we get $y = g(x) = -2x^2 + 4x + 3$.

And we can put the example above the way below, too.

We can have $y = f(x) = 2x^2 - 4x + 1 \Rightarrow y = 2x^2 - 4x + 1 \Rightarrow 2x^2 - 4x + 1 - y = 0$.
And we can put it this way: $h(x, y) = 2x^2 - 4x + 1 - y = 0$.

So suppose this time, the original equation is $h(x, y) = 2x^2 - 4x + 1 - y = 0$, and it gets transformed symmetrically about the line $y = 2$. Then, we can get the new equation the way as follows.

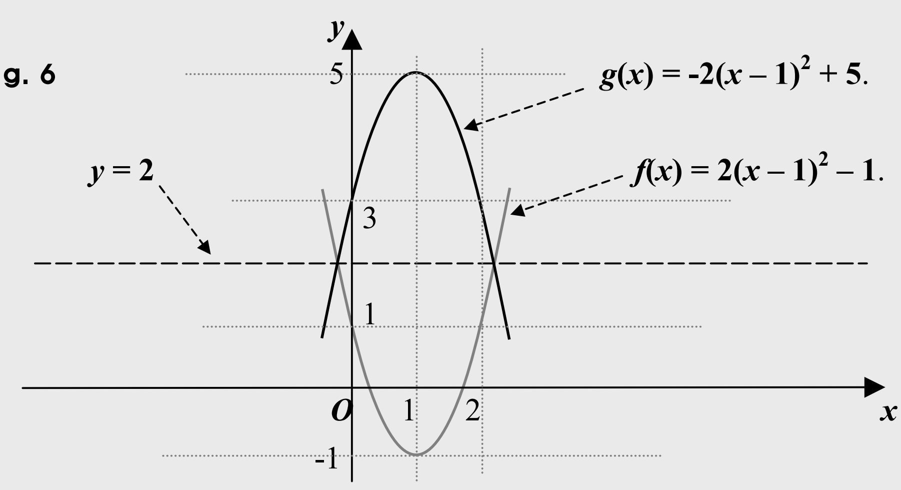
Since we apply the same transformation as the one above, we get $x = s$, and $y = 4 - t$.

So next, $h(x, y) = 0 \Rightarrow h(s, 4 - t) = 0 \Rightarrow 2s^2 - 4s + 1 - (4 - t) = 0 \Rightarrow t = -2s^2 + 4s + 3$, which is the equation connecting s and t. Replacing thus, s with x, and t with y, we get $y = -2x^2 + 4x + 3$, which is the new equation.

And assuming $k(x, y) = 0$ is the new equation above, we can set
$k(x, y) = 2x^2 - 4x - 3 + y = 0$.

And next, assuming g is the new function, we get $y = g(x) = -2x^2 + 4x + 3$.

Fig. 6



So in sum, transforming $\boldsymbol{y = f(x)}$ symmetrically about a line $\boldsymbol{y = b}$, we can begin with setting $\boldsymbol{(s, t) = (x, 2b - y)}$, where $\boldsymbol{(s, t)}$ is the arbitrary point in the new curve, that is, the new arbitrary point, and $\boldsymbol{(x, y)}$ is the arbitrary point in the original curve, the curve of $\boldsymbol{f}$, and thus, is the original arbitrary point.

Next, using $\boldsymbol{(s, t) = (x, 2b - y)}$, along with the original equation $\boldsymbol{y = f(x)}$, we can get the equation connecting the coordinates of the new point, that is, $\boldsymbol{s}$ and $\boldsymbol{t}$. And next, in the connective equation, replacing $\boldsymbol{s}$ with $\boldsymbol{x}$, and $\boldsymbol{t}$ with $\boldsymbol{y}$, we get the new curve.

Thus, getting the new equation, we begin with setting $\boldsymbol{s = x}$, and $\boldsymbol{t = 2b - y}$.
So next, we get $\boldsymbol{s = x}$, and $\boldsymbol{y = 2b - t}$. Thus, we get $\boldsymbol{y = f(x) \Rightarrow 2b - t = f(s)}$, which is the equation connecting the coordinates of $\boldsymbol{(s, t)}$, which is the arbitrary point in the new curve, which is in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane.

So replacing $\boldsymbol{s}$ with $\boldsymbol{x}$, and $\boldsymbol{t}$ with $\boldsymbol{y}$, we get $\boldsymbol{2b - t = f(s) \Rightarrow 2b - y = f(x)}$, which is the new equation, which indicates the curve of the new function. So assuming $\boldsymbol{y = g(x)}$ is the new function, we get $\boldsymbol{2b - y = f(x) \Rightarrow y = 2b - f(x) \Rightarrow y = g(x) = 2b - f(x)}$.

Thus, assuming for instance, the original function is $\boldsymbol{y = f(x) = x^3 - 2x^2 - x + 2}$,
we get $\boldsymbol{y = g(x) = 2b - f(x) = 2b - (x^3 - 2x^2 - x + 2) = 2b - x^3 + 2x^2 + x - 2}$.

So the new function is $\boldsymbol{y = g(x) = 2b - x^3 + 2x^2 + x - 2}$.

And we can put the same the way below, too.

Assuming $\boldsymbol{(x_f, y_f)}$ is the original arbitrary point, $\boldsymbol{(x_g, y_g)}$ is the new arbitrary point, and $\boldsymbol{B}$ is the transformation symmetric about a line $\boldsymbol{y = b}$, we can put $\boldsymbol{B}$ the way as follows.

$$\boldsymbol{B: (x_f, y_f) \longrightarrow (x_g, y_g) = (x_f, 2b - y_f)}.$$

That is, $\boldsymbol{f}$ is the original function, and $\boldsymbol{g}$ is the new.

So we get $\boldsymbol{x_g = x_f}$, and $\boldsymbol{y_g = 2b - y_f}$, and thus, we get $\boldsymbol{x_f = x_g}$, and $\boldsymbol{y_f = 2b - y_g}$.

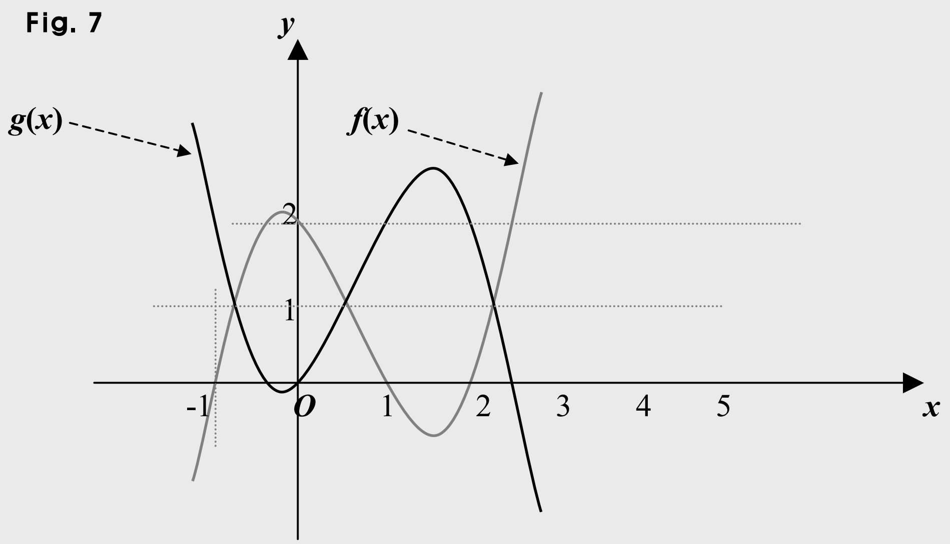
So next, we get $\boldsymbol{y_f = f(x_f) \Rightarrow 2b - y_g = f(x_g) \Rightarrow y_g = 2b - f(x_g) = g(x_g)}$.

Thus, we get $\boldsymbol{y = g(x) = 2b - f(x) = 2b - x^3 + 2x^2 + x - 2}$.

And assuming for instance, the axis of symmetry is $\boldsymbol{y = 1}$, that is, $\boldsymbol{b = 1}$, we get

$\boldsymbol{y = f(x) = x^3 - 2x^2 - x + 2}$, and $\boldsymbol{y = g(x) = 2 - f(x) = 2 - x^3 + 2x^2 + x - 2 = -x^3 + 2x^2 + x}$.

Fig. 7



What if we want the transformation to be symmetric about the $\boldsymbol{x}$-axis?

Setting $\boldsymbol{b = 0}$ in the equation that $\boldsymbol{y = b}$, we get the transformation symmetric about the $\boldsymbol{x}$-axis. And we have $\boldsymbol{B: (x, y) \longrightarrow (x, 2b - y)}$. So we want to set $\boldsymbol{b = 0}$ in $\boldsymbol{2b - y}$.

Thus, assuming the transformation is $\boldsymbol{X}$, we can put it the way below.

$\boldsymbol{X: (x, y) \longrightarrow (x, -y)}$.

So assuming a function $\boldsymbol{h}$ is the transformation of $\boldsymbol{f}$ by $\boldsymbol{X}$, we can get it the way below.

To begin with, we get $\boldsymbol{x_h = x_f}$, and $\boldsymbol{y_h = -y_f}$, and thus, we get $\boldsymbol{x_f = x_h}$, and $\boldsymbol{y_f = -y_h}$.

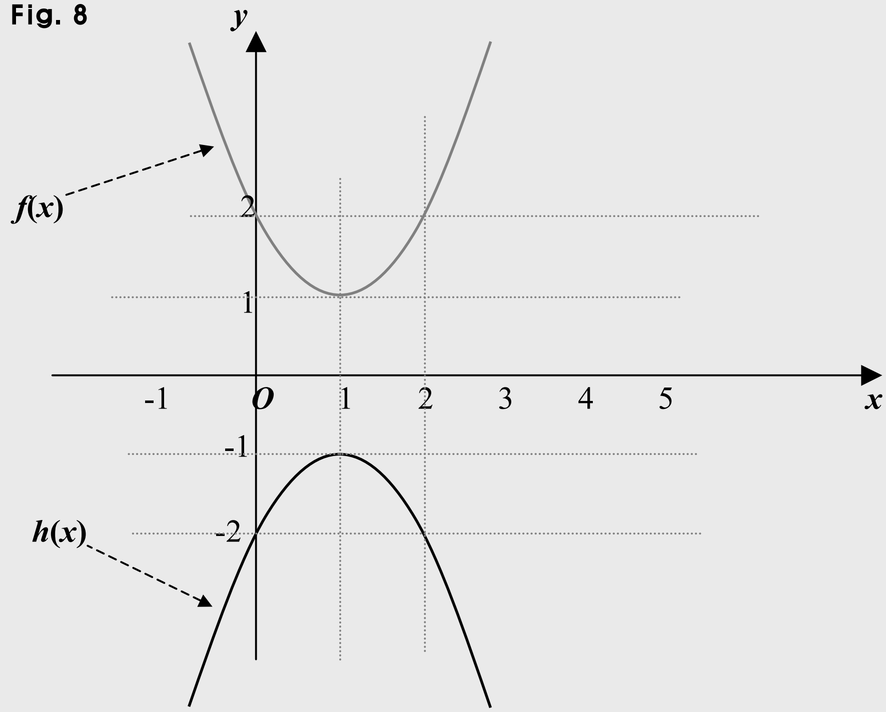
So next, we get $\boldsymbol{y_f = f(x_f) \Rightarrow -y_h = f(x_h) \Rightarrow y_h = -f(x_h) = h(x_h)}$.

Thus, we get $\boldsymbol{y = h(x) = -f(x)}$.

So for instance, assuming $\boldsymbol{y = f(x) = (x - 1)^2 + 1}$, we get

$\boldsymbol{y = h(x) = \text{-}f(x) = \text{-}\{(x - 1)^2 + 1\} = \text{-}(x - 1)^2 - 1 \Rightarrow y = h(x) = \text{-}(x - 1)^2 - 1}$.

Fig. 8



Let's next, for another instance, transform an equation $\boldsymbol{A(x, y) = \dfrac{y^2}{1 - x} + 2 = 0}$

symmetrically about a line $\boldsymbol{y = 1}$.

To begin with, we have $\boldsymbol{B}$: $\boldsymbol{(x, y) \longrightarrow (x, 2b - y)}$, which is the transformation symmetric about a line $\boldsymbol{y = b}$.

So setting $\boldsymbol{b = 1}$ in $\boldsymbol{2b - y}$, we get the transformation symmetric about the line $\boldsymbol{y = 1}$. Thus, assuming the transformation is $\boldsymbol{K}$, we can put it the way below.

$\boldsymbol{K}$: $\boldsymbol{(x, y) \longrightarrow (x, 2 - y)}$.

So assuming an equation $\boldsymbol{H(x, y) = 0}$ is the transformation of $\boldsymbol{A}$ by $\boldsymbol{K}$, we can get it the way below.

We get first, $\boldsymbol{x_H = x_A}$, and $\boldsymbol{y_H = 2 - y_A}$, and thus, we get $\boldsymbol{x_A = x_H}$, and $\boldsymbol{y_A = 2 - y_H}$.

So next, we get $A(x_A, y_A) = 0 \Rightarrow A(x_H, 2 - y_H) = 0$.

Thus, we get $H(x_H, y_H) = A(x_H, 2 - y_H) = 0$.

And we have $A(x, y) = \dfrac{y^2}{1-x} + 2 = 0$. That is, we have $A(x_A, y_A) = \dfrac{{y_A}^2}{1-x_A} + 2 = 0$.

So we get $A(x_A, y_A) = \dfrac{{y_A}^2}{1-x_A} + 2 = 0 \Rightarrow A(x_H, 2 - y_H) = \dfrac{(2-y_H)^2}{1-x_H} + 2 = 0$.

Thus, we get $H(x_H, y_H) = \dfrac{(2-y_H)^2}{1-x_H} + 2 = 0$.

That is, we get $H(x, y) = \dfrac{(2-y)^2}{1-x} + 2 = 0$.

So we can put $H(x_H, y_H) = A(x_H, 2 - y_H) = 0$ this way, too: $H(x, y) = A(x, 2 - y) = 0$.

Besides, using the transformation operator, the dot •, we can put it this way, too:

$H = K \bullet A = A(x, 2 - y) = \dfrac{(2-y)^2}{1-x} + 2 = 0$. And Let's now put the curves in a graph.

To begin with, converting the two equations, we get

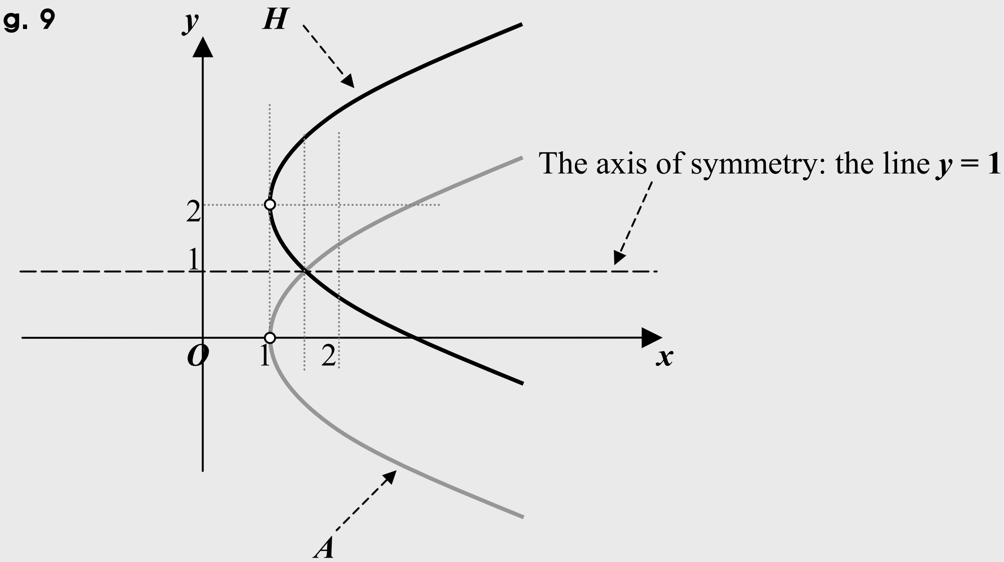
$$A(x, y) = \frac{y^2}{1-x} + 2 = 0 \Rightarrow y^2 + 2(1-x) = 0 \Rightarrow x = \frac{y^2}{2} + 1.$$

$$H(x, y) = \frac{(2-y)^2}{1-x} + 2 = 0 \Rightarrow (2-y)^2 + 2(1-x) = 0 \Rightarrow x = \frac{(2-y)^2}{2} + 1.$$

We want to note however, that the actual equations are $\dfrac{y^2}{1-x} + 2 = 0$ and $\dfrac{(2-y)^2}{1-x} + 2 = 0$.

So we can't get $x = 1$, since a denominator cannot be 0.

Fig. 9



And let's next, for another instance, transform the function below symmetrically about a line $\boldsymbol{y = 1}$.

$\boldsymbol{y = f(x) = 0.2x^5 - 2.25x^4 + 9x^3 - 15.5x^2 + 12x - 2}$.

Then, we can get the new equation, the way as follows.

First, we have $\boldsymbol{(x, y) \longrightarrow (x, 2b - y)}$. So we have $\boldsymbol{(x, y) \longrightarrow (x, 2 - y)}$.

So next, we can set $\boldsymbol{(s, t) = (x, 2 - y)}$.

Thus, we get $\boldsymbol{s = x}$, and $\boldsymbol{t = 2 - y}$, so we get $\boldsymbol{x = s}$, and $\boldsymbol{y = 2 - t}$.

Thus, we get $\boldsymbol{y = f(x) \Rightarrow 2 - t = f(s)}$

$\boldsymbol{\Rightarrow 2 - t = 0.2s^5 - 2.25s^4 + 9s^3 - 15.5s^2 + 12s - 2}$

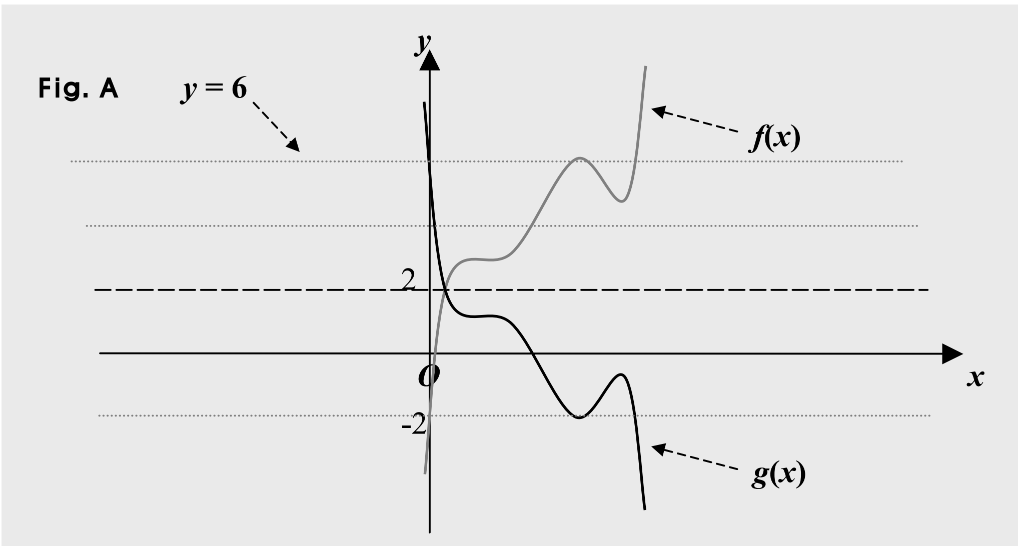
$\boldsymbol{\Rightarrow t = -(0.2s^5 - 2.25s^4 + 9s^3 - 15.5s^2 + 12s) + 4}$

$\boldsymbol{\Rightarrow t = 4 - 0.2s^5 + 2.25s^4 - 9s^3 + 15.5s^2 - 12s}$.

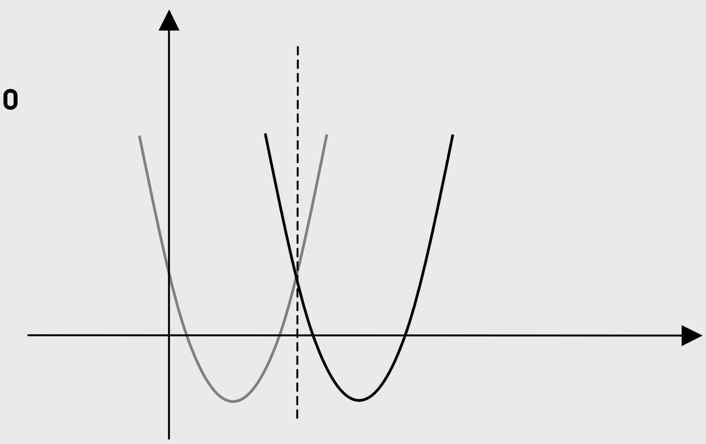
So we now have the equation connecting $\boldsymbol{s}$ and $\boldsymbol{t}$. Replacing thus, $\boldsymbol{s}$ with $\boldsymbol{x}$, and $\boldsymbol{t}$ with $\boldsymbol{y}$, we get $\boldsymbol{y = 4 - 0.2x^5 + 2.25x^4 - 9x^3 + 15.5x^2 - 12x}$, which is the new equation.

And next, assuming ***g*** is the new function, we get

$\boldsymbol{y = g(x) = 4 - 0.2x^5 + 2.25x^4 - 9x^3 + 15.5x^2 - 12x}$.



2.2. Symmetric about a line 2



The previous section covers the transformation symmetric about a line $\boldsymbol{y = b}$, and the line is parallel to the $\boldsymbol{x}$-axis. And if $\boldsymbol{B}$ is the transformation, we can put it the way below.

> $\boldsymbol{B: (x, y) \longrightarrow (s, t) = (x, 2b - y)}$, where $\boldsymbol{b}$ is constant.
>
> And briefly, we can put it this way, too: $\boldsymbol{B: (x, y) \longrightarrow (x, 2b - y)}$.

What then is the transformation symmetric about a line $\boldsymbol{x = a}$?

The line $\boldsymbol{x = a}$ is parallel to the $\boldsymbol{y}$-axis, and the nature of the symmetry is exactly the same as the transformation $\boldsymbol{B}$ above. So assuming $\boldsymbol{A}$ is the transformation symmetric about the line $\boldsymbol{x = a}$, we can put it the way below.

> $\boldsymbol{A: (x, y) \longrightarrow (s, t) = (2a - x, y)}$, where $\boldsymbol{a}$ is constant.
>
> And we can quickly put it this way, too: $\boldsymbol{A: (x, y) \longrightarrow (2a - x, y)}$.

If not quite sure of the idea behind it, follow the explanations and the steps shown below.

In a transformation symmetric about a line, every point in the original curve gets moved symmetrically about the line, and then, all the points moved form the new curve, which is thus, said to be symmetric to the original curve about the line.

What do we mean by then, transforming a function symmetrically about a line?

Transforming a function symmetrically about a line, we get a new curve, which is symmetric about the line to the curve of the function. So do we get a new function?

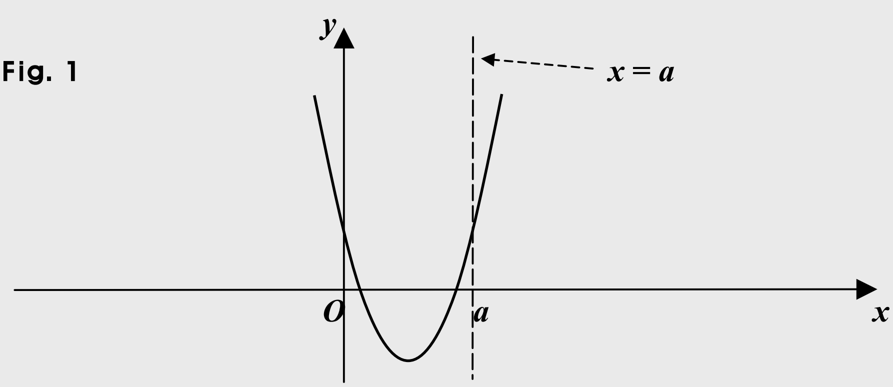
If the line taken as the axis of symmetry is horizontal or vertical, that is, parallel to or perpendicular to the $\boldsymbol{x}$-axis, we get a new function.

And what's covered in this section is a case where the axis of symmetry is a vertical line. So covered here is a transformation symmetric about a line parallel to the $\boldsymbol{y}$-axis.

So the two curves, the original and the new in such a transformation can be said to be symmetric about a line $\boldsymbol{x = a}$, where $\boldsymbol{a}$ is constant. And if $\boldsymbol{a = 0}$, the axis of symmetry is the $\boldsymbol{y}$-axis.

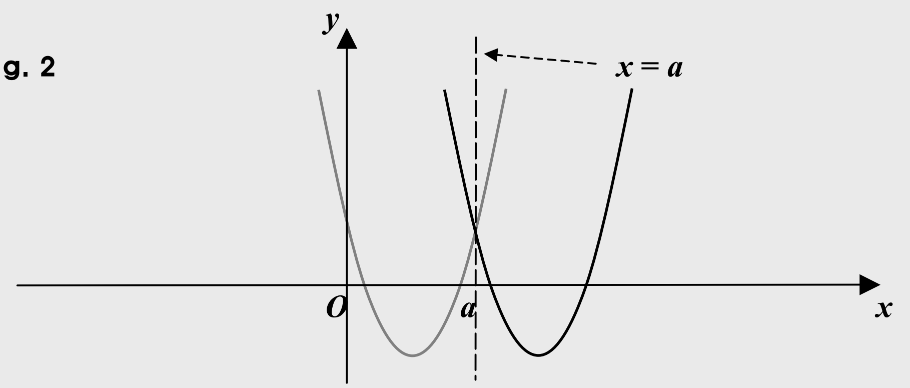
Suppose now that $\boldsymbol{F}$ is the original curve, and is the curve of a function $\boldsymbol{y = f(x)}$, and it is put in a graph the way below.

Fig. 1



Then, applying to $\boldsymbol{f}$ the symmetric transformation stated above, we get a new curve, which is symmetric to the curve of $\boldsymbol{f}$ about the line $\boldsymbol{x = a}$. Therefore, the new curve is the curve in black in the graph as follows.

Fig. 2

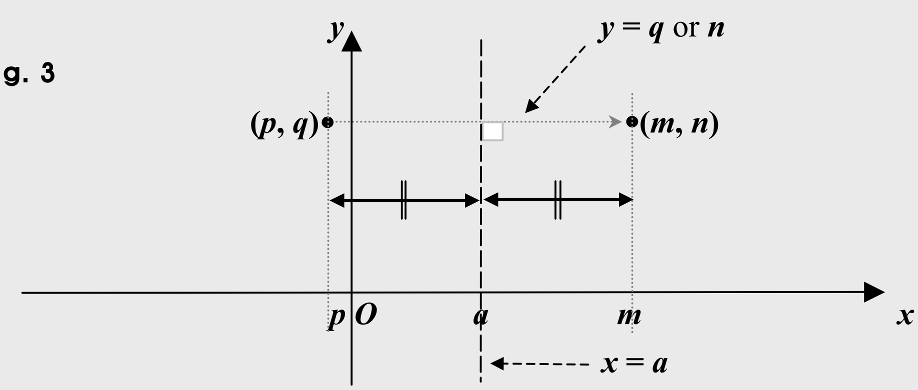


How then can we get the new curve, that is, the new equation?

A transformation definition shows the specifics of ***the way*** every point gets changed in the original curve. In other words, it specifies ***the way*** the original curve changes, that is, ***the way*** the new curve gets made. So let's now get the definition first.

To begin with, if two points are symmetric about a line, the two are equal distance away from the line, and both points are in a line perpendicular to the line.
So for instance, assuming two points (p, q) and (m, n) are symmetric about the line $x = a$, we can put the two points in a graph the way below.

Fig. 3



Then first, since the two points (p, q) and (m, n) are symmetric about the line $x = a$, their y-coordinates are the same, that is, we get first, $q = n$.

Next, the distance from (p, q) to the line equals the distance from (m, n) to the line.

In other words, $\boldsymbol{a}$ is right in the middle of $\boldsymbol{m}$ and $\boldsymbol{p}$. That is, $\boldsymbol{a}$ is the average of $\boldsymbol{m}$ and $\boldsymbol{p}$.

So we get $\boldsymbol{a = \frac{m+p}{2} \Rightarrow 2a = m + p \Rightarrow m = 2a - p}$. Thus, we get $\boldsymbol{n = q}$, and $\boldsymbol{m = 2a - p}$.

Thus, assuming $\boldsymbol{A}$ is the transformation symmetric about the line $\boldsymbol{x = a}$, and applying $\boldsymbol{A}$ to the point $\boldsymbol{(p, q)}$, we get $\boldsymbol{(m, n) = (2a - p, q)}$, which shows therefore, the correlation between $\boldsymbol{(p, q)}$ and $\boldsymbol{(m, n)}$ in the symmetry about the line $\boldsymbol{x = a}$.

What then is the correlation between the original arbitrary point and the new arbitrary point in the symmetric transformation?

The arbitrary point in a curve represents all the points in the curve. So assuming $\boldsymbol{(p, q)}$ is in the original curve, $\boldsymbol{(x, y)}$ is the original arbitrary point, $\boldsymbol{(s, t)}$ is the new arbitrary point, and $\boldsymbol{(m, n)}$ is in the new curve, we get a correlation as follows. $\boldsymbol{(s, t) = (2a - x, y)}$, because $\boldsymbol{(x, y)}$ behaves the way $\boldsymbol{(p, q)}$ does, and vice versa, and the same is true for $\boldsymbol{(s, t)}$ and $\boldsymbol{(m, n)}$, too.

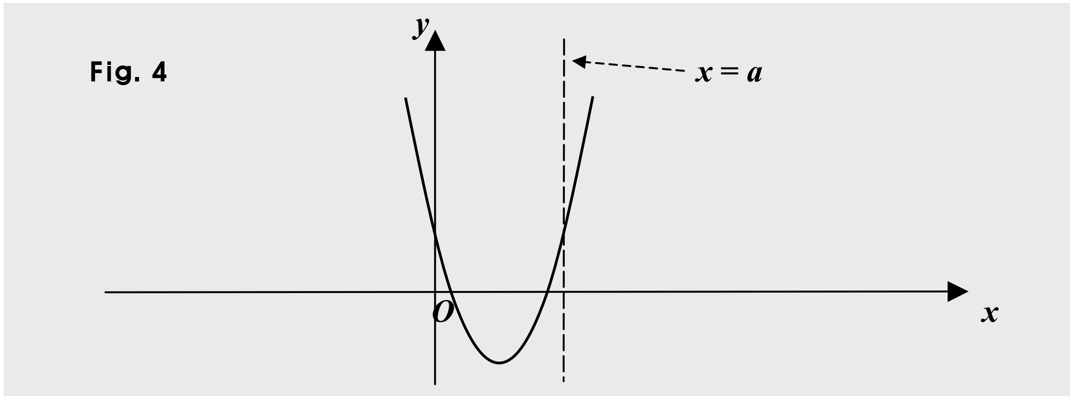
Thus, we can put the transformation $\boldsymbol{A}$ the way below.

> $\boldsymbol{A: (x, y) \longrightarrow (s, t) = (2a - x, y)}$, where $\boldsymbol{a}$ is constant.
>
> And quickly, we can put it this way, too: $\boldsymbol{B: (x, y) \longrightarrow (2a - x, y)}$.

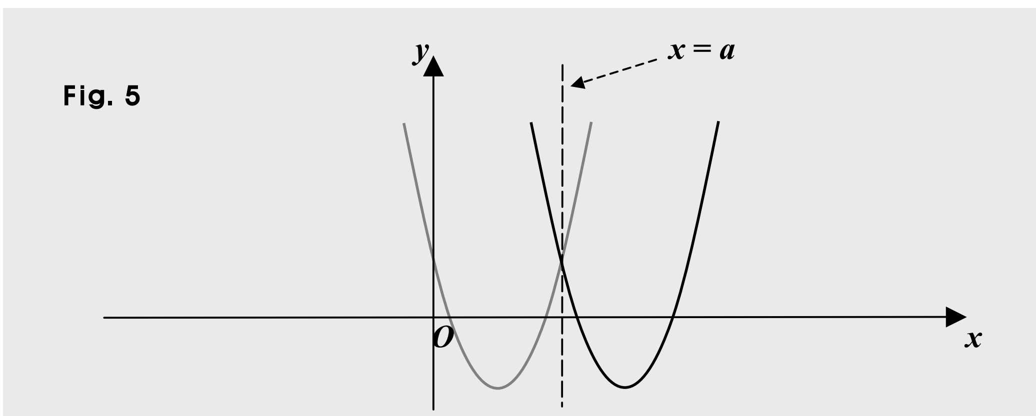
So we can call the expression above the definition of the transformation symmetric about a line $\boldsymbol{x = a}$. And the core of the transformation definition is $\boldsymbol{(2a - x, y)}$, which is the new arbitrary point, so we set $\boldsymbol{(s, t) = (2a - x, y)}$, and it shows how each and every point in the original curve gets changed, that is, the way every point in the new curve gets made.

Is the new curve then a curve of a function?

Yes. Applying to a function a transformation symmetric about a vertical line $\boldsymbol{x = a}$, we get another function. That is, if the original curve is a curve of a function, the new curve is a curve of a function, too. In this example, the original curve is as follows.



Applying the transformation ***A*** to the original curve above, we get a new curve, which is the curve in black below, and can be a curve of a function, which is a new function.



And the transformation of a function by ***A*** is a function, too.
How then do we get the new curve, that is, the equation of the new curve?

We can get it getting the equation connecting $\boldsymbol{s}$ and $\boldsymbol{t}$. And we can get the connective equation mixing the new arbitrary point into the original curve. And we can get them mixed using $\mathbf{(2\boldsymbol{a} - \boldsymbol{x}, \boldsymbol{y})}$. And using it, we begin with this: $\mathbf{(\boldsymbol{s}, \boldsymbol{t}) = (2\boldsymbol{a} - \boldsymbol{x}, \boldsymbol{y})}$.

Then, we can set $\boldsymbol{s} = \mathbf{2}\boldsymbol{a} - \boldsymbol{x}$, and $\boldsymbol{t} = \boldsymbol{y}$, which is the system of equations stated above. So solving it for $\boldsymbol{x}$ and $\boldsymbol{y}$, we get $\boldsymbol{x} = \mathbf{2}\boldsymbol{a} - \boldsymbol{s}$, and $\boldsymbol{y} = \boldsymbol{t}$.

Thus, we can now mix the new arbitrary point $\mathbf{(\boldsymbol{s}, \boldsymbol{t})}$ into the original curve. That is, we can mix now, $\mathbf{(\boldsymbol{s}, \boldsymbol{t})}$ into the original equation $\boldsymbol{y} = \boldsymbol{f}\mathbf{(\boldsymbol{x})}$ to get the connective equation.

And getting the mixture, that is, the connective equation, putting $x = 2a - s$ and $y = t$ into the original equation. That is, in $y = f(x)$, we replace x with $2a - s$, and y with t.

So doing the substitutions, we get $y = f(x) \Rightarrow t = f(2a - s)$, which is the connective equation between s and t, and thus, is the new equation indicating the new curve.

And we know that the new curve is in the x-y plane, and that s is another name for the variable x, and t is another name for the variable y.

So replacing s with x, and t with y, we get the new equation in the x-y system.

That is, we get $t = f(2a - s) \Rightarrow y = f(2a - x)$, which is the new equation.

So in sum, transforming an equation $y = f(x)$ symmetrically about a line $x = a$, we get a new equation, and the new equation is $y = f(2a - x)$.

And we can put it the way below, too.

Transforming an equation $h(x, y) = 0$ symmetrically about a line $x = a$, and assuming k is the new equation, we get $k(x, y) = h(2a - x, y) = 0$. How?

We have $(s, t) = (2a - x, y)$.

So we can get $s = 2a - x$, and $t = y$, and thus, we get $x = 2a - s$, and $y = t$.

Next, mixing (s, t) into the original equation $h(x, y) = 0$, we get the connective equation.

And getting the mixture, that is, the connective equation, putting $x = 2a - s$ and $y = t$ into the original equation. That is, in $h(x, y) = 0$, we replace x with $2a - s$, and y with t.

So doing the substitutions, we get $h(x, y) = 0 \Rightarrow h(2a - s, t) = 0$, which is the connective equation between s and t, and thus, is the new equation.

And we know that the new curve is in the x-y plane, and that s is another name for the variable x, and t is another name for the variable y.

So replacing s with x, and t with y, we get the new equation in the x-y system.

That is, we get $h(2a - s, t) = 0 \Rightarrow h(2a - x, y) = 0$, which is the new equation.

And thus, assuming k is the new equation, we get $k(x, y) = h(2a - x, y) = 0$.

And next, we know applying to a function such a symmetric transformation, we get a new function. How then can we get the new function?

Expressing a function, we put the output variable in terms of the input variable.
That is, we set y equal to an expression in terms of x, and the expression is the function.

Now, in this case, we have $y = f(2a - x)$. And assuming g is the new function, we can set $y = g(x)$. So we get $y = g(x) = f(2a - x)$.

Suppose for a concrete case, the original function is $y = f(x) = 2x^2 - 4x + 1$, and it gets transformed symmetrically about a line $x = 2$, that is, $a = 2$ in the axis of symmetry $x = a$.

Then, we can get the new equation, the way as follows.

First, we have $(x, y) \longrightarrow (2a - x, y)$. So we have $(x, y) \longrightarrow (4 - x, y)$.

So next, we can set $(s, t) = (4 - x, y)$.

Thus, we get $s = 4 - x$, and $t = y$, so we get $x = 4 - s$, and $y = t$.

Thus, we get $y = f(x) \Rightarrow t = f(4 - s) \Rightarrow t = 2(4 - s)^2 - 4(4 - s) + 1$.

So we now have the equation connecting s and t. Replacing thus, s with x, and t with y, we get $y = 2(4 - s)^2 - 4(4 - x) + 1$, which is the new equation.

And next, assuming g is the new function, we get $y = g(x) = 2(4 - x)^2 - 4(4 - x) + 1$.

And simplifying it, we can put it this way, too: $y = g(x) = 2x^2 - 12x + 17$.

And we can put the example above the way below, too.

We can have $\boldsymbol{y = f(x) = 2x^2 - 4x + 1 \Rightarrow y = 2x^2 - 4x + 1 \Rightarrow 2x^2 - 4x + 1 - y = 0}$.

And we can put it this way: $\boldsymbol{h(x, y) = 2x^2 - 4x + 1 - y = 0}$.

So suppose this time, the original equation is $\boldsymbol{h(x, y) = 2x^2 - 4x + 1 - y = 0}$, and it gets transformed symmetrically about the line $\boldsymbol{x = 2}$. Then, we can get the new equation the way as follows.

Since we apply the same transformation as the one above, we get $\boldsymbol{x = 4 - s}$, and $\boldsymbol{y = t}$.

So next, $\boldsymbol{h(x, y) = 0 \Rightarrow h(4 - s, t) = 0 \Rightarrow 2(4 - s)^2 - 4(4 - x) + 1 - t = 0}$

$\boldsymbol{\Rightarrow t = 2(4 - s)^2 - 4(4 - s) + 1}$, which is the equation connecting $\boldsymbol{s}$ and $\boldsymbol{t}$.

Replacing thus, $\boldsymbol{s}$ with $\boldsymbol{x}$, and $\boldsymbol{t}$ with $\boldsymbol{y}$, we get

$\boldsymbol{y = 2(4 - s)^2 - 4(4 - x) + 1}$, which is the new equation.

And assuming $\boldsymbol{k(x, y) = 0}$ is the new equation above, we can set

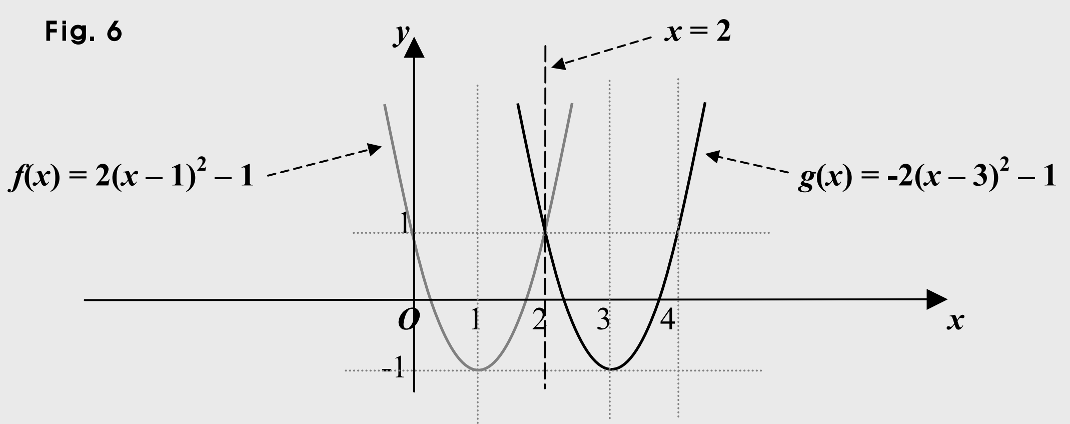
$\boldsymbol{k(x, y) = 2(4 - x)^2 - 4(4 - x) + 1 - y = 0}$.

And next, assuming $\boldsymbol{g}$ is the new function, we get $\boldsymbol{y = g(x) = 2(4 - x)^2 - 4(4 - x) + 1}$.

Simplifying it, we get $\boldsymbol{y = g(x) = 2x^2 - 12x + 17}$.

And putting it in the vertex form, we get $\boldsymbol{y = g(x) = 2(x - 3)^2 - 1}$.

Fig. 6



So in sum, transforming $y = f(x)$ symmetrically about a line $x = a$, we can begin with setting $(s, t) = (2a - x, y)$, where (s, t) is the new arbitrary point, and (x, y) is the original arbitrary point.

Next, using $(s, t) = (2a - x, y)$, along with the original equation $y = f(x)$, we can get the equation connecting s and t. And next, in the connective equation, replacing s with x, and t with y, we get the new curve.

Thus, getting the new equation, we begin with setting $s = 2a - x$, and $t = y$.
So next, we get $s = 2a - x$, and $y = t$. Thus, we get $y = f(x) \Rightarrow t = f(2a - s)$, which is the equation connecting the coordinates of (s, t), which is the arbitrary point in the new curve, which is in the x-y plane.

So replacing s with x, and t with y, we get $t = f(2a - s) \Rightarrow y = f(2a - x)$, which is the new equation, which indicates the curve of the new function. So assuming $y = g(x)$ is the new function, we get $y = f(2a - x) \Rightarrow y = g(x) = f(2a - x)$.

Thus, assuming for instance, the original function is $y = f(x) = x^3 - 2x^2 - x + 2$, we get $y = g(x) = f(2a - x) = (2a - x)^3 - 2(2a - x)^2 - (2a - x) + 2$.

So the new function is $y = g(x) = (2a - x)^3 - 2(2a - x)^2 - (2a - x) + 2$.

And we can put the same the way below, too.

Assuming (x_f, y_f) is the original arbitrary point, (x_g, y_g) is the new arbitrary point, and A is the transformation symmetric about a line $x = a$, we can put A the way below.

B: $(x_f, y_f) \longrightarrow (x_g, y_g) = (2a - x_f, y_f)$.

That is, f is the original function, and g is the new.

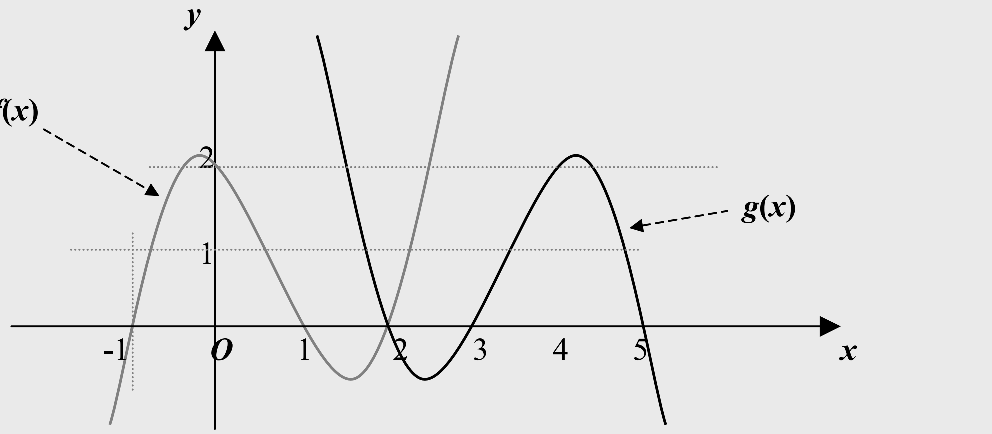
So we get $x_g = 2a - x_f$, and $y_g = y_f$, and thus, we get $x_f = 2a - x_g$, and $y_f = y_g$.

So next, we get $y_f = f(x_f) \Rightarrow y_g = f(2a - x_g) = g(x_g)$.

Thus, we get $y = g(x) = f(2a - x) = (2a - x)^3 - 2(2a - x)^2 - (2a - x) + 2$.

And assuming for instance, the axis of symmetry is $x = 2$, that is, $a = 2$, we get $y = f(x) = x^3 - 2x^2 - x + 2$, and $y = g(x) = f(4 - x) = (4 - x)^3 - 2(4 - x)^2 - (4 - x) + 2$.

Fig. 7



What if we want the transformation to be symmetric about the y-axis?

Setting $a = 0$ in the equation that $x = a$, we get the transformation symmetric about the y-axis. And we have $A: (x, y) \longrightarrow (2a - x, y)$. So we want to set $a = 0$ in $2a - x$.

Thus, assuming the transformation is Y, we can put it the way below.

$Y: (x, y) \longrightarrow (-x, y)$.

So assuming a function h is the transformation of f by Y, we can get it the way below.

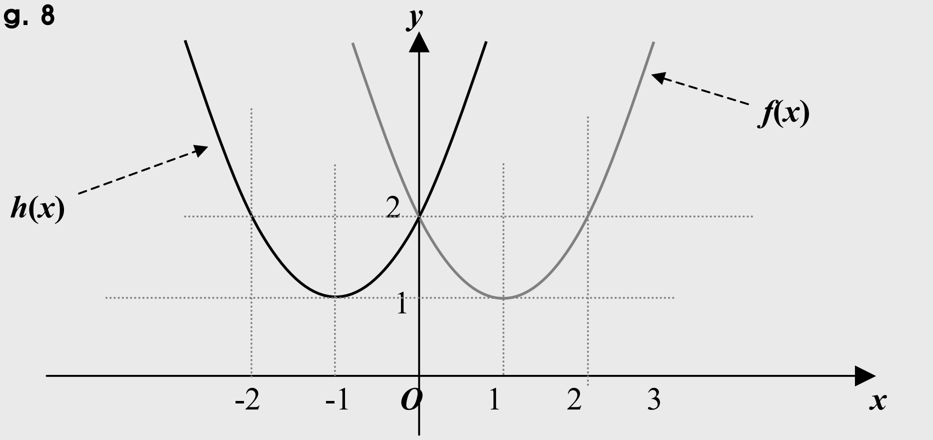
To begin with, we get $x_h = -x_f$, and $y_h = y_f$, and thus, we get $x_f = -x_h$, and $y_f = y_h$.

So next, we get $y_f = f(x_f) \Rightarrow y_h = f(-x_h) = h(x_h)$. Thus, we get $y = h(x) = f(-x)$.

So for instance, assuming $y = f(x) = (x - 1)^2 + 1$, we get

$y = h(x) = f(-x) = (-x - 1)^2 + 1 = (x + 1)^2 + 1 \Rightarrow y = h(x) = (x + 1)^2 + 1$.

Fig. 8



Let's next, for another instance, transform an equation $\boldsymbol{B(x,y)=\frac{y^2}{1-x}+2=0}$ symmetrically about a line $\boldsymbol{x = 2}$.

To begin with, we have $\boldsymbol{A}$**:** $\boldsymbol{(x, y) \longrightarrow (2a - x, y)}$, which is the transformation symmetric about a line $\boldsymbol{x = a}$.

So setting $\boldsymbol{a = 2}$ in $\boldsymbol{2a - x}$, we get the transformation symmetric about the line $\boldsymbol{x = 2}$. Thus, assuming the transformation is $\boldsymbol{K}$, we can put it the way below.

$\boldsymbol{K}$**:** $\boldsymbol{(x, y) \longrightarrow (4 - x, y)}$.

So assuming an equation $\boldsymbol{H(x, y) = 0}$ is the transformation of $\boldsymbol{B}$ by $\boldsymbol{K}$, we can get it the way below.

We get first $\boldsymbol{x_H = 4 - x_B}$, and $\boldsymbol{y_H = y_B}$, and thus, we get $\boldsymbol{x_B = 4 - x_H}$, and $\boldsymbol{y_B = y_H}$.

So next, we get $\boldsymbol{B(x_B, y_B) = 0 \Rightarrow B(4 - x_H, y_H) = 0}$.

Thus, we get $\boldsymbol{H(x_H, y_H) = B(4 - x_H, y_H) = 0}$.

And we have $\boldsymbol{B(x,y)=\frac{y^2}{1-x}+2=0}$. That is, we have $\boldsymbol{B(x_B,y_B)=\frac{{y_B}^2}{1-x_B}+2=0}$.

So we get $\boldsymbol{B(x_B,y_B)=\frac{{y_B}^2}{1-x_B}+2=0 \Rightarrow B(4-x_H,y_H)=\frac{{y_H}^2}{1-(4-x_H)}+2=0}$.

Ant thus, we get $\boldsymbol{H(x_H,y_H)=\frac{{y_H}^2}{x_H-3}+2=0}$. That is, we get $\boldsymbol{H(x,y)=\frac{y^2}{x-3}+2=0}$.

So we can put $\boldsymbol{H(x_H, y_H) = B(4 - x_H, y_H) = 0}$ this way, too: $\boldsymbol{H(x, y) = B(4 - x, y) = 0}$.

Besides, using the transformation operator, the dot •, we can put it this way, too:

$\boldsymbol{H = K \bullet B = B(4 - x, y) = \frac{y^2}{x-3}+2=0}$. And Let's now put the curves in a graph.

To begin with, converting the two equations, we get

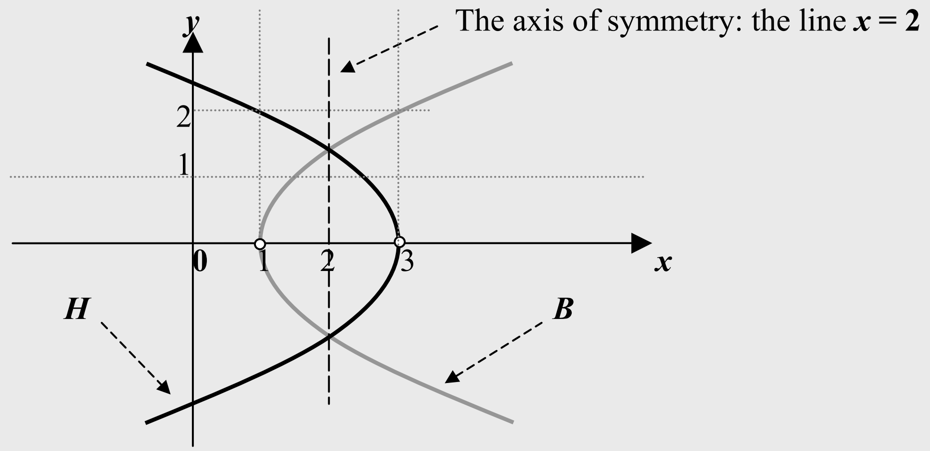
$$\boldsymbol{B(x,y)=\frac{y^2}{1-x}+2=0 \Rightarrow y^2+2(1-x)=0 \Rightarrow x=\frac{y^2}{2}+1}.$$

$$H(x,y)=\frac{y^2}{x-3}+2=0 \Rightarrow y^2+2(x-3)=0 \Rightarrow x=3-\frac{y^2}{2}.$$

We want to note however, that the actual equations are $\frac{y^2}{1-x}+2=0$ and $\frac{y^2}{x-3}+2=0$.

So we get $x \neq 3$, and $x \neq 1$, since a denominator cannot be 0.

Fig. 9



And let's next, for another instance, transform the function below symmetrically about a line $x = 4$.

$y = f(x) = 0.2x^5 - 2.25x^4 + 9x^3 - 15.5x^2 + 12x - 2$.

Then, we can get the new equation, the way as follows.

First, we have $(x, y) \longrightarrow (2a - x, y)$. So we have $(x, y) \longrightarrow (8 - x, y)$.

So next, we can set $(s, t) = (8 - x, y)$.

Thus, we get $s = 8 - x$, and $t = y$, so we get $x = 8 - s$, and $y = t$.

Thus, we get $y = f(x) \Rightarrow t = f(8 - s)$

$$\Rightarrow t = 0.2(8-s)^5 - 2.25(8-s)^4 + 9(8-s)^3 - 15.5(8-s)^2 + 12(8-s) - 2$$

So we now have the equation connecting s and t. Replacing thus, s with x, and t with y, we get $y = 0.2(8-x)^5 - 2.25(8-x)^4 + 9(8-x)^3 - 15.5(8-x)^2 + 12(8-x) - 2$, which is the new equation.

And next, assuming $\boldsymbol{g}$ is the new function, we get

$$y = g(x) = 0.2(8 - x)^5 - 2.25(8 - x)^4 + 9(8 - x)^3 - 15.5(8 - x)^2 + 12(8 - x) - 2.$$

And for another instance, transforming the function $\boldsymbol{f}$ symmetrically about a line $\boldsymbol{x = -1}$, we can set first, $(s, t) = (-2 - x, y)$.
Thus, we get $s = -2 - x$, and $t = y$, so we get $x = -2 - s$, and $y = t$.

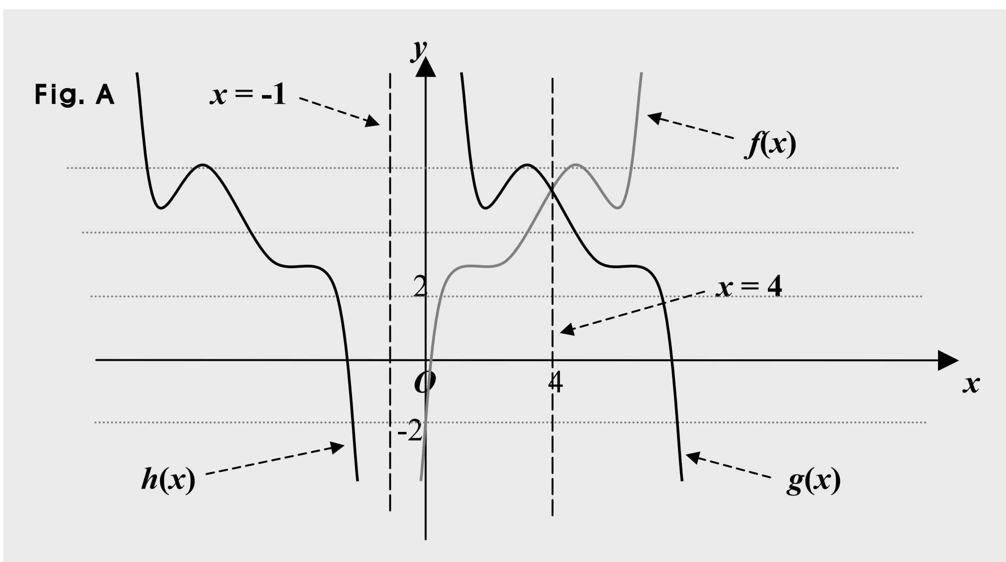
So next, assuming the new function is $\boldsymbol{h}$, we get

$$y = f(x) \Rightarrow t = f(-2 - s) \Rightarrow y = h(x) = f(-2 - x)$$

Thus, the new function $\boldsymbol{h}$ is as follows.

$$y = h(x) = 0.2(-2 - x)^5 - 2.25(-2 - x)^4 + 9(-2 - x)^3 - 15.5(-2 - x)^2 + 12(-2 - x) - 2.$$

And putting all the three functions in a graph, we get this:



2.3. Symmetric about a Point

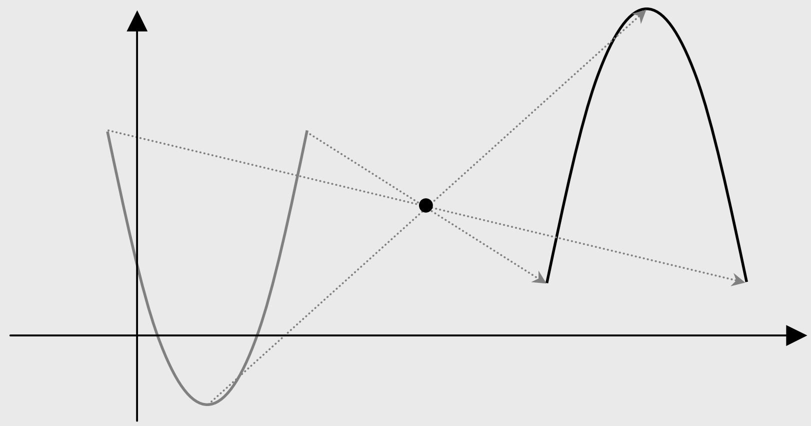


Fig. 0

• First, note that in this book, some important terminologies are used the way as follows.

The original curve: the curve of the function or equation a transformation is applied to
The new curve: the curve of the function or equation made by a transformation

The original function or equation: the function or equation a transformation is applied to
The new function or equation: the function or equation made by a transformation

The original arbitrary point or the original point: the arbitrary point in the original curve.
The new arbitrary point or the new point: the arbitrary point in the new curve.

Applying a transformation symmetric about a point, we get a new curve, which is symmetric to the original curve about the point, which is called therefore, the axis of symmetry. How then is the new curve made?

In a transformation symmetric about a point, every point in the original curve gets moved symmetrically about the point, and then, all the points moved form the new curve, which is thus, said to be symmetric to the original curve about the point.

What do we mean by then, transforming a function symmetrically about a point?

Transforming a function symmetrically about a point, we get a new curve, which is symmetric about the point to the curve of the function. So do we get a new function?

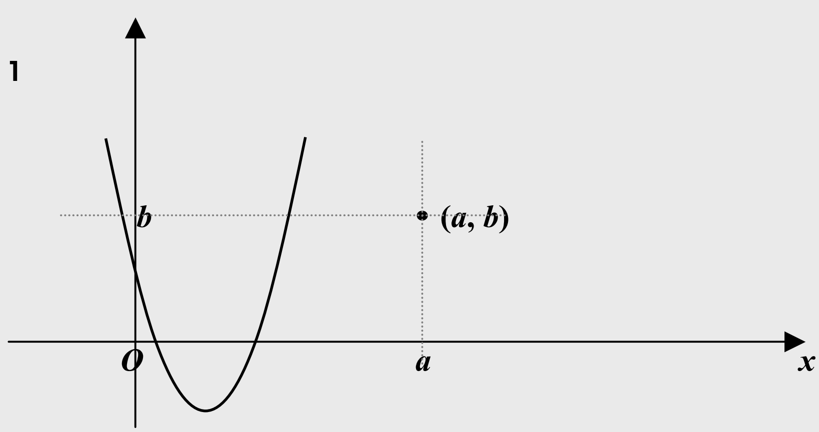
Yes, we do. So the new curve is the curve of the new function.

And what's covered in this section is a case where the axis of symmetry is a point (a, b). So covered here is a transformation symmetric about a point (a, b).

So the two curves, the original and the new in such a transformation can be said to be symmetric about a point (a, b). And if $a = b = 0$, the axis of symmetry is the origin.

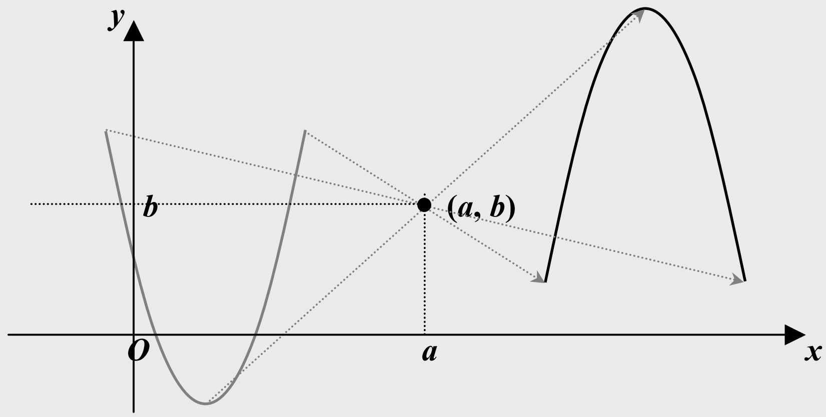
Suppose now that the original curve is the curve of a function $y = f(x)$, and is put in a graph the way below.

Fig. 1



Then, applying to the curve of the function f the symmetric transformation stated above, we get a new curve, which is of course, symmetric to the curve of the function f about the point (a, b). Thus, the new curve is the curve in black in the graph below.

Fig. 2



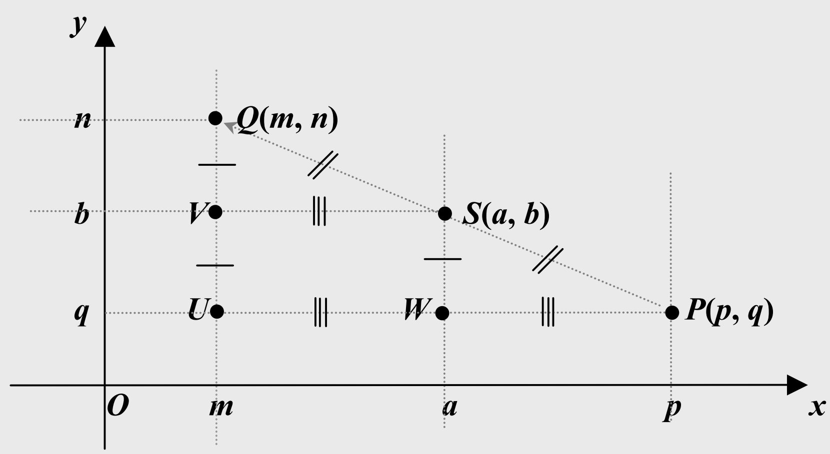
How then can we get the new curve, that is, the new equation?

A transformation definition shows specifically ***how*** every point gets changed in the original curve. So it says ***the way*** the original curve changes, that is, ***the way*** the new curve gets made. We want to get therefore, the definition first.

To begin with, if two points are symmetric about a point (a, b), the two are equal distance away from the point, and both points are in a line passing through the point.

So for instance, assuming two points (p, q) and (m, n) are symmetric about a point (a, b), we can put the two points in a graph the way below.

Fig. 3



Suppose now, given the point (p, q), we want to find the point (m, n).
That is, we want to find the point symmetric to the point P about the point S.

In other words, we want to put the point $\boldsymbol{Q(m, n)}$ in terms of the coordinates of $\boldsymbol{P(p, q)}$.

That is to say that using the fact that $\boldsymbol{(m, n)}$ and $\boldsymbol{(p, q)}$ are symmetric about $\boldsymbol{(a, b)}$, we want to put each of the coordinates of $\boldsymbol{(m, n)}$ in terms of the coordinates of $\boldsymbol{(p, q)}$.

To begin with, in the graph above, the distance from $\boldsymbol{U}$ to $\boldsymbol{W}$ equals the distance from $\boldsymbol{W}$ to $\boldsymbol{P}$. That's because $\boldsymbol{P}$ and $\boldsymbol{Q}$ are the same distance away from the point $\boldsymbol{S}$, and thus, the triangle $\boldsymbol{QVS}$ is the same as the triangle $\boldsymbol{SWP}$.

So we get $\boldsymbol{m - a = a - p}$. Thus, we get $\boldsymbol{m = 2a - p}$.

And next, the same is probably true for the other coordinate $\boldsymbol{n}$, too, for the same reason. In other words, since the triangle $\boldsymbol{QVS}$ is the same as the triangle $\boldsymbol{SWP}$, the distance from $\boldsymbol{Q}$ to $\boldsymbol{V}$ equals the distance from $\boldsymbol{V}$ to $\boldsymbol{U}$.

So we get $\boldsymbol{n - b = b - q}$. And thus, we get $\boldsymbol{n = 2b - q}$.

So expressing the point $\boldsymbol{Q(m, n)}$ in terms of the coordinates of the point $\boldsymbol{P(p, q)}$, we get

$\boldsymbol{(m, n) = (2a - p, 2b - q)}$.

And of course, we can get the same, too, using the fact that the point $\boldsymbol{S(a, b)}$, that is, the axis of symmetry is the midpoint between $\boldsymbol{P(p, q)}$ and $\boldsymbol{Q(m, n)}$. Then, we get

$\boldsymbol{a = \frac{(m+p)}{2} \Rightarrow 2a = m + p \Rightarrow m = 2a - p}$. $\qquad \boldsymbol{b = \frac{(n+q)}{2} \Rightarrow 2b = n + q \Rightarrow n = 2b - q}$.

And thus, assuming $\boldsymbol{C}$ is the transformation symmetric about a point $\boldsymbol{(a, b)}$, and applying $\boldsymbol{C}$ to the point $\boldsymbol{(p, q)}$, we get $\boldsymbol{(m, n) = (2a - p, 2b - q)}$, which shows therefore, the correlation between $\boldsymbol{(p, q)}$ and $\boldsymbol{(m, n)}$ in the symmetry about the point $\boldsymbol{(a, b)}$.

What then is the correlation between the original arbitrary point and the new arbitrary point in the symmetric transformation?

The arbitrary point in a curve represents all the points in the curve. So assuming $\boldsymbol{(p, q)}$ is in the original curve, $\boldsymbol{(x, y)}$ is the original arbitrary point, $\boldsymbol{(s, t)}$ is the new arbitrary point, and $\boldsymbol{(m, n)}$ is in the new curve, we get a correlation as follows. $\boldsymbol{(s, t) = (2a - x, 2b - y)}$, because $\boldsymbol{(x, y)}$ behaves the way $\boldsymbol{(p, q)}$ does, and vice versa, and the same is true for $\boldsymbol{(s, t)}$ and $\boldsymbol{(m, n)}$, too. So we can put the transformation $\boldsymbol{C}$ the way below.

$\boldsymbol{C}$**:** $(\boldsymbol{x}, \boldsymbol{y}) \longrightarrow (\boldsymbol{s}, \boldsymbol{t}) = (2\boldsymbol{a} - \boldsymbol{x}, 2\boldsymbol{b} - \boldsymbol{y})$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are constant.

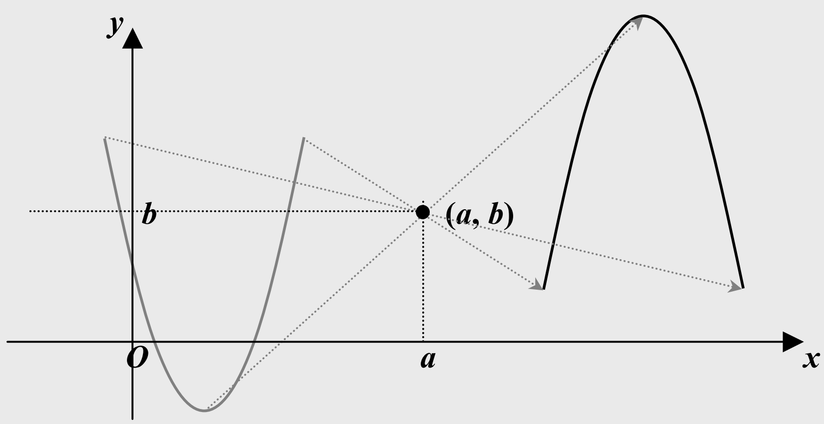
And briefly, we can put it this way, too: $\boldsymbol{C}$**:** $(\boldsymbol{x}, \boldsymbol{y}) \longrightarrow (2\boldsymbol{a} - \boldsymbol{x}, 2\boldsymbol{b} - \boldsymbol{y})$.

So we can call the expression above the definition of the transformation symmetric about a point $(\boldsymbol{a}, \boldsymbol{b})$. And the core of the transformation definition is $(2\boldsymbol{a} - \boldsymbol{x}, 2\boldsymbol{b} - \boldsymbol{y})$, which is the new arbitrary point, so we set $(\boldsymbol{s}, \boldsymbol{t}) = (2\boldsymbol{a} - \boldsymbol{x}, 2\boldsymbol{b} - \boldsymbol{y})$, and it shows how each and every point in the original curve gets changed, that is, the way every point in the new curve gets made.

Applying the transformation $\boldsymbol{C}$ to a function, we get another function. That is, if the original curve is a curve of a function, the new curve is a curve of a function, too.

In the figure below, the curve in gray is the original curve. Applying the transformation $\boldsymbol{C}$ to the original curve, we get a new curve, which is the curve in black below.

Fig. 4



How then do we get the new curve, that is, the equation of the new curve?

We can get it getting the equation connecting the coordinates of the new arbitrary point $(\boldsymbol{s}, \boldsymbol{t})$, that is, the connective equation between $\boldsymbol{s}$ and $\boldsymbol{t}$. And we can get the connective equation mixing the new arbitrary point into the original curve. And we can get them mixed using $(2\boldsymbol{a} - \boldsymbol{x}, 2\boldsymbol{b} - \boldsymbol{y})$. And using it, we begin with this: $(\boldsymbol{s}, \boldsymbol{t}) = (2\boldsymbol{a} - \boldsymbol{x}, 2\boldsymbol{b} - \boldsymbol{y})$.

Then, we can set $\boldsymbol{s} = 2\boldsymbol{a} - \boldsymbol{x}$, and $\boldsymbol{t} = 2\boldsymbol{b} - \boldsymbol{y}$, which is the system of equations stated above. So solving it for $\boldsymbol{x}$ and $\boldsymbol{y}$, we get $\boldsymbol{x} = 2\boldsymbol{a} - \boldsymbol{s}$, and $\boldsymbol{y} = 2\boldsymbol{b} - \boldsymbol{t}$.

Thus, we can now mix the new arbitrary point (s, t) into the original curve. That is, we can mix now, (s, t) into the original equation $y = f(x)$ to get the connective equation.

And getting the connective equation, putting $x = 2a - s$ and $y = 2b - t$ into the original equation. That is, in $y = f(x)$, we replace x with $2a - s$, and y with $2b - t$.

So doing the substitutions, we get $y = f(x) \Rightarrow 2b - t = f(2a - s)$, which is the connective equation between s and t, and thus, is the new equation indicating the new curve.

And we know that the new curve is in the x-y plane, and that s is another name for the variable x, and t is another name for the variable y.

So replacing s with x, and t with y, we get the new equation in the x-y system.
That is, we get $2b - t = f(2a - s) \Rightarrow 2b - y = f(2a - x)$, which is the new equation.

So in sum, transforming an equation $y = f(x)$ symmetrically about a line $y = b$, we get a new equation, and the new equation is $2b - y = f(2a - x)$.

And we can put it the way below, too.

Transforming an equation $h(x, y) = 0$ symmetrically about a line $y = b$, and assuming k is the new equation, we get $k(x, y) = h(2a - x, 2b - y) = 0$.   How?

We have $(s, t) = (2a - x, 2b - y)$.
So we can get $s = 2a - x$, and $t = 2b - y$, and thus, we get $x = 2a - s$, and $y = 2b - t$.

Next, mixing (s, t) into the original equation $h(x, y) = 0$, we get the connective equation.

And getting the connective equation, putting $x = 2a - s$ and $y = 2b - t$ into the original equation. That is, in $h(x, y) = 0$, we replace x with $2a - s$, and y with $2b - t$.

So doing the substitutions, we get $h(x, y) = 0 \Rightarrow h(2a - s, 2b - t) = 0$, which is the connective equation between s and t, and thus, is the new equation.

And we know that the new curve is in the x-y plane, and that s is another name for the variable x, and t is another name for the variable y.

So replacing s with x, and t with y, we get the new equation in the x-y system.
That is, we get $h(2a - s, 2b - t) = 0 \Rightarrow h(2a - x, 2b - y) = 0$, which is the new equation.

Thus, assuming k is the new equation, we get $k(x, y) = h(2a - x, 2b - y) = 0$.

Next, we know applying to a function such a symmetric transformation, we get a new function. How then can we get the new function?

Expressing a function, we put the output variable in terms of the input variable.
That is, we set y equal to an expression in terms of x, and the expression is the function.

In this case, we have $2b - y = f(2a - x)$. So we can set $y = 2b - f(2a - x)$.

Assuming thus, g is the new function, we can set $y = g(x) = 2b - f(2a - x)$.

Suppose for a concrete case, the original function is $y = f(x) = 2x^2 - 4x + 1$, and it gets transformed symmetrically about a point (4, 2), that is, $a = 4$, and $b = 2$ in the axis of symmetry (a, b).

Then, we can get the new equation, the way as follows.

First, we have $(x, y) \longrightarrow (2a - x, 2b - y)$. So we have $(x, y) \longrightarrow (8 - x, 4 - y)$.

So next, we can set $(s, t) = (8 - x, 4 - y)$.

Thus, we get $s = 8 - x$, and $t = 4 - y$, so we get $x = 8 - s$, and $y = 4 - t$.

Thus, we get $y = f(x) \Rightarrow 4 - t = f(8 - s) \Rightarrow 4 - t = 2(8 - s)^2 - 4(8 - s) + 1$.

So we now have the equation connecting s and t. Replacing thus, s with x, and t with y, we get $4 - y = 2(8 - x)^2 - 4(8 - x) + 1$, which is the new equation.

And simplifying it, we get $y = -2x^2 + 28x - 93$.

Putting it in the vertex form, we get $y = -2(x - 7)^2 + 5$.

And next, assuming g is the new function, we get $y = g(x) = -2(x - 7)^2 + 5$.

And we can put the example above the way below, too.

We can have $y = f(x) = 2x^2 - 4x + 1 \Rightarrow y = 2x^2 - 4x + 1 \Rightarrow 2x^2 - 4x + 1 - y = 0$.

And we can put it this way: $h(x, y) = 2x^2 - 4x + 1 - y = 0$.

So suppose this time, the original equation is $h(x, y) = 2x^2 - 4x + 1 - y = 0$, and it gets transformed symmetrically about the point (4, 2). Then, we can get the new equation the way as follows.

Since we apply the same transformation as the one above, we get $x = 8 - s$, and $y = 4 - t$.

So next, $h(x, y) = 0 \Rightarrow h(8 - s, 4 - t) = 0 \Rightarrow 2(8 - s)^2 - 4(8 - s) + 1 - (4 - t) = 0$

$\Rightarrow t = -2(8 - s)^2 + 4(8 - s) + 3$, which is the equation connecting s and t. Replacing thus, s with x, and t with y, we get $y = -2(8 - x)^2 + 4(8 - x) + 3$, which is the new equation.

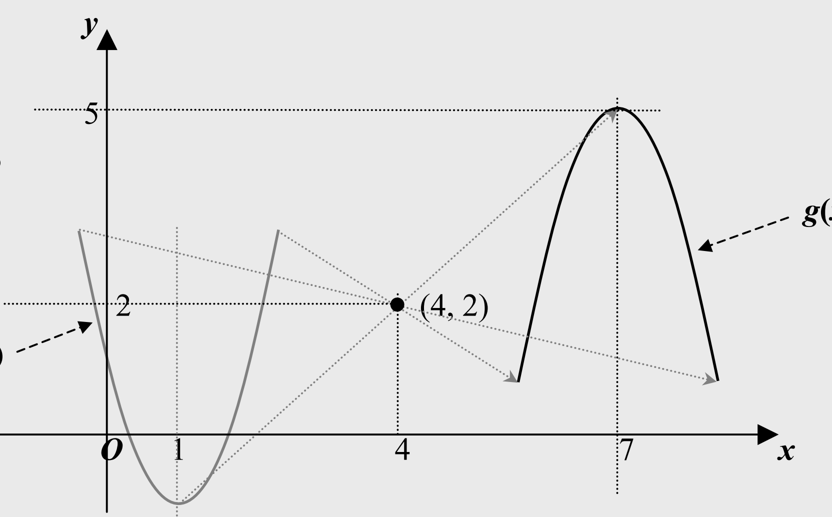
And simplifying it, we get $y = -2x^2 + 28x - 93$.

Putting it in the vertex form, we get $y = -2(x - 7)^2 + 5$.

And assuming $k(x, y) = 0$ is the new equation above, we can set
$k(x, y) = 2(x - 7)^2 - 5 + y = 0$.

And next, assuming g is the new function, we get $y = g(x) = -2(x - 7)^2 + 5$.

Fig. 5



So in sum, transforming $\boldsymbol{y = f(x)}$ symmetrically about a point $\boldsymbol{(a, b)}$,
we can begin with setting $\boldsymbol{(s, t) = (2a - x, 2b - y)}$,
where $\boldsymbol{(s, t)}$ is the new arbitrary point, and $\boldsymbol{(x, y)}$ is the original arbitrary point.

Next, using $\boldsymbol{(s, t) = (2a - x, 2b - y)}$, along with the original equation $\boldsymbol{y = f(x)}$, we can get the equation connecting the coordinates of the new point, i.e., $\boldsymbol{s}$ and $\boldsymbol{t}$. Next, in the connective equation, replacing $\boldsymbol{s}$ with $\boldsymbol{x}$, and $\boldsymbol{t}$ with $\boldsymbol{y}$, we get the new curve.

Thus, getting the new equation, we begin with setting $\boldsymbol{s = 2a - x}$, and $\boldsymbol{t = 2b - y}$.
So next, we get $\boldsymbol{x = 2a - s}$, and $\boldsymbol{y = 2b - t}$. Thus, we get $\boldsymbol{y = f(x) \Rightarrow 2b - t = f(2a - s)}$, which is the equation connecting the coordinates of $\boldsymbol{(s, t)}$, which is the arbitrary point in the new curve, which is in the $\boldsymbol{x\text{-}y}$ plane.

So replacing $\boldsymbol{s}$ with $\boldsymbol{x}$, and $\boldsymbol{t}$ with $\boldsymbol{y}$, we get $\boldsymbol{2b - t = f(2a - s) \Rightarrow 2b - y = f(2a - x)}$, which is the new equation, which indicates the curve of the new function. So assuming $\boldsymbol{y = g(x)}$ is the new function, we get
$\boldsymbol{2b - y = f(2a - x) \Rightarrow y = 2b - f(2a - x) \Rightarrow y = g(x) = 2b - f(2a - x)}$.

So for instance, assuming the original function is $\boldsymbol{y = f(x) = x^3 - 2x^2 - x + 2}$, we get

$$\boldsymbol{y = g(x) = 2b - f(2a - x) = 2b - \{(2a - x)^3 - 2(2a - x)^2 - (2a - x) + 2\}}$$
$$\boldsymbol{= 2b - (2a - x)^3 + 2(2a - x)^2 + (2a - x) - 2}.$$

So the new function is $\boldsymbol{y = g(x) = 2b - (2a - x)^3 + 2(2a - x)^2 + (2a - x) - 2}$.

And we can put the same the way below, too.

Assuming $\boldsymbol{(x_f, y_f)}$ is the original arbitrary point, $\boldsymbol{(x_g, y_g)}$ is the new arbitrary point, and $\boldsymbol{C}$ is the transformation symmetric about a point $\boldsymbol{(a, b)}$, we can put $\boldsymbol{C}$ the way below.

$$\boldsymbol{B: (x_f, y_f) \longrightarrow (x_g, y_g) = (2a - x_f, 2b - y_f)}.$$

That is, $\boldsymbol{f}$ is the original function, and $\boldsymbol{g}$ is the new.

So we get $x_g = 2a - x_f$, and $y_g = 2b - y_f$, and thus, we get $x_f = 2a - x_g$, and $y_f = 2b - y_g$.

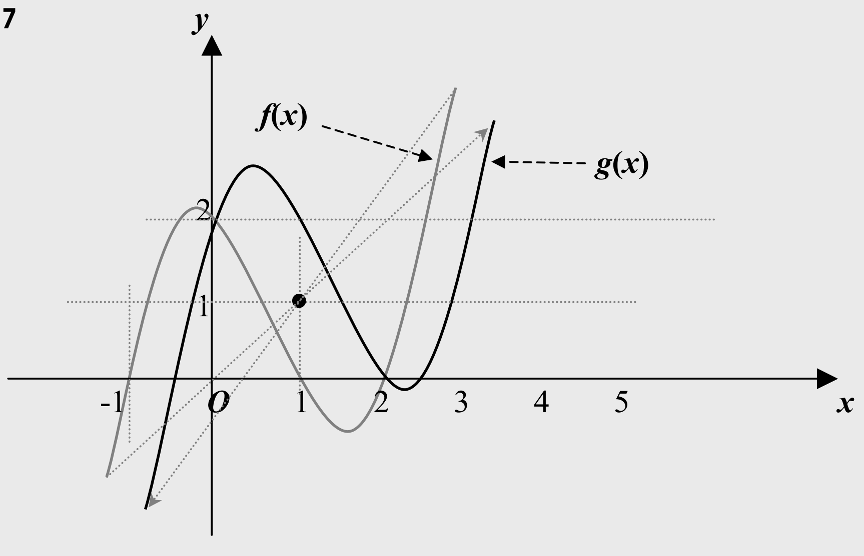
So next, we get $y_f = f(x_f) \Rightarrow 2b - y_g = f(2a - x_g) \Rightarrow y_g = 2b - f(2a - x_g) = g(x_g)$.

And thus, we get $y = g(x) = 2b - f(2a - x) = 2b - (2a - x)^3 + 2(2a - x)^2 + (2a - x) - 2$.

And assuming for instance, the axis of symmetry is a point (1, 1), that is, $a = 1$ and $b = 1$, we get $y = f(x) = x^3 - 2x^2 - x + 2$, and

$$y = g(x) = 2 - f(2 - x)$$
$$= 2 - (2 - x)^3 + 2(2 - x)^2 + (2 - x) - 2 = -(2 - x)^3 + 2(2 - x)^2 + (2 - x)$$
$$= -(2 - x)\{(2 - x)^2 - 2(2 - x) - 1\} = (2 - x)(x^2 - 2x - 1).$$

Fig. 7



What if we want the transformation to be symmetric about the origin?

Setting $a = 0$ and $b = 0$ in (a, b), we get the transformation symmetric about the origin.

And we have $C: (x, y) \longrightarrow (2a - x, 2b - y)$.

So assuming the transformation we want is X, we can put it the way below.

$X: (x, y) \longrightarrow (-x, -y)$.

So assuming a function $\boldsymbol{h}$ is the transformation of $\boldsymbol{f}$ by $\boldsymbol{X}$, we can get it the way below.

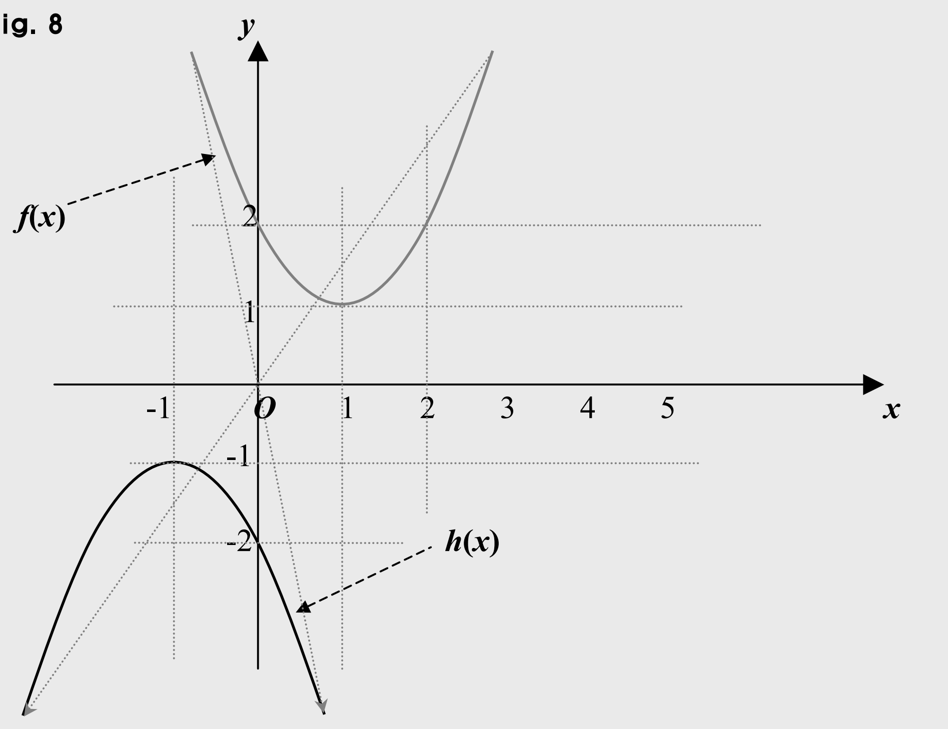
To begin with, we get $x_h = -x_f$, and $y_h = -y_f$, and thus, we get $x_f = -x_h$, and $y_f = -y_h$.

So next, we get $y_f = f(-x_f) \Rightarrow -y_h = f(-x_h) \Rightarrow y_h = -f(-x_h) = h(x_h)$.

Thus, we get $y = h(x) = -f(-x)$.
So for instance, assuming $y = f(x) = (x - 1)^2 + 1$, we get

$y = h(x) = -f(-x) = -\{(-x - 1)^2 + 1\} = -(x + 1)^2 - 1 \Rightarrow y = h(x) = -(x + 1)^2 - 1$.

Fig. 8



Let's next, for another instance, transform an equation $A(x, y) = \dfrac{y^2}{1-x} + 2 = 0$

symmetrically about a point (1, 2).

To begin with, we have $C: (x, y) \longrightarrow (2a - x, 2b - y)$, which is the transformation symmetric about a point (a, b).

So setting $a = 1$, and $b = 2$, we get the transformation symmetric about the point (1, 2).

Thus, assuming the transformation is $\boldsymbol{K}$, we can put it the way below.

$\boldsymbol{K}$: $(\boldsymbol{x}, \boldsymbol{y}) \longrightarrow (2-\boldsymbol{x}, 4-\boldsymbol{y})$.

So assuming an equation $\boldsymbol{H}(\boldsymbol{x}, \boldsymbol{y}) = 0$ is the transformation of $\boldsymbol{A}$ by $\boldsymbol{K}$, we can get it the way below.

We get first $x_H = 2 - x_A$, and $y_H = 4 - y_A$, and thus, we get $x_A = 2 - x_H$, and $y_A = 4 - y_H$.

So next, we get $A(x_A, y_A) = 0 \Rightarrow A(2 - x_H, 4 - y_H) = 0$.

Thus, we get $H(x_H, y_H) = A(2 - x_H, 2 - y_H) = 0$.

And we have $A(x, y) = \dfrac{y^2}{1-x} + 2 = 0$. That is, we have $A(x_A, y_A) = \dfrac{{y_A}^2}{1-x_A} + 2 = 0$.

So we get $A(x_A, y_A) = \dfrac{{y_A}^2}{1-x_A} + 2 = 0 \Rightarrow A(2-x_H, 4-y_H) = \dfrac{(4-y_H)^2}{1-(2-x_H)} + 2 = 0$.

Thus, we get $H(x_H, y_H) = \dfrac{(4-y_H)^2}{x_H - 1} + 2 = 0$. That is, we get $H(x, y) = \dfrac{(4-y)^2}{x-1} + 2 = 0$.

So we can put $H(x_H, y_H) = A(2 - x_H, 4 - y_H) = 0$ the way below, too.

$H(x, y) = A(2 - x, 4 - y) = 0$.

Besides, using the transformation operator, the dot •, we can put it the way below, too.

$H = K \bullet A = A(2 - x, 4 - y) = \dfrac{(4-y)^2}{x-1} + 2 = 0$. And Let's now put the curves in a graph.

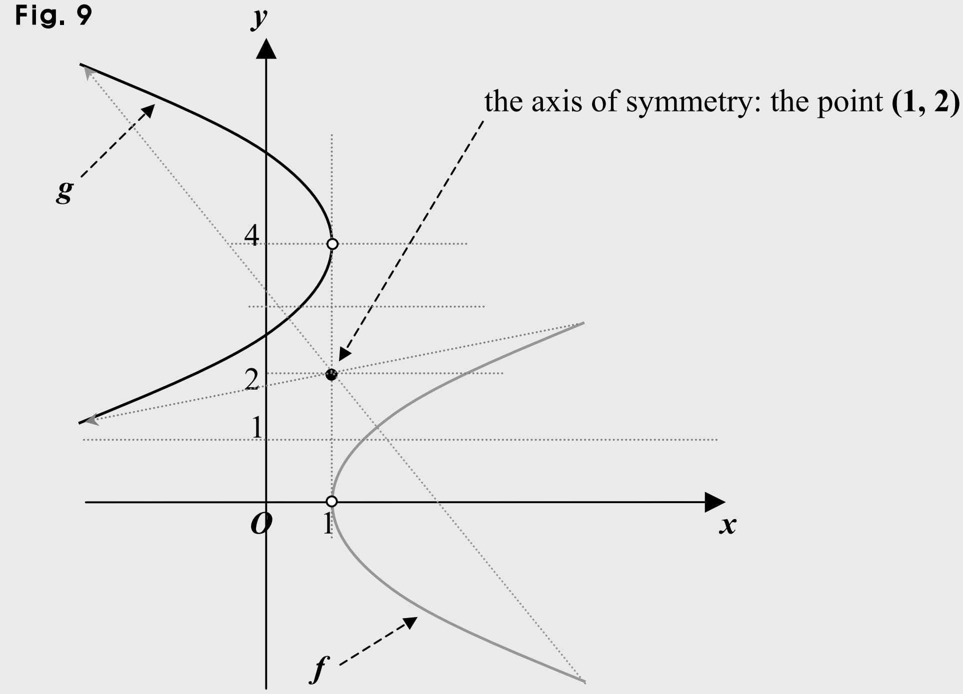
To begin with, converting the two equations, we get

$$A(x, y) = \frac{y^2}{1-x} + 2 = 0 \Rightarrow y^2 + 2(1-x) = 0 \Rightarrow x = \frac{y^2}{2} + 1.$$

$$H(x, y) = \frac{(4-y)^2}{x-1} + 2 = 0 \Rightarrow (4-y)^2 + 2(x-1) = 0 \Rightarrow x = 1 - \frac{(y-4)^2}{2}.$$

We want to note however, that the actual equations are $\dfrac{y^2}{1-x}+2=0$ and $\dfrac{(4-y)^2}{x-1}+2=0$. So we can't have $x = 1$, since a denominator cannot be 0.

Fig. 9



And let's next, for another instance, transform the function below symmetrically about a point (2, 2).

$y = f(x) = 0.2x^5 - 2.25x^4 + 9x^3 - 15.5x^2 + 12x - 2$.

Then, we can get the new equation, the way as follows.

First, we have $(x, y) \longrightarrow (2a - x, 2b - y)$. So we have $(x, y) \longrightarrow (4 - x, 4 - y)$.

So next, we can set $(s, t) = (4 - x, 4 - y)$.

Thus, we get $s = 4 - x$, and $t = 4 - y$, so we get $x = 4 - s$, and $y = 4 - t$.

Thus, we get $y = f(x) \Rightarrow 4 - t = f(4 - s)$

$$\Rightarrow 4 - t = 0.2(4 - s)^5 - 2.25(4 - s)^4 + 9(4 - s)^3 - 15.5(4 - s)^2 + 12(4 - s) - 2$$

$$\Rightarrow t = -\{0.2(4 - s)^5 - 2.25(4 - s)^4 + 9(4 - s)^3 - 15.5(4 - s)^2 + 12(4 - s) - 2\} + 4$$

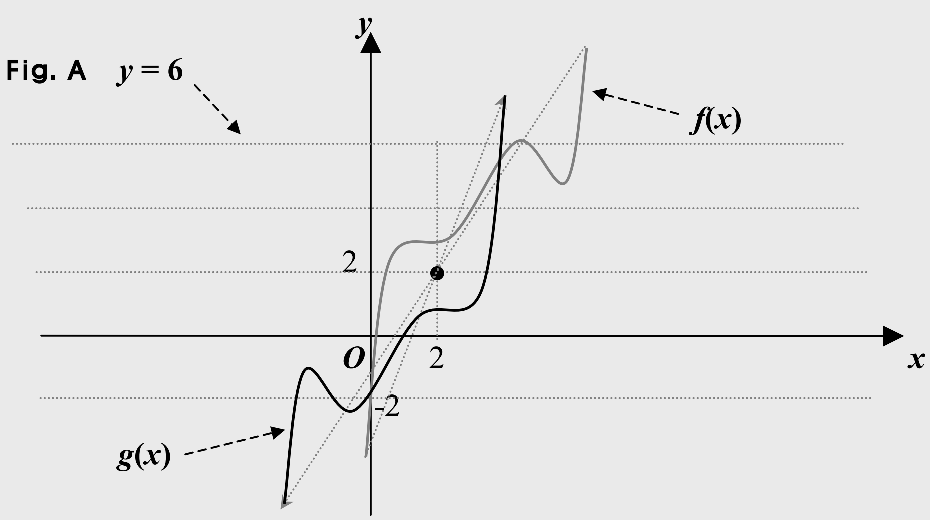
$$\Rightarrow t = 6 - 0.2(4 - s)^5 + 2.25(4 - s)^4 - 9(4 - s)^3 + 15.5(4 - s)^2 - 12(4 - s).$$

So we now have the equation connecting s and t. Replacing thus, s with x, and t with y, we get $y = 6 - 0.2(4 - x)^5 + 2.25(4 - x)^4 - 9(4 - x)^3 + 15.5(4 - x)^2 - 12(4 - x)$, which is the new equation.

And next, assuming g is the new function, we get

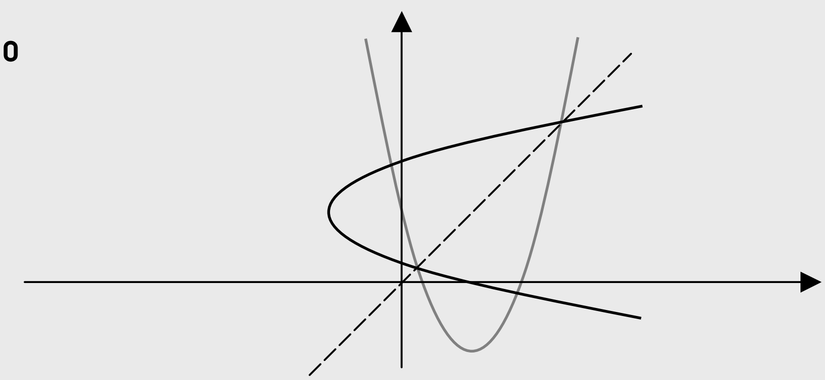
$$y = g(x) = 6 - 0.2(4 - x)^5 + 2.25(4 - x)^4 - 9(4 - x)^3 + 15.5(4 - x)^2 - 12(4 - x).$$

Fig. A



2.4. Symmetric about a line 3

Fig. 0



• First, note that in this book, some important terminologies are used the way as follows.

The original curve: the curve of the function or equation a transformation is applied to
The new curve: the curve of the function or equation made by a transformation

The original function or equation: the function or equation a transformation is applied to
The new function or equation: the function or equation made by a transformation

The original arbitrary point or the original point: the arbitrary point in the original curve.
The new arbitrary point or the new point: the arbitrary point in the new curve.

Applying a transformation symmetric about a line, we get a new curve, which is symmetric to the original curve about the line, which is called therefore, the axis of symmetry. How then is the new curve made?

In a transformation symmetric about a line, every point in the original curve gets moved symmetrically about the line, and then, all the points moved form the new curve, which is thus, said to be symmetric to the original curve about the line.

What do we mean by then, transforming a function symmetrically about a line?

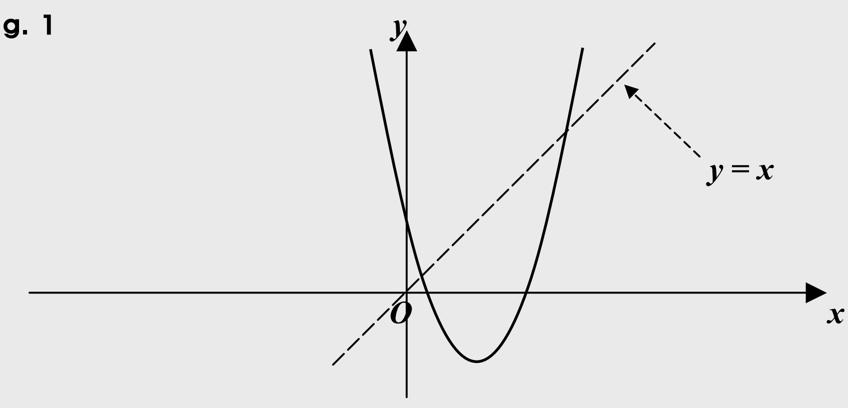
Transforming a function symmetrically about a line, we get a new curve, which is symmetric about the line to the curve of the function. So do we get a new function?

Maybe. It depends on the original function and the line taken as the axis of symmetry. If for instance, the line is horizontal or vertical, that is, parallel or perpendicular to a coordinate axis as the x-axis, we get a new function.

And what's covered in this section is a case the axis of symmetry is the line $y = x$. So covered here is a transformation symmetric about the line $y = x$.

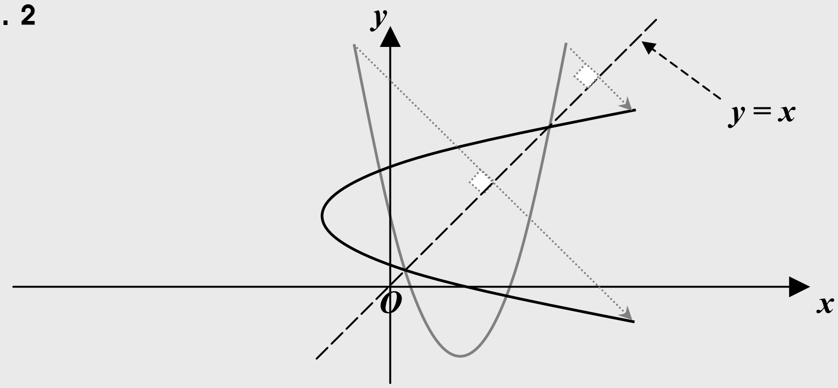
So the two curves, the original and the new in such a transformation can be said to be symmetric about the line $y = x$. Suppose now that the original curve is the curve of a function $y = f(x)$, and is put in a graph the way below.

Fig. 1



Then, applying to the curve of the function f the symmetric transformation stated above, we get a new curve, which is of course, symmetric to the curve of the function f about the line $y = x$. And thus, the new curve is the curve in black in the graph below.

Fig. 2



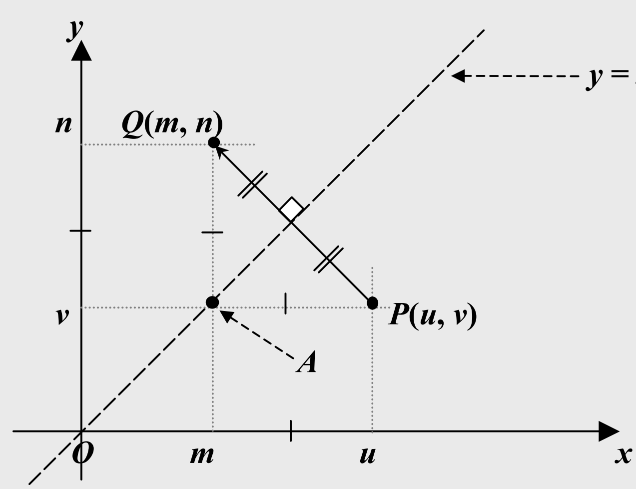
How then can we get the new curve, that is, the new equation?

A transformation definition shows specifically **how** every point gets changed in the original curve. So it says **the way** the original curve changes, that is, **the way** the new curve gets made. We want to get therefore, the definition first.

To begin with, if two points are symmetric about a line $\boldsymbol{y = ax + b}$, the two are equal distance away from the line, and both points are in a line perpendicular to the line.

So for instance, assuming two points $\boldsymbol{(u, v)}$ and $\boldsymbol{(m, n)}$ are symmetric about a point $\boldsymbol{(a, b)}$, we can put the two points in a graph the way below.

Fig. 3



Suppose now, given the point $\boldsymbol{(u, v)}$, we want to find the point $\boldsymbol{(m, n)}$.

That is, we want to find the point symmetric to the point $\boldsymbol{P}$ about the line $\boldsymbol{y = x}$.

In other words, we want to put the point $\boldsymbol{Q(m, n)}$ in terms of the coordinates of $\boldsymbol{P(u, v)}$.

That is to say that using the fact that $\boldsymbol{(m, n)}$ and $\boldsymbol{(u, v)}$ are symmetric about the line $\boldsymbol{y = x}$, we want to put each coordinate of $\boldsymbol{(m, n)}$ in terms of the coordinates of $\boldsymbol{(u, v)}$.

So to begin with, in the graph above, the distance from $\boldsymbol{m}$ to $\boldsymbol{u}$ equals the distance from $\boldsymbol{v}$ to $\boldsymbol{n}$. That's because $\boldsymbol{P}$ and $\boldsymbol{Q}$ are the same distance away from the line $\boldsymbol{y = x}$, the line segment $\boldsymbol{PQ}$ is perpendicular to the line $\boldsymbol{y = x}$, and thus, the triangle $\boldsymbol{QAP}$ is isosceles.

Next, we get $\boldsymbol{m = v}$, because the point $\boldsymbol{A}$ is in the line $\boldsymbol{y = x}$, $\boldsymbol{m}$ is the $\boldsymbol{x}$-coordinate of the point $\boldsymbol{A}$, and $\boldsymbol{v}$ is the $\boldsymbol{y}$-coordinate.

And next the distance from $\boldsymbol{m}$ to $\boldsymbol{u}$ equals the distance from $\boldsymbol{v}$ to $\boldsymbol{n}$, and we have $\boldsymbol{m = v}$. So we get $\boldsymbol{n = u}$.

Thus, putting the point $\boldsymbol{Q(m, n)}$ in terms of the coordinates of the point $\boldsymbol{P(u, v)}$, we get $\boldsymbol{(m, n) = (v, u)}$.

What then is the correlation between the original and the new arbitrary points?

The arbitrary point in a curve represents all the points in the curve. So assuming $\boldsymbol{(u, v)}$ is in the original curve, $\boldsymbol{(x, y)}$ is the original arbitrary point, $\boldsymbol{(s, t)}$ is the new arbitrary point, and $\boldsymbol{(m, n)}$ is in the new curve, we get this correlation: $\boldsymbol{(s, t) = (y, x)}$, because $\boldsymbol{(x, y)}$ behaves the way $\boldsymbol{(u, v)}$ does, and vice versa, and the same is true for $\boldsymbol{(s, t)}$ and $\boldsymbol{(m, n)}$, too.

Thus, assuming $\boldsymbol{R}$ is the transformation symmetric about the line $\boldsymbol{y = x}$, we can put $\boldsymbol{R}$ the way as follows. $\boldsymbol{R: (x, y) \longrightarrow (s, t) = (y, x)}$.

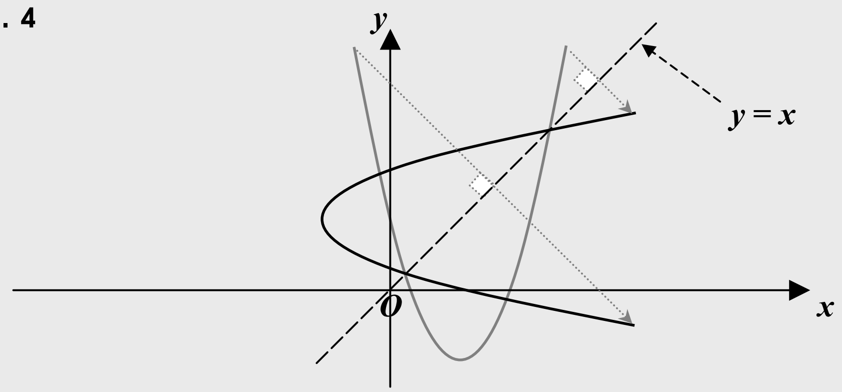
And briefly, we can just put it this way, too: $\boldsymbol{R: (x, y) \longrightarrow (y, x)}$.

So we can call the expression above the definition of the transformation <u>symmetric about the line $\boldsymbol{y = x}$</u>. And the core of the transformation definition is $\boldsymbol{(y, x)}$, which is the new arbitrary point, so we set $\boldsymbol{(s, t) = (y, x)}$. And it shows how each and every point in the original curve gets changed, that is, the way every point in the new curve gets made.

So applying such a transformation to a function, do we get another function? That is, if the original curve is a curve of a function, is the new curve a curve of a function, too?

It is not always the case. In the figure below, the curve in gray is the original curve. Applying the transformation $\boldsymbol{R}$ to the original curve, we get a new curve, which is the curve in black, which cannot be however, a curve of a function.

Fig. 4



That's because the new curve can meet a vertical line at more than one point, and thus, cannot be a curve of a function. In the graph above, when $\boldsymbol{x} = \mathbf{0}$, $\boldsymbol{y}$ gets two values. A function has to have one output only for one input. So the new curve in the graph above, that is, the curve in black is not a curve of a function but just a curve of an equation.

How then do we get the new curve, that is, the equation of the new curve?

We can get it getting the equation connecting $\boldsymbol{s}$ and $\boldsymbol{t}$. And we can get the connective equation mixing $(\boldsymbol{s}, \boldsymbol{t})$ into the original curve. And we can get them mixed using $(\boldsymbol{y}, \boldsymbol{x})$.

And using $(\boldsymbol{y}, \boldsymbol{x})$, since we have $(\boldsymbol{s}, \boldsymbol{t}) = (\boldsymbol{y}, \boldsymbol{x})$. we can get $\boldsymbol{x} = \boldsymbol{t}$, and $\boldsymbol{y} = \boldsymbol{s}$.

And thus, we can now get the mixture, that is, the connective equation, putting $\boldsymbol{x} = \boldsymbol{t}$ and $\boldsymbol{y} = \boldsymbol{s}$ into the original equation. That is, in the original equation $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$, we replace $\boldsymbol{x}$ with $\boldsymbol{t}$, and $\boldsymbol{y}$ with $\boldsymbol{s}$. So doing the substitutions, we get $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x}) \Rightarrow \boldsymbol{s} = \boldsymbol{f}(\boldsymbol{t})$, which is the equation connecting $\boldsymbol{s}$ and $\boldsymbol{t}$, and thus, is the new equation indicating the new curve.

And we know that the new curve is in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane, and that $\boldsymbol{s}$ is another name for the variable $\boldsymbol{x}$, and $\boldsymbol{t}$ is another name for the variable $\boldsymbol{y}$. So next, replacing $\boldsymbol{s}$ with $\boldsymbol{x}$, and $\boldsymbol{t}$ with $\boldsymbol{y}$, we get $\boldsymbol{s} = \boldsymbol{f}(\boldsymbol{t}) \Rightarrow \boldsymbol{x} = \boldsymbol{f}(\boldsymbol{y})$, which is the new curve, that is, the new equation.

So in sum, transforming an equation $y = f(x)$ symmetrically about a line $y = x$, we get a new equation, and the new equation is $x = f(y)$.

And we can put it the way below, too.

Transforming an equation $h(x, y) = 0$ symmetrically about a line $y = x$, and assuming k is the new equation, we get $k(x, y) = h(y, x) = 0$. How?

We have $(s, t) = (y, x)$. So we can get $s = y$, and $t = x$.

Next, mixing (s, t) into the original equation $h(x, y) = 0$, we get the connective equation. And getting the connective equation, putting $x = t$ and $y = s$ into the original equation. That is, in $h(x, y) = 0$, we replace x with t, and y with s.

So doing the substitutions, we get $h(x, y) = 0 \Rightarrow h(t, s) = 0$, which is the connective equation between s and t, and thus, is the new equation.

And we know that the new curve is in the x-y plane, and that s is another name for the variable x, and t is another name for the variable y.

So replacing s with x, and t with y, we get the new equation in the x-y system.
That is, we get $h(t, s) = 0 \Rightarrow h(y, x) = 0$, which is the new equation.

And thus, assuming k is the new equation, we get $k(x, y) = h(y, x) = 0$.

Suppose for a concrete case, the original function is $y = f(x) = 2x^2 - 4x + 1$, and it gets transformed symmetrically about the line $y = x$.

Then, we can get the new equation, the way as follows.

First, we have $(x, y) \longrightarrow (s, t) = (y, x)$. So we get $s = y$, and $t = x$.

Thus, we get $y = f(x) \Rightarrow s = f(t) \Rightarrow s = 2t^2 - 4t + 1$.

So we now have the equation connecting s and t. Replacing thus, s with x, and t with y, we get $x = 2y^2 - 4y + 1$, which is the new equation, which is a horizontal parabola.

And putting it in the vertex form, we get $\boldsymbol{x = 2(y-1)^2 - 1}$.

And we can put the example above the way below, too.

We can have $\boldsymbol{y = f(x) = 2x^2 - 4x + 1 \Rightarrow y = 2x^2 - 4x + 1 \Rightarrow 2x^2 - 4x + 1 - y = 0}$.
And we can put it this way: $\boldsymbol{h(x, y) = 2x^2 - 4x + 1 - y = 0}$.

So suppose this time, the original equation is $\boldsymbol{h(x, y) = 2x^2 - 4x + 1 - y = 0}$, and it gets transformed symmetrically about the line $\boldsymbol{y = x}$. Then, we can get the new equation, the way as follows.

Since we apply the same transformation as the one above, we get $\boldsymbol{x = t}$, and $\boldsymbol{y = s}$.

So next, $\boldsymbol{h(x, y) = 0 \Rightarrow h(t, s) = 0 \Rightarrow 2t^2 - 4t + 1 - s = 0}$, which is the equation connecting $\boldsymbol{s}$ and $\boldsymbol{t}$. Replacing thus, $\boldsymbol{s}$ with $\boldsymbol{x}$, and $\boldsymbol{t}$ with $\boldsymbol{y}$, we get $\boldsymbol{2y^2 - 4y + 1 - x = 0}$, which is the new equation.

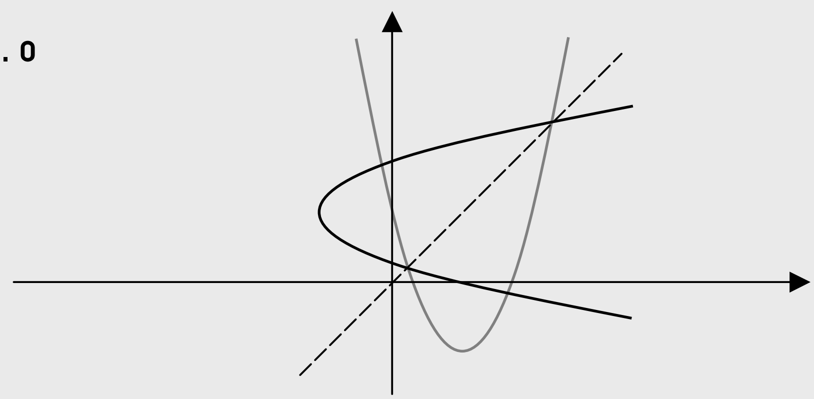
And putting it in the vertex form, we get $\boldsymbol{x = 2(y-1)^2 - 1}$.
Also, assuming $\boldsymbol{k(x, y) = 0}$ is the new equation above, we can set
$\boldsymbol{k(x, y) = 2(y-1)^2 - 1 - x = 0}$.

What then about the new function?

For the example taken above, we don't get a new function. That's because the original is not one-to-one. And we will continue this discussion, together with some important facts about quadratic equations in the next section.

2.5. Symmetric about a line 4

Fig. 0



The previous section covers the transformation symmetric about the line $\boldsymbol{y = x}$. And if $\boldsymbol{R}$ is the transformation, we can put it the way below.

$\boldsymbol{R}$**:** $\boldsymbol{(x, y)} \longrightarrow \boldsymbol{(s, t) = (y, x)}$.

And briefly, we can put it this way, too: $\boldsymbol{R}$**:** $\boldsymbol{(x, y)} \longrightarrow \boldsymbol{(y, x)}$.

What then is the transformation symmetric about the line $\boldsymbol{y = x}$?

The definition is saying that in the original equation, $\boldsymbol{x}$ takes the position of $\boldsymbol{y}$, and $\boldsymbol{y}$ takes the position of $\boldsymbol{x}$. For instance, we get $(3, 7) \longrightarrow (7, 3)$. That is, at each and every point in the curve of the original equation, the $\boldsymbol{x}$-coordinate and the $\boldsymbol{y}$-coordinate swap their positions.

So the $\boldsymbol{y}$ in the new equation is the $\boldsymbol{x}$ in the original equation, and of course, the $\boldsymbol{x}$ in the new is the $\boldsymbol{y}$ in the original. And we can call the new equation, the inverse of the original.

So solving the inverse for y, that is, solving the new equation for y, and then, putting a value into x, we get x-value in the original equation.

So for instance, assuming (3, 7) is a point in the original curve, and putting 7 into x in the new equation, we get 3, that is, we get (7, 3) in the new curve.

Assuming for instance, $y = 2x$ is the original, and applying R to it, we get $x = 2y$, which is the new equation, and solving it for y, we get $y = \frac{x}{2}$, which is called the inverse.

In short, the original $y = 2x \Rightarrow x = \frac{y}{2}$, and the new $x = 2y \Rightarrow y = \frac{x}{2}$.
So the y in the new is the same as the x in the original. That is, swapping x and y in the original, we get the new.

What do we mean by then transforming a function symmetrically about the line $y = x$?

Transforming a function symmetrically about a line, we get a new curve, which is symmetric about the line to the curve of the function. So do we get a new function?

If the line taken as the axis of symmetry is horizontal or vertical, that is, parallel to or perpendicular to the x-axis, we get a new function.

What's covered in this section is however, a case where <u>the axis of symmetry is neither vertical nor horizontal</u>. So it may not be the case, we get a new function.

If however, the original is one-to-one, we get a new function, and the new is called the inverse of the original. The example taken above, that is, the original function $y = 2x$ is one-to-one, so taking the transformation of it by R, that is,
<u>symmetrically about the line $y = x$</u>, we get a new function, and the new function is $y = \frac{x}{2}$.

A typical example of a function not one-to-one is a function as $y = f(x) = 2x^2 - 5x + 3$, called a quadratic function, the curve of which is a parabola. So taking the inverse of it, that is, taking the transformation of it by R, we don't get a function, that is, the inverse is not a function. We can force however, such a function (or equation) to be one-to-one.

That is, we can change the domain so that it can be one-to-one. Then, of course, taking the transformation of it by $\boldsymbol{R}$, we get a new function called the inverse. And a function or equation quadratic is often used in many areas of study as well as math. So let's now take a close look at it.

Solving for instance, a quadratic equation $\boldsymbol{ax^2 + bx + c = 0}$,

we can use the quadratic formula, which is $\boldsymbol{x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$.

Solving for $\boldsymbol{y}$ therefore, $\boldsymbol{ax^2 + bx + c = y}$, we can put it this way first, $\boldsymbol{ax^2 + bx + c - y = 0}$,

and then, can use this formula: $\boldsymbol{x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$.

The formula above can give us solutions to equations for all values of $\boldsymbol{y}$. For instance, it can give us solutions to $\boldsymbol{ax^2 + bx + c = 1}$, $\boldsymbol{ax^2 + bx + c = 2}$, $\boldsymbol{ax^2 + bx + c = 5}$, etc.

Swapping now, $\boldsymbol{x}$ and $\boldsymbol{y}$ in the formula above, we get $\boldsymbol{y = \dfrac{-b \pm \sqrt{b^2 - 4a(c - x)}}{2a}}$.

Then, we get two $\boldsymbol{y}$-values for each $\boldsymbol{x}$-value.

For instance, solving $\boldsymbol{y = x^2 - 4x + 5}$ for a value of $\boldsymbol{y}$, we can put it this way first: $\boldsymbol{x^2 - 4x + 5 - y = 0}$. And then, using the quadratic formula above, we can get

$$\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4a(c - y)}}{2a} = \frac{4 \pm \sqrt{4^2 - 4(5 - y)}}{2} = \frac{4 \pm 2\sqrt{4 - (5 - y)}}{2}}$$

$$\boldsymbol{= 2 \pm \sqrt{y - 1} \Rightarrow x = 2 \pm \sqrt{y - 1}}.$$

So using the equation $\boldsymbol{x = 2 \pm \sqrt{y - 1}}$, we can get all the solutions to $\boldsymbol{y = x^2 - 4x + 5}$ for all values of $\boldsymbol{y}$. For instance, solving $\boldsymbol{5 = x^2 - 4x + 5}$, we get

$\boldsymbol{x = 2 \pm \sqrt{y - 1} = 2 \pm 2 \Rightarrow x = 0}$ or $\boldsymbol{4}$. So what can we do with this: $\boldsymbol{x = 2 \pm \sqrt{y - 1}}$?

Swapping now, $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\boldsymbol{x = 2 \pm \sqrt{y - 1}}$, we get $\boldsymbol{y = 2 \pm \sqrt{x - 1}}$.

Then, we get two $\boldsymbol{y}$-values for each $\boldsymbol{x}$-value. For instance, we get $\boldsymbol{y = 0}$ or $\boldsymbol{4}$ for $\boldsymbol{x = 5}$.

What then are the two $\boldsymbol{y}$-values for the $\boldsymbol{x}$-value?

They are the two roots of the equation $x^2 - 4x + 5 = 5$, which is, $x^2 - 4x = 0$.

Now, what then is this equation: $y = \dfrac{-b \pm \sqrt{b^2 - 4a(c-x)}}{2a}$?

It is the new equation we get transforming by R the equation $ax^2 + bx + c = y$.

In other words, assuming E is the equation $y = ax^2 + bx + c$, and S is the new equation $y = \dfrac{-b \pm \sqrt{b^2 - 4a(c-x)}}{2a}$, we get $S = R \bullet E$. That is, S is the transformation of E by R.
And in fact, the new equation S is equivalent to $x = ay^2 + by + c$. How?

$$y = \frac{-b \pm \sqrt{b^2 - 4a(c-x)}}{2a} \Rightarrow 2ay = -b \pm \sqrt{b^2 - 4a(c-x)} \Rightarrow 2ay + b = \pm\sqrt{b^2 - 4a(c-x)}$$

$$\Rightarrow (2ay + b)^2 = b^2 - 4a(c - x) \Rightarrow 4a^2y^2 + 4aby + b^2 = b^2 - 4a(c - x)$$

$$\Rightarrow 4a^2y^2 + 4aby = 4a(x - c) \Rightarrow ay^2 + by = x - c \Rightarrow x = ay^2 + by + c.$$

So transforming by R the equation $y = ax^2 + bx + c$, we get $x = ay^2 + by + c$.

Suppose now, A is an equation $y = \dfrac{-b + \sqrt{b^2 - 4a(c-x)}}{2a}$,

and B is an equation $y = \dfrac{-b - \sqrt{b^2 - 4a(c-x)}}{2a}$.

Then, each of the two equations A and B is made of x and y, which are also, from the equation $y = ax^2 + bx + c$. Also, we can put it in two parts.

One is $y = ax^2 + bx + c$ for $x \geq -\dfrac{b}{2a}$, which indicates a right half parabola,

and the other is $y = ax^2 + bx + c$ for $x < -\dfrac{b}{2a}$, which indicates a left half parabola.

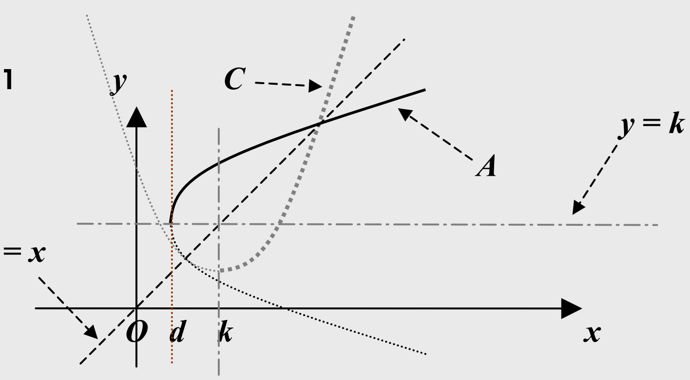
So suppose next, $\boldsymbol{C}$ is $\boldsymbol{y = ax^2 + bx + c}$ for $\boldsymbol{x \geq k}$,

and $\boldsymbol{D}$ is $\boldsymbol{y = ax^2 + bx + c}$ for $\boldsymbol{x < k}$, where $\boldsymbol{k = -\dfrac{b}{2a}}$.

Then, what's important is the fact that applying to the equation $\boldsymbol{C}$ the transformation symmetric about the line $\boldsymbol{y = x}$, that is, the transformation $\boldsymbol{R}$, we get the equation $\boldsymbol{A}$. That is, if $\boldsymbol{C}$ is the original in the case of the transformation $\boldsymbol{R}$, $\boldsymbol{A}$ is the new, and vice versa. What curve then does $\boldsymbol{A}$ have?

The curve of $\boldsymbol{A}$ is an upper half parabola, which is in black shown below.

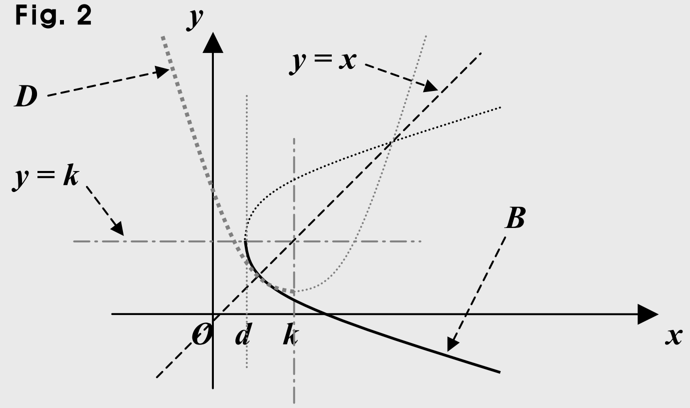
Fig. 1



And the same is true, also, for the other equations $\boldsymbol{B}$ and $\boldsymbol{D}$.
So applying the transformation $\boldsymbol{R}$ to the equation $\boldsymbol{D}$, we get the equation $\boldsymbol{B}$. That is, if $\boldsymbol{D}$ is the original in the case of the transformation $\boldsymbol{R}$, $\boldsymbol{B}$ is the new, and vice versa.

And the curve of $\boldsymbol{B}$ is a lower half parabola as shown below.

Fig. 2



And another important fact is that if we set $y = A(x) = \dfrac{-b+\sqrt{b^2-4a(c-x)}}{2a}$, and set $y = C(x) = ax^2 + bx + c$ for $x \geq -\dfrac{b}{2a}$, the function A can be called the inverse of the function C. And vice versa.

(If not sure of inverse functions, refer to the book, **BASIC FUNCTIONS**.)

And also, setting $y = B(x) = \dfrac{-b-\sqrt{b^2-4a(c-x)}}{2a}$,

and $y = D(x) = ax^2 + bx + c$ for $x < -\dfrac{b}{2a}$, we can call the function B the inverse of the function D. And vice versa, of course.

What then about the domains of the two inverse functions A and B?

Finding the domain in this case, we want to follow the rule as follows.

• What's inside the square root sign is greater than or equal to 0.

(If not sure of the rule above, refer to the book, **Powers and Logarithms**.)

And looking at the original curve in the figure above, we can assume that $a > 0$, because the curve, that is, the parabola is concave-up. So we can get the domain the way below.

$$b^2 - 4a(c-x) \geq 0 \Rightarrow 4a(c-x) \leq b^2 \Rightarrow c - x \leq \frac{b^2}{4a} \Rightarrow x \geq c - \frac{b^2}{4a} = \frac{-(b^2-4ac)}{4ac}.$$

Thus, the domain is $x \geq \dfrac{-(b^2-4ac)}{4ac}$.

So more specifically, we can put the two functions A and B the way as follows.

$$y = A(x) = \frac{-b+\sqrt{b^2-4a(c-x)}}{2a} \text{ for } x \geq \frac{-(b^2-4ac)}{4ac}.$$

$$y = B(x) = \frac{-b-\sqrt{b^2-4a(c-x)}}{2a} \text{ for } x \geq \frac{-(b^2-4ac)}{4ac}.$$

Thus, d in Fig. 2 above is $\dfrac{-(b^2-4ac)}{4ac}$.

And of course, we can get the domain using $(s, t) = (y, x)$, too. We should in fact, use it finding or checking the new domain.

We know (s, t) is the new arbitrary point, and (x, y) is the original arbitrary point.

And we know, y is the output variable in the original function, and thus, gets its value that belongs to the range of the original function. Also, s is the input variable in the new function, and thus, gets its value in the domain of the new function. So the domain of the new function is the same as the range of the original function. That is, the new domain is the same as the original range.

And in fact, the range of the original function is $y \geq \dfrac{-(b^2-4ac)}{4ac}$. How?

Putting in the vertex form the original equation $y = ax^2 + bx + c$, where $a > 0$, we get

$$y = ax^2 + bx + c = a(x^2 + \frac{b}{a}x) + c = a(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}) + c = a(x + \frac{b}{2a})^2 - \frac{b^2}{4a} + c$$

$$= a(x + \frac{b}{2a})^2 - \frac{(b^2-4ac)}{4a} \Rightarrow y = a(x + \frac{b}{2a})^2 - \frac{(b^2-4ac)}{4a}.$$

So since we have $y = C(x) = ax^2 + bx + c$ for $x \geq -\dfrac{b}{2a}$, and $a > 0$,

we get $y \geq \dfrac{-(b^2-4ac)}{4ac}$.

And the same is true for the other function D, too.

We have $y = D(x) = ax^2 + bx + c$ for $x < -\dfrac{b}{2a}$. And we can put it the way below, too.

$$y = D(x) = a(x + \frac{b}{2a})^2 - \frac{(b^2-4ac)}{4a} \text{ for } x < -\frac{b}{2a}. \text{ And } a > 0.$$

So we get $y \geq \dfrac{-(b^2-4ac)}{4ac}$, also.

And we know the new domain, that is, the domain of the new function is the same as the range of the original function. So the new domain is $x \geq \dfrac{-(b^2-4ac)}{4ac}$.

What then about the new range, the range in the new function?

We have $(s, t) = (y, x)$. And we know, x is the input variable in the original function, and thus, gets its value in the domain of the original function. Also, t is the output variable in the new function, and thus, gets its value that belongs to the range of the new function. So the range in the new function is the same as the domain in the original function. That is, the new range is the same as the original domain.

So first, since the original function C is $y = C(x) = ax^2 + bx + c$ for $x \geq -\dfrac{b}{2a}$, the domain of the original C is $x \geq -\dfrac{b}{2a}$. Thus, the range of the new function A is $y \geq -\dfrac{b}{2a}$.

Next, since the original function D is $y = D(x) = ax^2 + bx + c$ for $x < -\dfrac{b}{2a}$, the domain of the original D is $x < -\dfrac{b}{2a}$. So the range of the new function B is $y < -\dfrac{b}{2a}$.

What if we just apply R to a function $y = h(x) = ax^2 + bx + c$ for x real? That is, what is the new curve if the original curve is the entire parabola $y = ax^2 + bx + c$?

Let's now actually take the transformation of h by R, and see what the new curve is.
We have $(s, t) = (y, x)$.
That is, we get $x = t$, and $y = s$. Thus, we get $y = h(x) \Rightarrow s = h(t)$. So we get

$s = h(t) = at^2 + bt + c \Rightarrow s = at^2 + bt + c$, which is the equation connecting s and t.

And we know s is another name for the variable x in the new equation, and t is another name for the variable y.
So in $s = at^2 + bt + c$, replacing s with x, and t with y, we get $x = ay^2 + by + c$, which is the new curve.

Thus, setting $Q(x, y) = ay^2 + by + c - x = 0$, we get $Q = R \bullet h$, where

R: $(x, y) \longrightarrow (y, x)$, and $y = h(x) = ax^2 + bx + c$.

Also, solving for y the new equation $x = ay^2 + by + c$, we can express y in terms of x. That is, assuming the new equation is $y = g(x)$, we can get $g(x)$ solving for y the new equation above. And we can readily get the solution using the quadratic formula.

So using the formula, we get $y = \dfrac{-b \pm \sqrt{b^2 - 4a(c-x)}}{2a}$, which is therefore, the expression $g(x)$, and is equivalent to the new equation $x = ay^2 + by + c$.

And of course, we can solve for t the connective expression $s = h(t)$ before doing the replacements stated above. So doing it that way, we begin with $s = h(t) = at^2 + bt + c$.

So next, we solve for t the equation $s = at^2 + bt + c$, that is, $at^2 + bt + c - s = 0$.

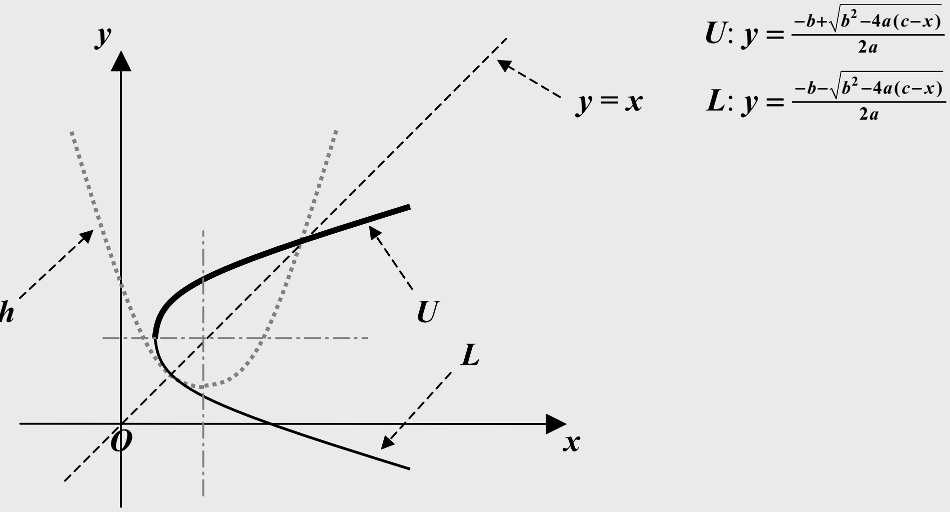
Then again, using the quadratic formula, we get $t = \dfrac{-b \pm \sqrt{b^2 - 4a(c-s)}}{2a}$, which is the very connective equation we want. So next, replacing s with x, and t with y, we get $y = \dfrac{-b \pm \sqrt{b^2 - 4a(c-s)}}{2a}$, which is the new equation. What does its curve look like though?

We know $y = \dfrac{-b + \sqrt{b^2 - 4a(c-x)}}{2a}$ indicates an upper half parabola,

and $y = \dfrac{-b - \sqrt{b^2 - 4a(c-x)}}{2a}$ indicates a lower half parabola.

So $y = \dfrac{-b \pm \sqrt{b^2 - 4a(c-x)}}{2a}$ indicates an entire parabola.

Fig. 3



What then about the extents of the variables?

If not specified, the extent of a variable is the set of all numbers the variable can take. So beginning with the extent of the variable $\boldsymbol{x}$, that is, the domain, we can use the rule as follows. What's inside the square root sign is $\geq \mathbf{0}$.

And using the rule, we have already found that the domain is $\boldsymbol{x} \geq \frac{-(b^2-4ac)}{4ac}$.

And next, moving on to the extent of $\boldsymbol{y}$, that is, the range, we have two cases as below.

$$y = \frac{-b+\sqrt{b^2-4a(c-x)}}{2a} \text{ for } x \geq -\frac{(b^2-4ac)}{4ac}.$$
$$y = \frac{-b-\sqrt{b^2-4a(c-x)}}{2a} \text{ for } x \geq -\frac{(b^2-4ac)}{4ac}.$$

So to begin with, the $\boldsymbol{y}$-values in both cases are the same when $x = -\dfrac{(b^2-4ac)}{4ac}$.

And next, the $\boldsymbol{y}$-value in $y = \dfrac{-b+\sqrt{b^2-4a(c-x)}}{2a}$ keeps increasing, and increases infinitely as the $\boldsymbol{x}$-value increases.

However, the $\boldsymbol{y}$-value in $y = \dfrac{-b-\sqrt{b^2-4a(c-x)}}{2a}$ keeps decreasing, and decreases infinitely as the $\boldsymbol{x}$-value increases.

So we can say that the extent of $\boldsymbol{y}$ is the entire $\boldsymbol{y}$-axis, that is, a set of all real numbers.

Thus, it can be said that applying to the function $\boldsymbol{y = h(x) = ax^2 + bx + c}$ for $\boldsymbol{x}$ real, the transformation symmetric about the line $\boldsymbol{y = x}$, we get two equations.

One is $y = \dfrac{-b+\sqrt{b^2-4a(c-x)}}{2a}$ for $x \geq -\dfrac{(b^2-4ac)}{4ac}$,

and the other is $y = \dfrac{-b-\sqrt{b^2-4a(c-x)}}{2a}$ for $x \geq -\dfrac{(b^2-4ac)}{4ac}$.

Normally though, we don't specify the extent of $\boldsymbol{x}$ in the case above.

So in sum, we say that applying to the function $\boldsymbol{y = h(x) = ax^2 + bx + c}$, we get

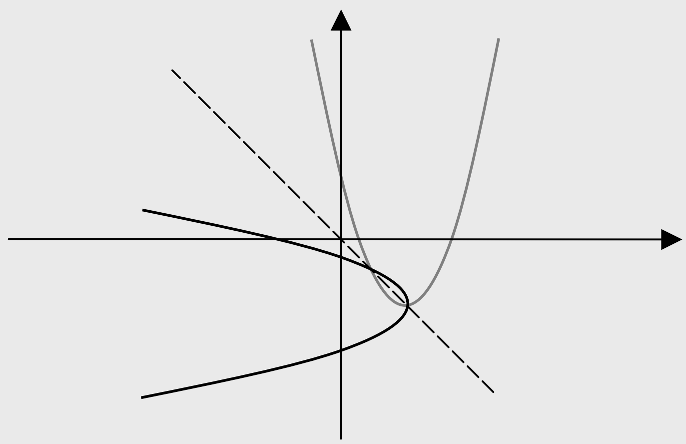
$$\boldsymbol{y = \frac{-b \pm \sqrt{b^2 - 4a(c-x)}}{2a}}.$$

And assuming the new equation is $\boldsymbol{G(x, y) = 0}$,

we get $\boldsymbol{G(x, y) = y + \frac{b \pm \sqrt{b^2 - 4a(c-x)}}{2a} = 0}$.

2.6. Symmetric about a line 5

Fig. 0



The previous section covers the transformation symmetric about the line $y = x$. And if R is the transformation, we can put it the way below.

R: $(x, y) \longrightarrow (s, t) = (y, x)$.

And briefly, we can put it this way, too: R: $(x, y) \longrightarrow (y, x)$.

What then, about the transformation symmetric about the line $y = -x$?

Assuming N is the transformation symmetric about the line $y = -x$, we can put N the way as follows. N: $(x, y) \longrightarrow (s, t) = (-y, -x)$.

And briefly, we can put it this way, too: N: $(x, y) \longrightarrow (-y, -x)$.

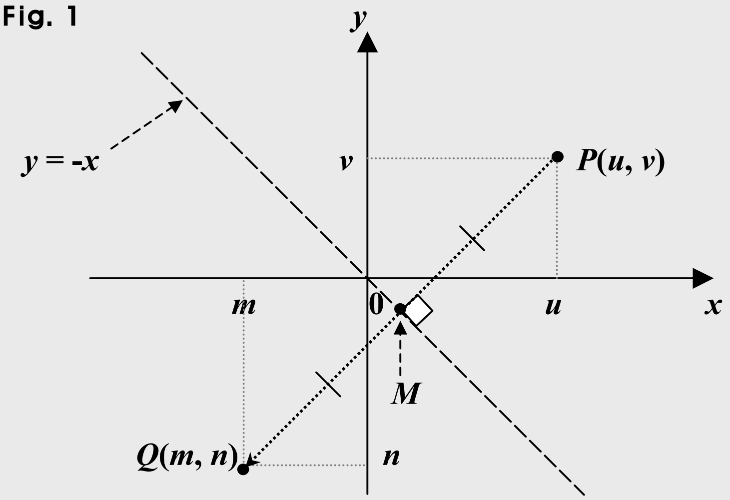
And we can call the expression above the definition of the transformation symmetric about the line $y = -x$. How then, can we get the definition?

A transformation definition shows specifically **how** every point gets changed in the original curve. So it says **the way** the original curve changes, that is, **the way** the new curve gets made. We want to get therefore, **the way** the original arbitrary point changes to the new arbitrary point.

To begin with, if two points are symmetric about a line $\boldsymbol{y = -x}$, the two are equal distance away from the line, and both points are in a line perpendicular to the line.

So for instance, if two points $\boldsymbol{(u, v)}$ and $\boldsymbol{(m, n)}$ are symmetric about the line $\boldsymbol{y = -x}$, we can put the two points in a graph the way below.

Fig. 1



Suppose now, given the point $\boldsymbol{(u, v)}$, we want to find the point $\boldsymbol{(m, n)}$.
That is, we want to find the point symmetric to the point $\boldsymbol{P}$ about the line $\boldsymbol{y = -x}$.

In other words, we want to put the point $\boldsymbol{Q(m, n)}$ in terms of the coordinates of $\boldsymbol{P(u, v)}$.

That is to say that using the fact that $\boldsymbol{(m, n)}$ and $\boldsymbol{(u, v)}$ are symmetric about the line $\boldsymbol{y = -x}$, we want to put each coordinate of $\boldsymbol{(m, n)}$ in terms of the coordinates of $\boldsymbol{(u, v)}$.

So to begin with, in the graph above, since the two points $\boldsymbol{P}$ and $\boldsymbol{Q}$ are symmetric about the axis of symmetry $\boldsymbol{y = -x}$, the line $\boldsymbol{y = -x}$ passes through the midpoint between the two points, and the midpoint is of course, in the line segment $\boldsymbol{PQ}$. So assuming the midpoint is $\boldsymbol{M(a, b)}$, we can put each of its coordinates this way: $\boldsymbol{a = (u + m)/2}$ and $\boldsymbol{b = (v + n)/2}$.

And next, since the midpoint $\boldsymbol{M(a, b)}$ is in the axis of symmetry $\boldsymbol{y = -x}$, we get $\boldsymbol{b = -a}$.

That is, we get $(v + n)/2 = -(u + m)/2$. Thus, we get

$(v + n)/2 = -(u + m)/2 \Rightarrow v + n = -(u + m) \Rightarrow u + v + m + n = 0$.

Also, since the line $y = -x$ is perpendicular to the line segment PQ, the product of the slope of the line and the slope of the line segment is -1. And we know the slope of the line $y = -x$ is -1. So the slope of the line segment is 1. And we know P is at (u, v), and Q is at (m, n). And thus, we get $(n - v)/(m - u) = 1 \Rightarrow n - v = m - u \Rightarrow n + u = m + v$.

And we have $u + v + m + n = 0$, and can put it this way, too: $n + u + m + v = 0$.

So we get $n + u = m + v \Rightarrow n + u + m + v = m + v + m + v = 2(m + v) = 0$.

Thus, we get $m = -v$.

Then again, going back to $n + u + m + v = 0$, we get $n + u - v + v = n + u = 0$.

So we get $n = -u$.

And thus, putting each of the coordinates of (m, n) in terms of the coordinates of (u, v), we get $(m, n) = (-v, -u)$. What then, is the correlation between the original arbitrary point and the new arbitrary point in this symmetric transformation?

The arbitrary point in a curve represents all the points in the curve. So assuming (u, v) is in the original curve, (x, y) is the original arbitrary point, (s, t) is the new arbitrary point, and (m, n) is in the new curve, we get this correlation: $(s, t) = (-y, -x)$, because (x, y) behaves the way (u, v) does, and vice versa, and the same is true for (s, t) and (m, n), too.

And thus, assuming N is the transformation symmetric about the line $y = x$, we can put N the way as follows. N: $(x, y) \longrightarrow (s, t) = (-y, -x)$.

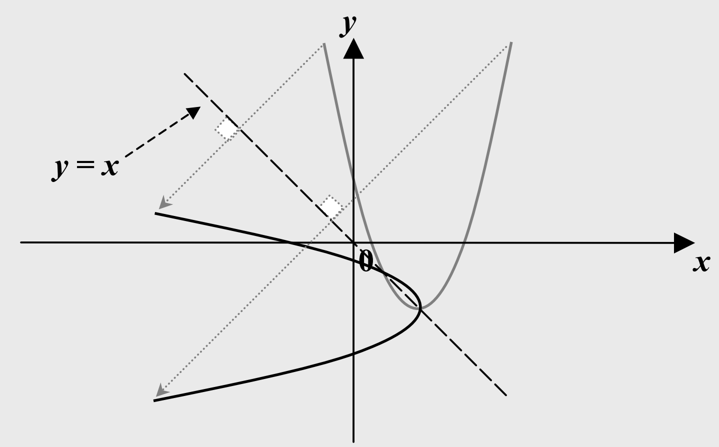
And briefly, we can just put it this way, too: N: $(x, y) \longrightarrow (-y, -x)$.

So we can call the expression above the definition of the transformation <u>symmetric about the line $y = -x$</u>. And the core of the transformation definition is $(-y, -x)$, which is the new arbitrary point, so we set $(s, t) = (-y, -x)$. It shows how each and every point in the original curve gets changed, that is, the way every point in the new curve gets made.

So applying such a transformation to a function, do we get another function? That is, if the original curve is a curve of a function, is the new curve a curve of a function, too?

It is not always the case. In the figure below, the curve in gray is the original curve. Applying the transformation ***N*** to the original curve, we get a new curve, which is the curve in black, which cannot be however, a curve of a function.

Fig. 2



That's because the new curve can meet a vertical line at more than one point, and thus, cannot be a curve of a function. In the graph above, when $x = 0$, y gets two values. A function has to have one output only for one input. So the new curve in the graph above, that is, the curve in black is not a curve of a function but just a curve of an equation.

How then, do we get the new curve, that is, the equation of the new curve?

We can get it getting the equation connecting s and t. And we can get the connective equation mixing (s, t) into the original curve. And we can get them mixed using $(-y, -x)$.

And using (y, x), since we have $(s, t) = (-y, -x)$. we can get $x = -t$, and $y = -s$.

And thus, we can now get the mixture, that is, the connective equation, putting $x = -t$ and $y = -s$ into the original equation. That is, in the original equation $y = f(x)$, we replace x with $-t$, and y with $-s$. So doing the substitutions, we get $y = f(x) \Rightarrow -s = f(-t)$, which is the equation connecting s and t, and thus, is the new equation indicating the new curve.

And we know that s is another name for the variable x, and t is another name for the variable y. So next, replacing s with x, and t with y, we get $-s = f(-t) \Rightarrow -x = f(-y)$, which is the new curve, that is, the new equation.

So in sum, transforming an equation $y = f(x)$ symmetrically about a line $y = -x$, we get a new equation, and the new equation is $x = -f(-y)$.

And we can put it the way below, too.

Transforming an equation $h(x, y) = 0$ symmetrically about a line $y = -x$, and assuming k is the new equation, we get $k(x, y) = h(-y, -x) = 0$. Why?

We have $(s, t) = (-y, -x)$. So we can get $s = -y$, and $t = -x$, that is, $y = -s$, and $x = -t$,

Next, mixing (s, t) into the original equation $h(x, y) = 0$, we get the connective equation. And getting the connective equation, putting $x = -t$ and $y = -s$ into the original equation. That is, in $h(x, y) = 0$, we replace x with $-t$, and y with $-s$.

So doing the substitutions, we get $h(x, y) = 0 \Rightarrow h(-t, -s) = 0$, which is the connective equation between s and t, and thus, is the new equation.

And we know that the new curve is in the x-y plane, and that s is another name for the variable x, and t is another name for the variable y.

So replacing s with x, and t with y, we get the new equation in the x-y system.
That is, we get $h(-t, -s) = 0 \Rightarrow h(-y, -x) = 0$, which is the new equation.

And thus, assuming k is the new equation, we get $k(x, y) = h(-y, -x) = 0$.

Suppose for a concrete case, the original function is $y = f(x) = 2x^2 – 4x + 1$, and it gets transformed symmetrically about the line $y = -x$.

Then, we can get the new equation, the way as follows.

First, we have $(x, y) \longrightarrow (s, t) = (-y, -x)$. So we get $y = -s$, and $x = -t$.

Thus, we get $y = f(x) \Rightarrow -s = f(-t) \Rightarrow -s = 2t^2 + 4t + 1 \Rightarrow s = -2t^2 – 4t – 1$.

So we now have the equation connecting s and t. Replacing thus, s with x, and t with y, we get $x = -2y^2 – 4y – 1$, which is the new equation, which is a horizontal parabola.
And putting it in the vertex form, we get $x = -2(y + 1)^2 + 1$.

And we can put the example above the way below, too.

We can have $\boldsymbol{y = f(x) = 2x^2 - 4x + 1 \Rightarrow y = 2x^2 - 4x + 1 \Rightarrow 2x^2 - 4x + 1 - y = 0}$.
And we can put it this way: $\boldsymbol{h(x, y) = 2x^2 - 4x + 1 - y = 0}$.

So suppose this time, the original equation is $\boldsymbol{h(x, y) = 2x^2 - 4x + 1 - y = 0}$, and it gets transformed symmetrically about the line $\boldsymbol{y = -x}$. Then, we can get the new equation, the way as follows.

Since we apply the same transformation as the one above, we get $\boldsymbol{x = -t}$, and $\boldsymbol{y = -s}$.

So next, $\boldsymbol{h(x, y) = 0 \Rightarrow h(-t, -s) = 0 \Rightarrow 2t^2 + 4t + 1 + s = 0}$, which is the equation connecting $\boldsymbol{s}$ and $\boldsymbol{t}$. Replacing thus, $\boldsymbol{s}$ with $\boldsymbol{x}$, and $\boldsymbol{t}$ with $\boldsymbol{y}$, we get $\boldsymbol{2y^2 + 4y + 1 + x = 0}$, which is the new equation.

And putting it in the vertex form, we get $\boldsymbol{x = -2(y + 1)^2 + 1}$.

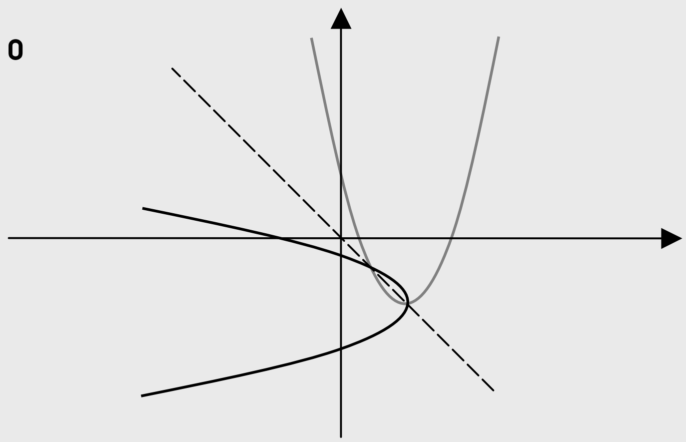
And also, assuming $\boldsymbol{k(x, y) = 0}$ is the new equation above, we can set
$\boldsymbol{k(x, y) = -2(y + 1)^2 + 1 - x = 0}$.

What then, about the new function?

For the example taken above, we don't get a new function. That's because the original is not one-to-one. And we will continue this discussion, together with some important facts about quadratic equations in the next section.

2.7. Symmetric about a line 6

Fig. 0



The previous section covers the transformation symmetric about the line $y = -x$. And if N is the transformation, we can put it the way below.

N: $(x, y) \longrightarrow (s, t) = (-y, -x)$.

And briefly, we can put it this way, too: N: $(x, y) \longrightarrow (-y, -x)$.

What then is the transformation symmetric about the line $y = -x$?

The definition says that in the original equation, x takes the position of y, and changes its sign, and y takes the position of x, and changes its sign, too.

For instance, (3, 7) changes to (-7, -3). That is, at each and every point in the curve of the original equation, both coordinates change their signs, as well as swap their positions.

So the y in the new equation is $-x$ in the original equation, and the x in the new is $-y$ in the original. So is the new equation the negative of the inverse?

No, it isn't. It is however, the inverse of the original only if the original is an equation of a line passing through the origin, except for the lines vertical or horizontal.

Assuming for instance, $\boldsymbol{y = 3x}$ is the original, and applying $\boldsymbol{N}$ to it, we get $\boldsymbol{-x = -3y \Rightarrow x = 3y}$, which is the new equation, and solving it for $\boldsymbol{y}$, we get $\boldsymbol{y = \frac{x}{3}}$, which is the new equation, too, and is called the inverse, too.

If however, the original is not a line as $\boldsymbol{y = ax}$ where $\boldsymbol{a \neq 0}$, the new is not the inverse.

Assuming thus, $\boldsymbol{y = ax + b}$ is the original, and applying $\boldsymbol{N}$ to it, we get
$\boldsymbol{-x = -ay + b \Rightarrow x = ay - b}$, which is the new equation, but is not the inverse.

It's because the $\boldsymbol{x}$ in the new is not the same as the $\boldsymbol{y}$ in the original.
That is, even though swapping $\boldsymbol{x}$ and $\boldsymbol{y}$ in the original, we don't get the new.

That's because $\boldsymbol{x}$ and $\boldsymbol{y}$ change not only roles but signs, too.

So for instance, assuming (3, 7) is a point in the original curve, and putting -7 into $\boldsymbol{x}$ in the new equation, we get -3, that is, we get (-7, -3) in the new curve.

And thus, in general, taking the transformation of a function by $\boldsymbol{N}$, we don't get the inverse even if the function is one-to-one.

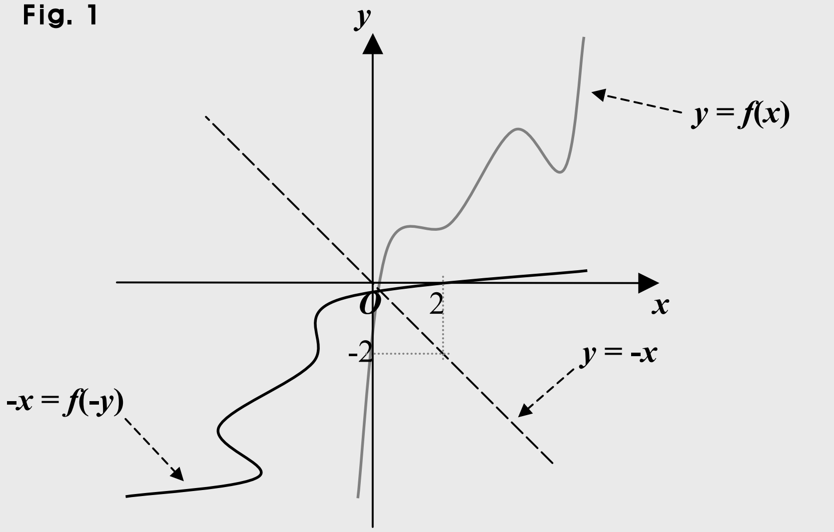
And as explained in the section 2.5, if we take the inverse of a function, the function is one-to-one, and we can take the inverse taking the transformation of the function by $\boldsymbol{R}$, where $\boldsymbol{R: (x, y) \longrightarrow (y, x)}$, which is symmetric about the line $\boldsymbol{y = x}$.

What do we mean by then the transformation symmetric about the line $\boldsymbol{y = -x}$?

Transforming symmetrically about the line $\boldsymbol{y = -x}$, we get a new equation. And looking at both curves, the new and the original, we can see that as $\boldsymbol{x}$ decreases in the new, the new curve behaves exactly the way the original behaves as $\boldsymbol{x}$ increases in the original.

And let's now take a close look at how the new equation behaves.
Assuming first, the original is $\boldsymbol{y = f(x) = 0.2x^5 - 2.25x^4 + 9x^3 - 15.5x^2 + 12x - 2}$, we can put in a graph the original and the new the way below.

Fig. 1



And in the transformation, the way the new gets made is $\boldsymbol{(x, y) \longrightarrow (s, t) = (-y, -x)}$.

So we simply get $\boldsymbol{s = -y}$ and $\boldsymbol{t = -x}$, that is, we get $\boldsymbol{y = -s}$ and $\boldsymbol{x = -t}$.

So getting the new, we get $\boldsymbol{y = f(x) \Rightarrow -s = f(-t)}$, which is the new.

And since $\boldsymbol{s}$ is another name for the variable $\boldsymbol{x}$, and $\boldsymbol{t}$ is another name for the variable $\boldsymbol{y}$, replacing $\boldsymbol{s}$ with $\boldsymbol{x}$, and $\boldsymbol{t}$ with $\boldsymbol{y}$, we get the new in the $\boldsymbol{x}$-$\boldsymbol{y}$ system.

Then, we get $\boldsymbol{-s = f(-t) \Rightarrow -x = f(-y) \Rightarrow x = -f(-y)}$.

And we know $\boldsymbol{f(x) = 0.2x^5 - 2.25x^4 + 9x^3 - 15.5x^2 + 12x - 2}$. So we get

$$\boldsymbol{x = -f(-y) = -\{0.2(-y)^5 - 2.25(-y)^4 + 9(-x)^3 - 15.5(-y)^2 + 12(-y) - 2\}}$$

$$\boldsymbol{= -(-0.2y^5 - 2.25y^4 - 9x^3 - 15.5y^2 - 12x - 2) = 0.2y^5 + 2.25y^4 + 9x^3 + 15.5y^2 + 12x + 2}.$$

So the new is $\boldsymbol{x = 0.2y^5 + 2.25y^4 + 9x^3 + 15.5y^2 + 12x + 2}$, which is $\boldsymbol{-x = f(-y)}$.

And putting together all the processes above, we get the new the way below.

To begin with, the original is $f(x) = 0.2x^5 - 2.25x^4 + 9x^3 - 15.5x^2 + 12x - 2$.

And in the transformation, the way the new gets made is $(s, t) = (-y, -x)$.
So we can simply get $y = -s$ and $x = -t$. So getting the new now, we get

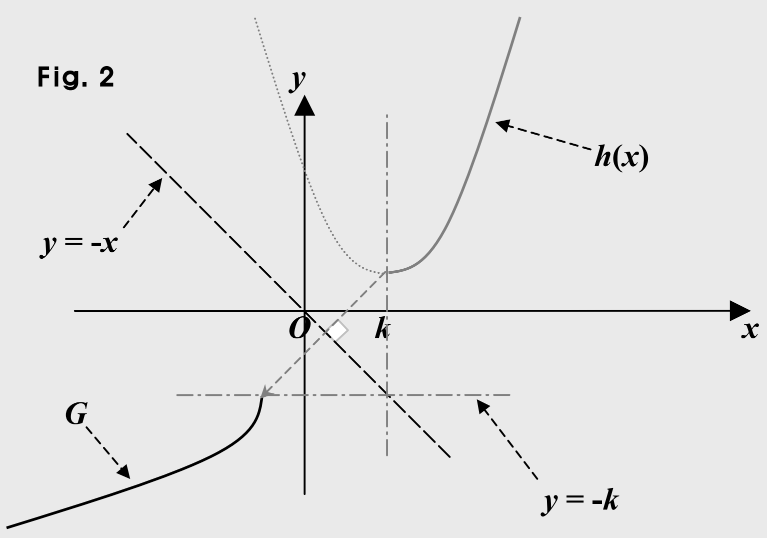
$y = f(x) \Rightarrow -s = f(-t) \Rightarrow -x = f(-y) \Rightarrow x = -f(-y)$
$= -(-0.2y^5 - 2.25y^4 - 9x^3 - 15.5y^2 - 12x - 2)$
$= 0.2y^5 + 2.25y^4 + 9x^3 + 15.5y^2 + 12x + 2$.

So the new is $x = 0.2y^5 + 2.25y^4 + 9x^3 + 15.5y^2 + 12x + 2$, which is $-x = f(-y)$.

Let's next, apply the transformation N to a function $y = h(x) = ax^2 + bx + c$ for $x \geq k$ where $k = -\dfrac{b}{2a}$, and $a > 0$, and see what we get. The transformation N is of course, symmetric about the line $y = -x$.

The original curve, the curve of h is a right half parabola. And taking the transformation of h by N, we get a new curve, symmetric to h about the line $y = -x$. So assuming the new is G, and putting both curves in one graph, we can put the two the way below.

Fig. 2



So let's now get the equation of the new curve G.

Looking at first, the new curve G, we can see that G can be a curve of a function.
Applying thus, N to the function h, we can get a new function, the curve of which is G.
To begin with, the original is $y = h(x) = ax^2 + bx + c$ for $x \ge -\frac{b}{2a}$, where $a > 0$.

And in the transformation N, the way the new gets made is $(s, t) = (-y, -x)$.
So we can simply get $y = -s$ and $x = -t$. So getting the new now, we get

$$y = h(x) \Rightarrow -s = h(-t) \Rightarrow -x = h(-y) \Rightarrow x = -h(-y) = -(ay^2 - by + c) = -ay^2 + by - c.$$

So the new is $x = -ay^2 + by - c$, which is $-x = f(-y)$, and is the equation of G.

> And in a transformation symmetric about a line $y = -x$, the extents of the variables in the original equation can change, too. So specifying the new equation, we need to specify the new extents, that is, the new domain and range, at least the new domain if the original equation is specified with its domain.

So next, moving on to the extents of the variables in the new curve, and beginning with the extent of s, we can use $s = -y$, and $y = h(x) = ax^2 + bx + c$ for $x \ge -\frac{b}{2a}$, where $a > 0$.

Next, we can put h this way: $y = h(x) = a(x + \frac{b}{2a})^2 - \frac{b^2 - 4ac}{4a}$ for $x \ge -\frac{b}{2a}$, where $a > 0$.

So we get $y \ge -\frac{b^2 - 4ac}{4a}$. Thus, we get $-y \le \frac{b^2 - 4ac}{4a}$. So we get $s \le \frac{b^2 - 4ac}{4a}$.
In other words, we get $s \le d$, where $d = \frac{b^2 - 4ac}{4a}$.

And next, moving on to the extent of t, we can use $x = -t$, where $x \ge -\frac{b}{2a}$.

That is, we get $x = -t \Rightarrow t = -x \le \frac{b}{2a} \Rightarrow t \le \frac{b}{2a}$. In other words, $t \le -k$, where $k = -\frac{b}{2a}$.

And we know that s is x in the new equation, and t is y in the new.
So in the new equation, we have $x \le d$, where $d = \frac{b^2 - 4ac}{4a}$, and $y \le -k$, where $k = -\frac{b}{2a}$.

Thus, the new equation is $x = -ay^2 + by - c$ for $x \le d$, and $y \le -k$, which indicates the new curve G. And we can put it the way below, too.

Assuming the new equation is $G(x, y) = 0$, we can put it the way below.

$G(x, y) = x + ay^2 - by + c = 0$ for $x \le d$, and $y \le -k$, where $k = -\dfrac{b}{2a}$, and $d = \dfrac{b^2 - 4ac}{4a}$.

And we can put it the way below, too.

$G = N \bullet h$, where $y = h(x) = ax^2 + bx + c$ for $x \ge k$, and N: $(x, y) \longrightarrow (-y, -x)$.
And next, solving for y the new equation $x = -ay^2 + by - c$ for $x \le d$, and $y \le -k$, we can express y in terms of x.
That is, assuming the new function is $y = g(x)$, we can get $g(x)$ solving for y the new equation above. And we can readily get the solution using the quadratic formula.

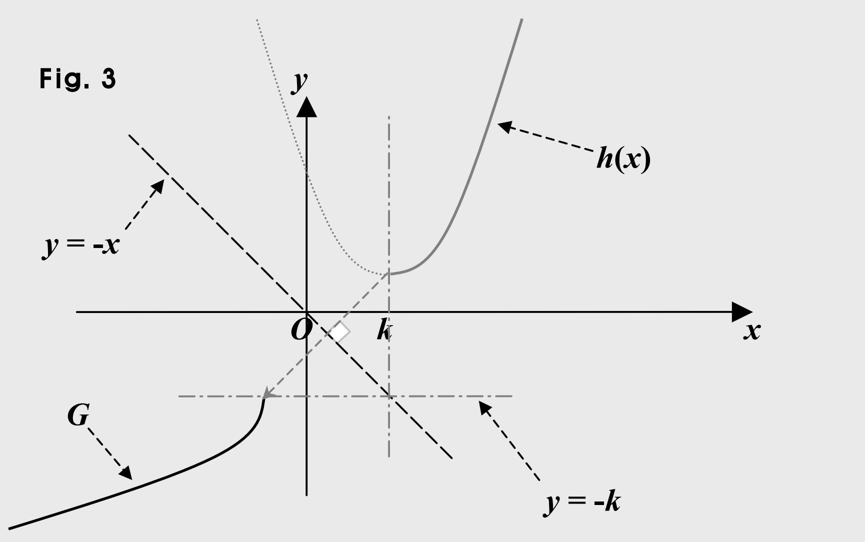
First, we can put the new equation this way, too: $ay^2 - by + c + x = 0$.

And the quadratic formula for $ax^2 + bx + c = 0$ is $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

So using the formula for $ay^2 - by + c + x = 0$, we get $y = \dfrac{b \pm \sqrt{b^2 - 4a(c + x)}}{2a}$, which is however, a set of two equations. Which one then do we want to pick?

Going back to the graph where the curve of h and the new curve G is put, we have this:

Fig. 3



Looking at the new curve G in the graph above, we can see y-value increases as x-value increases. So we want to pick $y = \frac{b+\sqrt{b^2-4a(c+x)}}{2a}$, which is therefore, the expression $g(x)$, and is equivalent to the new equation $x = -ay^2 + by - c$.

Thus, we can put the new function g the way below.

$y = g(x) = \frac{b+\sqrt{b^2-4a(c+x)}}{2a}$ for $x \le \frac{b^2-4ac}{4a}$. Also, we can put g the way below, too.

$g = N \bullet h$ where $y = h(x) = ax^2 + bx + c$ for $x \ge -\frac{b}{2a}$, and N: $(x, y) \longrightarrow (-y, -x)$.

And let's now take a look at this transformation N at a different angle.

First, in the transformation symmetric about the line $y = x$, we have $(s, t) = (y, x)$.

Next, in the transformation symmetric about a point (a, b), it is $(s, t) = (2a - x, 2b - y)$.
So in the transformation symmetric about the origin $(0, 0)$, we get $(s, t) = (-x, -y)$.

What transformation then, can we get combining the two symmetric transformations above? More specifically, applying the transformation symmetric about the line $y = x$ first, and then, the transformation symmetric about the origin, what transformation do we apply after all?

In the transformation symmetric about the line $y = x$, the coordinates of the original point get exchanged. That is, the new point is (y, x). In other words, the x-coordinate in the new point is y, which is the y-coordinate in the original point, and the y-coordinate in the new point is x, which is the x-coordinate in the original point.

And in the transformation symmetric about the origin, we have $(s, t) = (-x, -y)$. So in the transformation symmetric about the origin, only the signs of the coordinates of the original point get exchanged. What then do we get applying the transformation symmetric about the origin to the new point (y, x)?

The x-coordinate in the original point is y now, so the x-coordinate in the new point is $-y$. And the y-coordinate in the original point is x now, so the y-coordinate in the new point is $-x$. Applying therefore, to the point (y, x) the transformation symmetric about the origin, we get $(-y, -x)$. What then do we get?

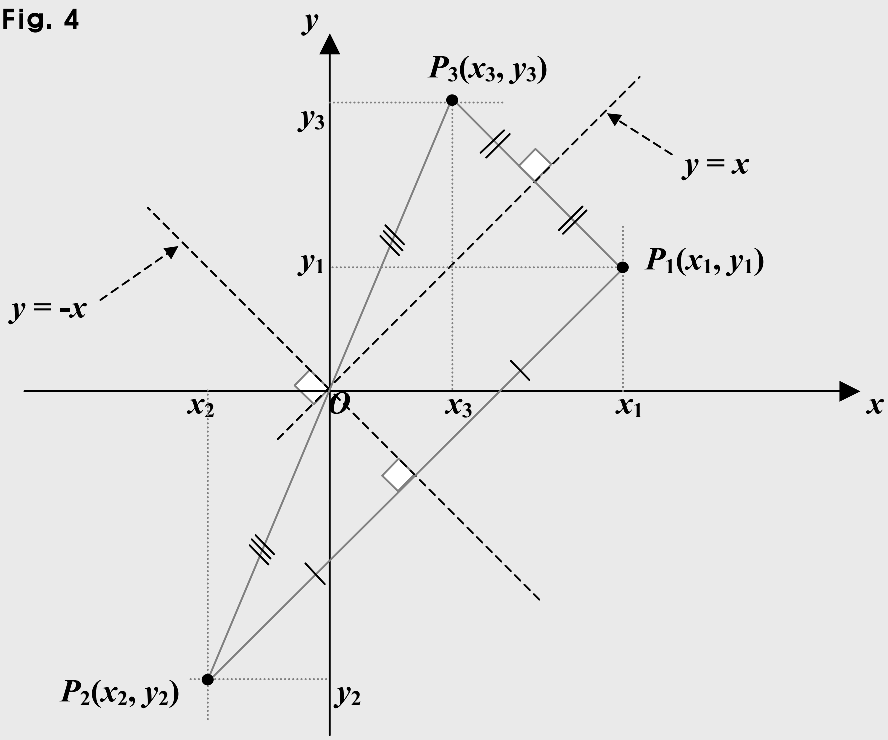
It is of course, the transformation symmetric about the line $y = -x$. And we can get the same, too, applying first, the second transformation, then the first one.

So applying in sequence the two symmetric transformations, we apply N.

And in fact, looking at the transformation N the way above, we call N a composite transformation, where the two symmetric transformations above are combined.
Let's for instance, put in a graph three points P_1, P_2, and P_3 the way below, and check to see if the two points P_1 and P_2 are symmetric about the line $y = -x$.

The two points P_1 and P_3 are symmetric about the line $y = x$.
And the two points P_3 and P_2 are symmetric about the origin $(0, 0)$.

Fig. 4



Then, to begin with, since the two points $P_1(x_1, y_1)$ and $P_3(x_3, y_3)$ are symmetric about the line $y = x$, we get $x_3 = y_1$, and $y_3 = x_1$. That is, $(x_3, y_3) = (y_1, x_1)$.

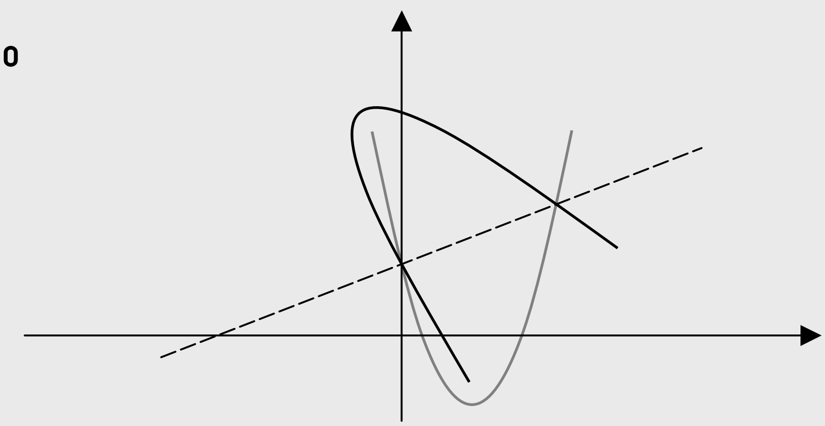
And next, since the two points $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$ are symmetric about the origin, we get $x_2 = -x_3$ and $y_2 = -x_3$. That is, $(x_2, y_2) = (-x_3, -y_3)$.

So next, since we have $x_3 = y_1$, and $y_3 = x_1$, we get $(-x_3, -y_3) = (-y_1, -x_1)$.

Thus, we get $(x_2, y_2) = (-x_3, -y_3) \Rightarrow (x_2, y_2) = (-y_1, -x_1)$, which means therefore, P_1 and P_2 are symmetric about the line $y = -x$.

2.8. Symmetric about a line 7

Fig. 0



• First, note that in this book, some important terminologies are used the way as follows.

The original curve: the curve of the function or equation a transformation is applied to
The new curve: the curve of the function or equation made by a transformation

The original function or equation: the function or equation a transformation is applied to
The new function or equation: the function or equation made by a transformation

The original arbitrary point or the original point: the arbitrary point in the original curve.
The new arbitrary point or the new point: the arbitrary point in the new curve.

Applying a transformation <u>symmetric about a line</u>, we get a new curve, which is symmetric to the original curve about <u>the line</u>, which is called therefore, <u>the axis of symmetry</u>. How then is the new curve made?

In a transformation symmetric about a line, every point in the original curve gets moved symmetrically about the line, and then, all the points moved form the new curve, which is thus, said to be symmetric to the original curve about the line.

What do we mean by then transforming a function symmetrically about a line?

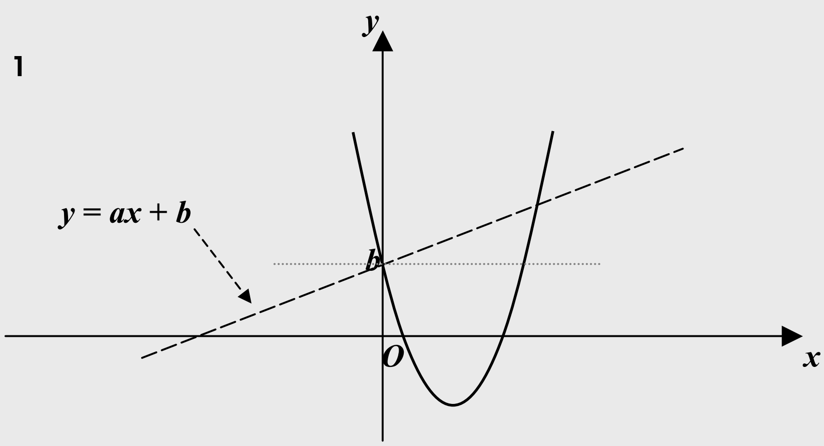
Transforming a function symmetrically about a line, we get a new curve, which is symmetric about the line to the curve of the function. So do we get a new function?

Maybe. It depends on the original function and the line taken as the axis of symmetry. If for instance, the line is horizontal or vertical, that is, parallel or perpendicular to a coordinate axis as the $\boldsymbol{x}$-axis, we get a new function.

And what's covered in this section is a case <u>the axis of symmetry is a line $\boldsymbol{y = ax + b}$</u>. So covered here is a transformation <u>symmetric about a line $\boldsymbol{y = ax + b}$</u>.

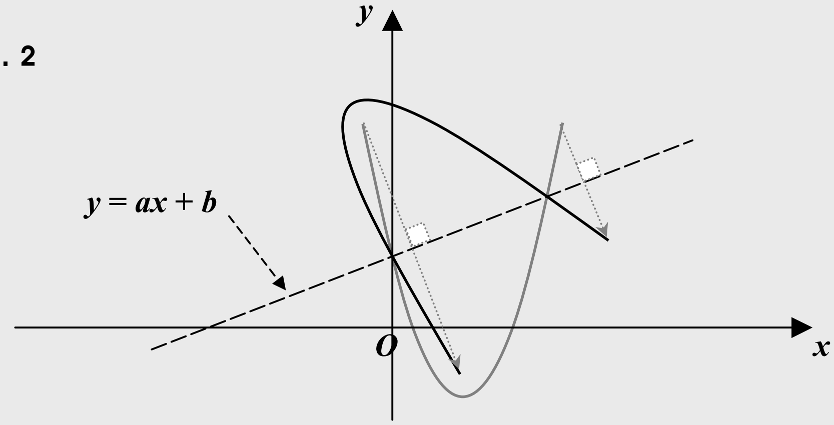
So the two curves, the original and the new in such a transformation can be said to be symmetric about a line $\boldsymbol{y = ax + b}$. Suppose now that the original curve is the curve of a function $\boldsymbol{y = f(x)}$, and is put in a graph the way below.

Fig. 1



Then, applying to the curve of the function $\boldsymbol{f}$ the symmetric transformation stated above, we get a new curve, which is of course, symmetric to the curve of the function $\boldsymbol{f}$ about the line $\boldsymbol{y = ax + b}$. Thus, the new curve is the curve in black in the graph below.

Fig. 2



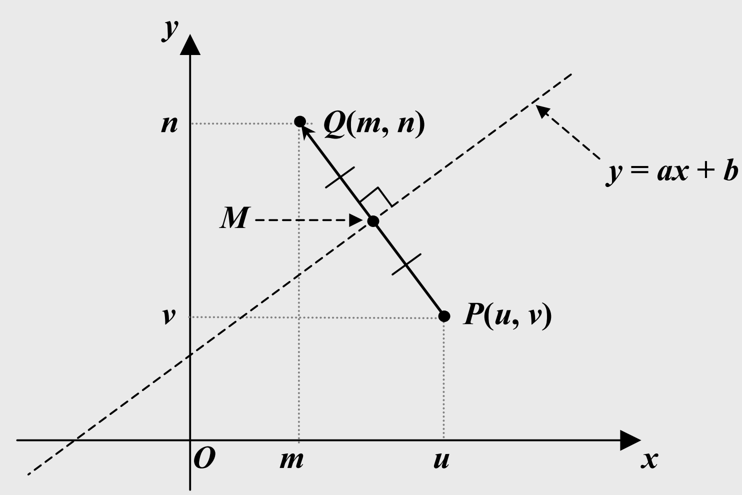
How then can we get the new curve, that is, the new equation?

A transformation definition shows specifically ***how*** every point gets changed in the original curve. So it says ***the way*** the original curve changes, that is, ***the way*** the new curve gets made. We want to get therefore, the definition first.

To begin with, if two points are symmetric about a line $y = ax + b$, the two are equal distance away from the line, and both points are in a line perpendicular to the line.

So for instance, assuming two points (u, v) and (m, n) are symmetric about a point (a, b), we can put the two points in a graph the way below.

Fig. 3



Suppose now, given the point (u, v), we want to find the point (m, n).
That is, we want to find the point symmetric to the point P about the line $y = ax + b$.
In other words, we want to put the point $Q(m, n)$ in terms of the coordinates of $P(u, v)$.

That is to say that using the fact that (m, n) and (u, v) are symmetric about $y = ax + b$, we want to put each of the coordinates of (m, n) in terms of u and v.

To begin with, the two points P and Q are symmetric about the line $y = ax + b$, so the line $y = ax + b$ passes through the midpoint between the two points, and the midpoint is of course, in the line segment PQ. Assuming thus, the midpoint is $M(c, d)$, we can put each of its coordinates this way: $c = \dfrac{u+m}{2}$ and $d = \dfrac{v+n}{2}$.

And next, since the midpoint $M(c, d)$ is in the line $y = ax + b$, we get $d = ac + b$.

That is, we get $\dfrac{v+n}{2} = \dfrac{a(u+m)}{2} + b$. Thus, we get

$\dfrac{v+n}{2} = \dfrac{a(u+m)}{2} + b \Rightarrow v + n = a(u + m) + 2b \Rightarrow v + n = au + am + 2b$.

Also, since the line $y = ax + b$ is perpendicular to the line segment PQ, the product of the slope of the line and the slope of the line segment is -1. And we know the slope of the line $y = ax + b$ is a. So the slope of the line segment is $-\dfrac{1}{a}$.

And we know P is at (u, v), and Q is at (m, n). So we get

$\dfrac{n-v}{m-u} = -\dfrac{1}{a} \Rightarrow a(n - v) = u - m \Rightarrow an - av = u - m$.

Thus, we get a system of two equations for m and n as follows.

$n - am = au - v + 2b$…… (1), and $an + m = u + av$……(2)

Let's find m first. Multiplying by a the equation (1), and then, subtracting the product from the equation (2), we eliminate n. In other words, we get first

$n - am = au - v + 2b \Rightarrow a(n - am) = a(au - v + 2b) \Rightarrow an - a^2m = a^2u - av + 2ab$…(3)

So next subtracting the equation (3) from the equation (2), we get
$\boldsymbol{m + a^2m = u - a^2u + 2av - 2ab}$.

Thus, we get $\boldsymbol{m(1 + a^2) = u(1 - a^2) + 2av - 2ab \Rightarrow m = \frac{(1-a^2)u+2av-2ab}{1+a^2}}$.

Next, finding $\boldsymbol{n}$, we get first, $\boldsymbol{n - am = au - v + 2b \Rightarrow n = am + au - v + 2b}$.
So next, putting into the equation above, the value of $\boldsymbol{m}$ found above, we get

$$\boldsymbol{n = am + au - v + 2b = a\frac{(1-a^2)u+2av-2ab}{1+a^2} + au - v + 2b}$$

$$\boldsymbol{= \frac{a(1-a^2)u+2a^2v-2a^2b}{1+a^2} + \frac{(1+a^2)(au-v+2b)}{1+a^2}}$$

$$\boldsymbol{= \frac{\{a(1-a^2)u+2a^2v-2a^2b\}+(1+a^2)au-(1+a^2)v+(1+a^2)2b}{1+a^2}}$$

$$\boldsymbol{= \frac{a(1-a^2)u+2a^2v-2a^2b+(1+a^2)au-(1+a^2)v+(1+a^2)2b}{1+a^2}}$$

$$\boldsymbol{= \frac{(a-a^3)u+2a^2v-2a^2b+(a+a^3)u-(1+a^2)v+2b+2a^2b}{1+a^2}}$$

$$\boldsymbol{= \frac{(a-a^3+a+a^3)u+(2a^2-1-a^2)v+2b}{1+a^2} = \frac{2au+(a^2-1)v+2b}{1+a^2}}.$$

And thus, putting each of the coordinates of $\boldsymbol{(m, n)}$ in terms of the coordinates of $\boldsymbol{(u, v)}$, we get $\boldsymbol{m = \frac{(1-a^2)u+2av-2ab}{1+a^2}}$, and $\boldsymbol{n = \frac{2au+(a^2-1)v+2b}{1+a^2}}$.

What then is the correlation between the original arbitrary point and the new arbitrary point in this symmetric transformation?

We know that the arbitrary point in a curve represents all the points in the curve, which means that every point in the curve acts the way the arbitrary point does, and vice versa.

So assuming $\boldsymbol{(u, v)}$ is in the original curve, and $\boldsymbol{(x, y)}$ is the original arbitrary point, we can say that $\boldsymbol{(u, v)}$ acts the way $\boldsymbol{(x, y)}$ does, and $\boldsymbol{(x, y)}$ acts the way $\boldsymbol{(u, v)}$ does, too.

Also, assuming $\boldsymbol{(m, n)}$ is in the new curve, and $\boldsymbol{(s, t)}$ is the new arbitrary point, we can say that $\boldsymbol{(m, n)}$ acts the way $\boldsymbol{(s, t)}$ does, and $\boldsymbol{(s, t)}$ acts the way $\boldsymbol{(m, n)}$ does, too.

Thus, we get a correlation as follows.

$$(s, t) = \left(\frac{(1-a^2)x+2ay-2ab}{1+a^2}, \frac{2ax+(a^2-1)y+2b}{1+a^2}\right).$$

That is to say that putting each of the coordinates of (s, t) in terms of the coordinates of (x, y) using the fact that (s, t) and (x, y) are symmetric about the line $y = ax + b$, we get the correlation where $(s, t) = \left(\frac{(1-a^2)x+2ay-2ab}{1+a^2}, \frac{2ax+(a^2-1)y+2b}{1+a^2}\right)$.

Thus, assuming L is the transformation symmetric about a line $y = ax + b$, we can put L the way as follows.

$$L: (x, y) \longrightarrow (s, t) = \left(\frac{(1-a^2)x+2ay-2ab}{1+a^2}, \frac{2ax+(a^2-1)y+2b}{1+a^2}\right).$$

And briefly, we can put it this way, too:

$$L: (x, y) \longrightarrow \left(\frac{(1-a^2)x+2ay-2ab}{1+a^2}, \frac{2ax+(a^2-1)y+2b}{1+a^2}\right).$$

So we can call the expression above the definition of the transformation symmetric about the line $y = ax + b$.

Thus, the core of the transformation is $\left(\frac{(1-a^2)x+2ay-2ab}{1+a^2}, \frac{2ax+(a^2-1)y+2b}{1+a^2}\right)$,

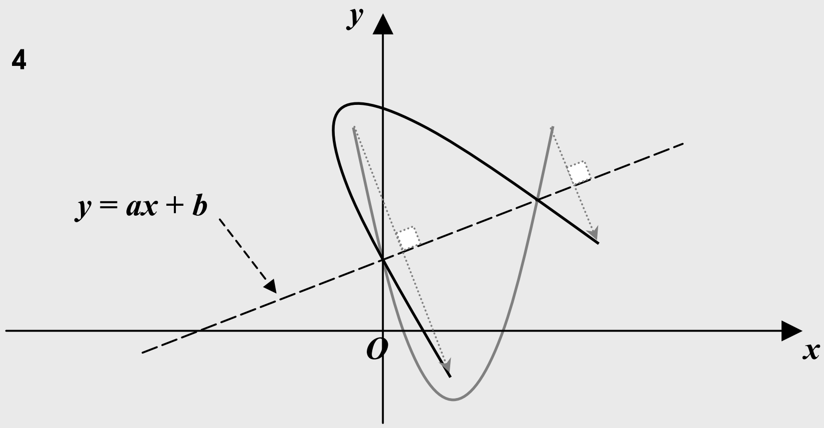
which is the new arbitrary point,
so we set $(s, t) = \left(\frac{(1-a^2)x+2ay-2ab}{1+a^2}, \frac{2ax+(a^2-1)y+2b}{1+a^2}\right)$.

And it explains **the way** every point in the original curve is changed, that is, **how** every point in the new curve gets made.

So applying such a transformation to a function, do we get another function? That is, if the original curve is a curve of a function, is the new curve a curve of a function, too?

Usually not. So it's not often the case. In the figure below, the curve in gray is the original curve, and applying the transformation $\boldsymbol{L}$ to the original curve, we get a new curve, which is the curve in black, which cannot be however, a curve of a function.

Fig. 4



That's because the new curve can meet a vertical line at more than one point, and thus, cannot be a curve of a function. In the new curve above, when $\boldsymbol{x = 0}$, $\boldsymbol{y}$ gets two values.

A function has to have one output only for one input. So the new curve in the graph above is not a curve of a function but just a curve of an equation.
How then can we get the new curve, that is, the new equation?

We can get it getting the equation connecting the coordinates of the new arbitrary point $\boldsymbol{(s, t)}$, that is, the connective equation between $\boldsymbol{s}$ and $\boldsymbol{t}$. And we can get the connective equation mixing the new arbitrary point into the original curve.

And we can get them mixed using this pair of expressions:

$$\left(\frac{(1-a^2)x+2ay-2ab}{1+a^2}, \frac{2ax+(a^2-1)y+2b}{1+a^2}\right).$$

And using it, we begin with this: $(s, t) = \left(\frac{(1-a^2)x+2ay-2ab}{1+a^2}, \frac{2ax+(a^2-1)y+2b}{1+a^2}\right)$.

Then, we can set $s = \frac{(1-a^2)x+2ay-2ab}{1+a^2}$, and $t = \frac{2ax+(a^2-1)y+2b}{1+a^2}$,
which is the system of equations stated above. So do we have to solve for $\boldsymbol{x}$ and $\boldsymbol{y}$ the system of two equations above?

The system above is in fact, the solution to the system below.
$t - as = ax - y + 2b$, and $at + s = x + ay$.

And the system above is no other than the system we've solved earlier for m and n. How?

Below is the solution to the system where $n - am = au - v + 2b$ and $an + m = u + av$.
$m = \dfrac{(1-a^2)u + 2av - 2ab}{1+a^2}$, and $n = \dfrac{2au + (a^2-1)v + 2b}{1+a^2}$

And replacing m and n with s and t, and also, u and v with x and y, we get

$s = \dfrac{(1-a^2)x + 2ay - 2ab}{1+a^2}$, and $t = \dfrac{2ax + (a^2-1)y + 2b}{1+a^2}$, which is the system we wanted to solve, and is the solution to the system where $t - as = ax - y + 2b$, and $at + s = x + ay$.

And of course, solving for x and y the system above, we express x and y in terms of s and t. We don't have to however, solve the system above. Why not?

We can put the system this way, too: $y - ax = as - t + 2b$, and $ay + x = s + at$, which is no other than the system where $n - am = au - v + 2b$, and $an + m = u + av$.

And note that
in the point (s, t), s is the x-coordinate, and t is the y-coordinate.
In the point (m, n), m is the x-coordinate, and n is the y-coordinate.
In the point (u, v), u is the x-coordinate, and v is the y-coordinate.
The two points (x, y) and (s, t) are symmetric about the line $y = ax + b$.
And the two points (u, v) and (m, n) are symmetric about the line $y = ax + b$.

So we can express each of x and y in terms of s and t the way below.
$x = \dfrac{(1-a^2)s + 2at - 2ab}{1+a^2}$, and $y = \dfrac{2as + (a^2-1)t + 2b}{1+a^2}$.

That's because the two points (s, t) and (x, y) are symmetric about the line $y = ax + b$.

In other words, finding (s, t) using the fact that it is symmetric to (x, y) about the line is no other than finding (x, y) using the fact that it is symmetric to (s, t) about the line.

And in fact, the same is true, too, for all the other symmetric transformations. So you may want to go back to each of all the other ones, and give it a check.

Let's now though, for practice purposes, solve for s and t the system we wanted to solve, and see if it really is the case.

The system of equations is $s = \dfrac{(1-a^2)x + 2ay - 2ab}{1+a^2}$, and $t = \dfrac{2ax + (a^2-1)y + 2b}{1+a^2}$.

To begin with, rewriting the first equation, we get

$(1 + a^2)s = (1 - a^2)x + 2ay - 2ab \Rightarrow (1 + a^2)s + 2ab = (1 - a^2)x + 2ay$.

And next, rewriting the second equation, we get

$(1 + a^2)t = 2ax + (a^2 - 1)y + 2b \Rightarrow (1 + a^2)t - 2b = 2ax + (a^2 - 1)y$. So we now have

$(1 - a^2)x + 2ay = (1 + a^2)s + 2ab$……(4), and $2ax + (a^2 - 1)y = (1 + a^2)t - 2b$……(5)

And next, eliminating x first, we multiply by $2a$ the equation (4), and also, multiply by $(1 - a^2)$ the equation (5), and then, subtract the second product from the first product. So to begin with, multiplying by $2a$ the equation (4), we get

$2a(1 - a^2)x + 4a^2y = 2a(1 + a^2)s + 4a^2b$……(6)

And next, multiplying by $(1 - a^2)$ the equation (5), we get

$2a(1 - a^2)x + (1 - a^2)(a^2 - 1)y = (1 - a^2)(1 + a^2)t - 2b(1 - a^2)$

$\Rightarrow 2a(1 - a^2)x - (a^2 - 1)^2y = (1 - a^4)t - 2b(1 - a^2)$……(7)

So next, doing the subtraction, that is, subtracting (7) from (6), we get

$\{2a(1 - a^2)x + 4a^2y\} - \{2a(1 - a^2)x - (a^2 - 1)^2y\}$

$= \{2a(1 + a^2)s + 4a^2b\} - \{(1 - a^4)t - 2b(1 - a^2)\}$

$\Rightarrow 4a^2y + (a^2 - 1)^2y = 2a(1 + a^2)s + 4a^2b - (1 - a^4)t + 2b(1 - a^2)$

$\Rightarrow (a^4 - 2a^2 + 1 + 4a^2)y = 2a(1 + a^2)s + 2a^2b - (1 - a^4)t + 2b$

$\Rightarrow (a^4 + 2a^2 + 1)y = 2a(1 + a^2)s + 2a^2b - (1 - a^4)t + 2b$

$\Rightarrow (a^2 + 1)^2y = 2a(1 + a^2)s - (1 - a^2)(1 + a^2)t + 2b(1 + a^2)$

$\Rightarrow (a^2 + 1)^2y = (1 + a^2)\{2as - (1 - a^2)t + 2b\}$

$\Rightarrow (a^2 + 1)y = 2as - (1 - a^2)t + 2b \Rightarrow y = \dfrac{2as + (a^2 - 1)t + 2b}{1 + a^2}$ ……(8)

And next, replacing the expression (8) with y in $(1 - a^2)x + 2ay = (1 + a^2)s + 2ab$, that is, $(1 - a^2)x = -2ay + (1 + a^2)s + 2ab$, we get

$$(1 - a^2)x = \frac{-2a\{2as - (1 - a^2)t + 2b\}}{1 + a^2} + (1 + a^2)s + 2ab$$

$$= \frac{-4a^2s + 2a(1 - a^2)t - 4ab}{1 + a^2} + (1 + a^2)s + 2ab$$

$$= \frac{-4a^2s + 2a(1 - a^2)t - 4ab + (1 + a^2)^2s + 2ab(1 + a^2)}{a^2 + 1}$$

$$= \frac{(-4a^2 + 1 + 2a^2 + a^4)s + 2a(1 - a^2)t - 4ab + 2ab(1 + a^2)}{1 + a^2}$$

$$= \frac{(a^4 - 2a^2 + 1)s + 2a(1 - a^2)t + 2ab(-2 + 1 + a^2)}{1 + a^2}$$

$$= \frac{(a^2 - 1)^2s + 2a(1 - a^2)t + 2ab(a^2 - 1)}{1 + a^2} = \frac{(1 - a^2)^2s + 2a(1 - a^2)t - 2ab(1 - a^2)}{1 + a^2}$$

$$\Rightarrow (1 - a^2)x = \frac{(1 - a^2)\{(1 - a^2)s + 2at - 2ab\}}{1 + a^2} \Rightarrow x = \frac{(1 - a^2)s + 2at - 2ab}{1 + a^2}.$$

So we get the same solution, which takes however, quite a few steps in algebra.

And thus, using the fact that (s, t) and (x, y) are symmetric about the line $y = ax + b$, we can easily get each of x and y expressed in terms of s and t.
So let's now get the new curve.

Getting first, back to L, the transformation symmetric about a line $y = ax + b$, we can put L the way as follows.

$$L: (x, y) \longrightarrow (s, t) = \left(\frac{(1 - a^2)x + 2ay - 2ab}{1 + a^2},\ \frac{2ax + (a^2 - 1)y + 2b}{1 + a^2}\right)$$

Then, we get $s = \dfrac{(1-a^2)x + 2ay - 2ab}{1+a^2} \Rightarrow x = \dfrac{(1-a^2)s + 2at - 2ab}{1+a^2}$,

and $t = \dfrac{2ax + (a^2-1)y + 2b}{1+a^2} \Rightarrow y = \dfrac{2as + (a^2-1)t + 2b}{1+a^2}$.

So we can now mix the new arbitrary point (s, t) into the original curve. That is, we can mix now, (s, t) into the original equation $y = f(x)$ to get the connective equation. How?

In $y = f(x)$, we replace x with $\dfrac{(1-a^2)s + 2at - 2ab}{1+a^2}$, and y with $\dfrac{2as + (a^2-1)t + 2b}{1+a^2}$.

So doing the substitutions,

we get $y = f(x) \Rightarrow \dfrac{2as + (a^2-1)t + 2b}{1+a^2} = f\left(\dfrac{(1-a^2)s + 2at - 2ab}{1+a^2}\right)$, which is the equation connecting s and t, that is, the new equation indicating the new curve.

And we know that the new curve is in the x-y plane, and that s is another name for the variable x, and t is another name for the variable y.

So next, replacing s with x, and t with y,

we get $\dfrac{2ax + (a^2-1)y + 2b}{1+a^2} = f\left(\dfrac{(1-a^2)x + 2ay - 2ab}{1+a^2}\right)$, which is the new equation.

So in sum, transforming an equation $y = f(x)$ symmetrically about a line $y = ax + b$, we get a new equation,

and the new equation is $\dfrac{2ax + (a^2-1)y + 2b}{1+a^2} = f\left(\dfrac{(1-a^2)x + 2ay - 2ab}{1+a^2}\right)$,

Also, assuming the new equation is $g(x, y) = 0$, we can put g the way as follows, too.

$$g(x, y) = f\left(\frac{(1-a^2)x + 2ay - 2ab}{1+a^2}\right) - \frac{2ax + (a^2-1)y + 2b}{1+a^2} = 0.$$

And we can notice that in this case, we can just do direct substitutions.

It's because the expressions for s and x are the same, and the same is true for t and y, too, as we can see below.

$$s = \frac{(1-a^2)x + 2ay - 2ab}{1+a^2} \Rightarrow x = \frac{(1-a^2)s + 2at - 2ab}{1+a^2},$$

and $t = \dfrac{2ax + (a^2-1)y + 2b}{1+a^2} \Rightarrow y = \dfrac{2as + (a^2-1)t + 2b}{1+a^2}$.

So given a function or equation as $y = f(x)$, we can get the new one just replacing x with $\dfrac{(1-a^2)s + 2at - 2ab}{1+a^2}$, and y with $\dfrac{2as + (a^2-1)t + 2b}{1+a^2}$.

That is, the new equation is $\dfrac{2ax + (a^2-1)y + 2b}{1+a^2} = f\left(\dfrac{(1-a^2)x + 2ay - 2ab}{1+a^2}\right)$.

And assuming $g(x, y) = 0$ is the new equation, we get

$$g(x, y) = f\left(\frac{(1-a^2)x + 2ay - 2ab}{1+a^2}\right) - \frac{2ax + (a^2-1)y + 2b}{1+a^2} = 0.$$

Or if the original equation is $f(x, y) = 0$,

the new one is $f\left(\dfrac{(1-a^2)x + 2ay - 2ab}{1+a^2}, \dfrac{2ax + (a^2-1)y + 2b}{1+a^2}\right) = 0$.

That is, assuming again, $g(x, y) = 0$ is the new equation, we get

$$g(x, y) = f\left(\frac{(1-a^2)x + 2ay - 2ab}{1+a^2}, \frac{2ax + (a^2-1)y + 2b}{1+a^2}\right) = 0.$$

Suppose for instance, the original function is $y = f(x) = 2x^2 - 4x + 1$.
Then, the new equation is

$$\frac{2ax + (a^2-1)y + 2b}{1+a^2} = 2\left(\frac{(1-a^2)x + 2ay - 2ab}{1+a^2}\right)^2 - 4\left(\frac{(1-a^2)x + 2ay - 2ab}{1+a^2}\right) + 1,$$

which is therefore, the transformation of f by L.

Suppose for a concrete case, the original function is $\boldsymbol{y = f(x) = 2x^2 - 4x + 1}$, and it gets transformed symmetrically about a line $\boldsymbol{y = \frac{1}{2}x + 1}$, that is, $\boldsymbol{a = \frac{1}{2}}$, and $\boldsymbol{b = 1}$ in the axis of symmetry $\boldsymbol{y = ax + b}$. Then, first, we get

$$\frac{(1-a^2)x + 2ay - 2ab}{1+a^2} = \frac{(1-\frac{1}{4})x + y - 1}{1+\frac{1}{4}} = \frac{4\{(\frac{3}{4})x + y - 1\}}{5} = \frac{3x+4y-4}{5}.$$

$$\frac{2ax + (a^2-1)y + 2b}{1+a^2} = \frac{x + (\frac{1}{4}-1)y + 2}{1+\frac{1}{4}} = \frac{4(x - \frac{3}{4}y + 2)}{5} = \frac{4x-3y+8}{5}.$$

And next, the original equation is $\boldsymbol{y = f(x)}$, which means $\boldsymbol{y = 2x^2 - 4x + 1}$.

So in the original equation $\boldsymbol{y = f(x)}$,

replacing $\boldsymbol{x}$ with $\frac{3x+4y-4}{5}$, and $\boldsymbol{y}$ with $\frac{4x-3y+8}{5}$, we get

$$\frac{4x-3y+8}{5} = 2\left(\frac{3x+4y-4}{5}\right)^2 - 4\left(\frac{3x+4y-4}{5}\right) + 1$$

$$\Rightarrow \frac{4x-3y+8}{5} = \frac{2(3x+4y-4)^2}{25} - \frac{4(3x+4y-4)}{5} + 1$$

$$\Rightarrow 5(4x - 3y + 8) = 2(3x + 4y - 4)^2 - 20(3x + 4y - 4) + 25.$$

And we can have $\boldsymbol{(3x + 4y - 4)^2 = 9x^2 + 16y^2 + 24xy - 16y - 24x}$.

So we get $\boldsymbol{5(4x - 3y + 8) = 2(3x + 4y - 4)^2 - 20(3x + 4y - 4) + 25}$

$$\Rightarrow 20x - 15y + 40 = 18x^2 + 32y^2 + 48xy - 32y - 48x - 60x - 80y + 80 + 25$$

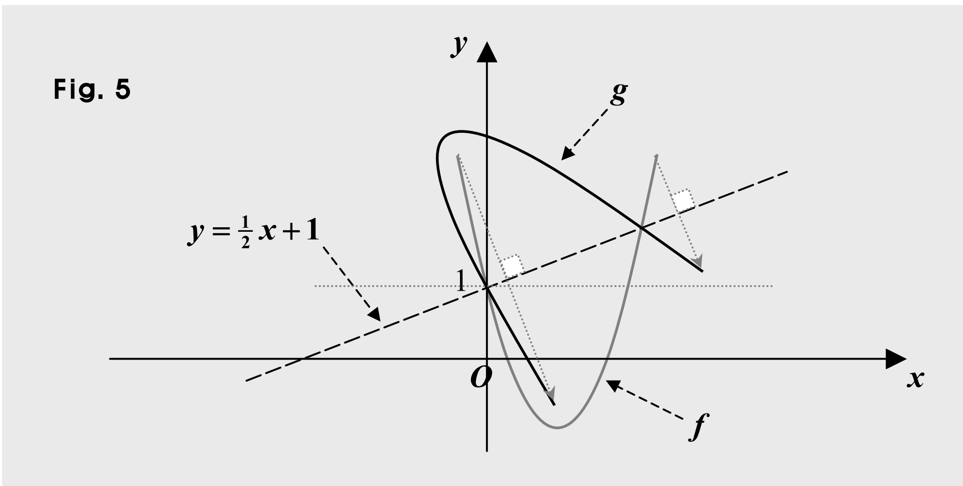
$\Rightarrow \boldsymbol{18x^2 + 32y^2 + 48xy - 97y - 128x + 65 = 0}$, which is the new equation.

So assuming $\boldsymbol{g}$ is the new equation above, we can say that the curve of $\boldsymbol{g}$ is symmetric about the line $\boldsymbol{y = \frac{1}{2}x + 1}$ to the curve of the original equation $\boldsymbol{y = 2x^2 - 4x + 1}$.

And we can put ***g*** and ***f*** the way below.

$\boldsymbol{g(x, y) = 18x^2 + 32y^2 + 48xy - 97y - 128x + 65 = 0}$, and $\boldsymbol{f(x, y) = 2x^2 - 4x + 1 - y = 0}$.

And both curves are in fact, parabolas, and putting both in a graph, we get this:



So the parabola ***g*** is symmetric to the parabola ***f*** about the line $y = \frac{1}{2}x + 1$.

And for a simpler example, assuming the original function is $\boldsymbol{y = h(x) = -2x + 3}$, and the axis of symmetry is $\boldsymbol{y = 2x + 1}$, we have $\boldsymbol{a = 2}$, and $\boldsymbol{b = 1}$, so we get first,

$$\frac{(1-a^2)x + 2ay - 2ab}{1+a^2} = \frac{-3x+4y-4}{5}, \text{ and } \frac{2ax + (a^2-1)y + 2b}{1+a^2} = \frac{4x+3y+2}{5}.$$

And next, getting the new equation, we get

$$\frac{4x+3y+2}{5} = h\left(\frac{-3x+4y-4}{5}\right) = -2\left(\frac{-3x+4y-4}{5}\right) + 3$$

$$= \frac{6x-8y+8+15}{5} = \frac{6x-8y+23}{5} \Rightarrow \frac{4x+3y+2}{5} = \frac{6x-8y+23}{5}$$

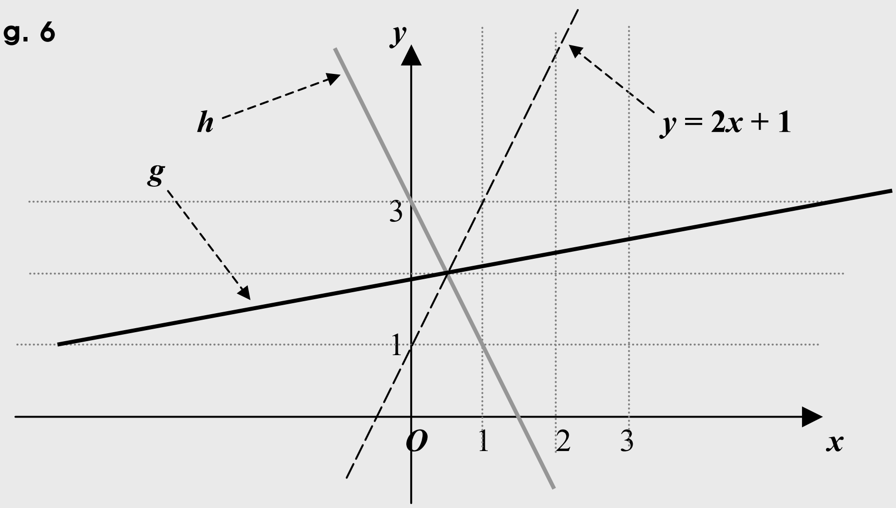
$$\Rightarrow 4x + 3y + 2 = 6x - 8y + 23 \Rightarrow 11y - 2x - 21 = 0 \Rightarrow y = \frac{1}{11}(2x + 21)\text{, the new curve.}$$

And since the original curve is a line, the new curve is a line, too, and also, we get a new function, too. And assuming g is the new function, we get $y = g(x) = \dfrac{1}{11}(2x + 21)$

And we can put g this way, too, of course: $g = T \bullet h$ where $y = h(x) = -2x + 3$, and

T: $(x, y) \longrightarrow \left(\dfrac{-3x+4y-4}{5}, \dfrac{4x+3y+2}{5}\right)$

Fig. 6



And for another instance, let's assume that the original function is $y = f(x) = x^2 - 2x + 2$ for $x \geq 1$, and the axis of symmetry is a line $y = -2x + 6$, and see what we get.

To begin with, assuming L is a transformation symmetric about a line $y = ax + b$, we can put L this way: L: $(x, y) \longrightarrow \left(\dfrac{(1-a^2)x+2ay-2ab}{1+a^2}, \dfrac{2ax+(a^2-1)y+2b}{1+a^2}\right)$

And in this case, we have $a = -2$, and $b = 6$.

So we get $\dfrac{(1-a^2)x+2ay-2ab}{1+a^2} = \dfrac{-3x-4y+24}{5}$,

and $\dfrac{2ax+(a^2-1)y+2b}{1+a^2} = \dfrac{-4x+3y+12}{5}$.

Thus, assuming the transformation we apply in this case is Q, we can put Q this way: Q: $(x, y) \longrightarrow \left(\dfrac{-3x-4y+24}{5}, \dfrac{-4x+3y+12}{5}\right)$

Then, applying $\boldsymbol{Q}$ to $\boldsymbol{f}$ above, we get $\boldsymbol{y = f(x)} \Rightarrow \dfrac{-4x+3y+12}{5} = f\left(\dfrac{-3x-4y+24}{5}\right)$.

which is the new equation.

> And in a transformation symmetric about a line $\boldsymbol{y = ax + b}$,
> the extents of the variables in the original equation can change, too.
> So specifying the new equation, we need to specify the new extents,
> that is, the new domain and range, at least the new domain
> if the original equation is specified with its domain.

And getting the new extents, we can begin with this:

$$(s,\ t) = \left(\frac{(1-a^2)x+2ay-2ab}{1+a^2},\ \frac{2ax+(a^2-1)y+2b}{1+a^2}\right)$$

So getting the extents in this case, we can begin with this:

$$(s,\ t) = \left(\frac{-3x-4y+24}{5},\ \frac{-4x+3y+12}{5}\right)$$

Finding the extent of $\dfrac{-3x-4y+24}{5}$, we find the extent of $\boldsymbol{s}$, which is $\boldsymbol{x}$ in the new curve, that is, new equation, and of course, finding the extent of $\dfrac{-4x+3y+12}{5}$, we find the extent of $\boldsymbol{t}$, which is $\boldsymbol{y}$ in the new curve.

And in this example, the original function is $\boldsymbol{y = f(x) = x^2 - 2x + 2}$ for $\boldsymbol{x \geq 1}$.
And putting it in the vertex form, we get $\boldsymbol{y = f(x) = (x-1)^2 + 1}$ for $\boldsymbol{x \geq 1}$.

So we get $\boldsymbol{x \geq 1}$, and $\boldsymbol{y \geq 1}$. And also, we can see that $\boldsymbol{y}$-value keeps increasing as $\boldsymbol{x}$-value increases from 1.

Thus next, finding the extent of $\boldsymbol{s}$ first, we want to get the extent of $\dfrac{-3x-4y+24}{5}$.

Then, we can get first, $\boldsymbol{x \geq 1 \Rightarrow 3x \geq 3 \Rightarrow -3x \leq -3}$, and $\boldsymbol{y \geq 1 \Rightarrow 4y \geq 4 \Rightarrow -4y \leq -4}$.

So next, we get $\mathbf{-3x - 4y \le -7 \Rightarrow -3x - 4y + 24 \le -7 + 24 = 17 \Rightarrow -3x - 4y + 24 \le 17}$.

Note that we can do the operations the way above because of the fact that $\boldsymbol{y}$-value keeps increasing as $\boldsymbol{x}$-value increases from 1, and the fact that $\mathbf{-3x}$ and $\mathbf{-4y}$ have the same signs.

Normally though, we may want to do the operations the way below.

To begin with, we have $\boldsymbol{y = f(x) = x^2 - 2x + 2}$, that is, we have $\boldsymbol{y = x^2 - 2x + 2}$.

So we get $\mathbf{-3x - 4y + 24 = -3x - 4(x^2 - 2x + 2) + 24 = -4x^2 + 5x + 16}$.

Suppose now, $\boldsymbol{h(x) = -4x^2 + 5x + 16}$. Then, the curve of $\boldsymbol{h}$ is a parabola concave-down, and the axis of symmetry of the parabola is $\boldsymbol{x = \frac{5}{8}}$. And we have $\boldsymbol{x \ge 1}$. So the axis of symmetry of the parabola is on the left of the line $\boldsymbol{x = 1}$.

So we get $\boldsymbol{h(x) = -4x^2 + 5x + 16 \le h(1) = 17}$. In other words, $\mathbf{-3x - 4y + 24 \le 17}$.

Thus next, we get $\boldsymbol{\frac{-3x - 4y + 24}{5} \le \frac{17}{5}}$. So we get $\boldsymbol{s \le \frac{17}{5}}$.

Let's next, move on to the extent of $\boldsymbol{t}$. To begin with, we have $\boldsymbol{t = \frac{-4x + 3y + 12}{5}}$.

And we have $\boldsymbol{y = f(x) = x^2 - 2x + 2}$ for $\boldsymbol{x \ge 1}$. That is, we have $\boldsymbol{y = x^2 - 2x + 2}$ for $\boldsymbol{x \ge 1}$.

So we get $\mathbf{-4x + 3y + 12 = -4x + 3(x^2 - 2x + 2) + 12 = 3x^2 - 10x + 18}$

$$\boldsymbol{= 3(x^2 - \frac{10}{3}x) + 18 = 3\left(x^2 - \frac{10}{3}x + \left(\frac{5}{3}\right)^2 - \left(\frac{5}{3}\right)^2\right) + 18 = 3(x - \frac{5}{3})^2 - \frac{25}{3} + 18}$$

$$\boldsymbol{= 3(x - \frac{5}{3})^2 + \frac{29}{3} \ge \frac{29}{3}}, \text{ since } \boldsymbol{x \ge 1}. \quad \text{Why, though?}$$

Suppose $q(x) = 3(x - \frac{5}{3})^2 + \frac{29}{3}$ for $x \geq 1$.

Then, the curve of q is a parabola concave-up, and the axis of symmetry of the parabola is $x = \frac{5}{3}$, and therefore, is on the right of the line $x = 1$. So we get $q(x) \geq \frac{29}{3}$.

That's because it is concave-up, $q(1) > \frac{29}{3}$, and $x \geq 1$.

And we know $-4x + 3y + 12 = 3(x - \frac{5}{3})^2 + \frac{29}{3}$.

So we get $\frac{-4x+3y+12}{5} = \frac{3(x-\frac{5}{3})^2 + \frac{29}{3}}{5} \geq \frac{\frac{29}{3}}{5} = \frac{29}{15}$, which means we get $t \geq \frac{29}{15}$.

Now, we know s is the x-coordinate, t is the y-coordinate, and (s, t) is the arbitrary point in the curve of g. So in the equation of the curve of g, we have $x \leq \frac{17}{5}$, and $y \geq \frac{29}{15}$.

Thus, the new equation of g is

$\frac{-4x+3y+12}{5} = f\left(\frac{-3x-4y+24}{5}\right)$ for $x \leq \frac{17}{5}$, and $y \geq \frac{29}{15}$.

And getting the expression of the new equation g, we begin with the original equation, that is, $y = f(x) = x^2 - 2x + 2$. Then, we get

$$\frac{-4x+3y+12}{5} = f\left(\frac{-3x-4y+24}{5}\right) = \left(\frac{-3x-4y+24}{5}\right)^2 - 2\left(\frac{-3x-4y+24}{5}\right) + 2$$

$$= \frac{9x^2+16y^2+24^2+2(12xy-96y-72x)}{25} - \frac{-6x-8y+48}{5} + 2$$

$$= \frac{9x^2+16y^2+24^2+24xy-192y-144x+30x+40y-240+50}{25}$$

$$= \frac{9x^2+16y^2+24xy-152y-114x+386}{25}$$

$$\Rightarrow \frac{-4x+3y+12}{5} = \frac{9x^2+16y^2+24xy-152y-114x+386}{25}.$$

So we get $\mathbf{5(-4x + 3y + 12) = 9x^2 + 16y^2 + 24xy - 152y - 114x + 386}$

$\Rightarrow \mathbf{-20x + 15y + 60 = 9x^2 + 16y^2 + 24xy - 152y - 114x + 386}$

$\Rightarrow \mathbf{9x^2 + 16y^2 + 24xy - 94x - 167y + 326 = 0}$.

And as mentioned earlier, specifying the new equation, we need to specify the new extents, that is, the new domain and range, at least the new domain if the original equation is specified with its domain. So the new equation is as follows.

$\mathbf{9x^2 + 16y^2 + 24xy - 94x - 167y + 326 = 0}$ for $x \le \frac{17}{5}$, and $y \ge \frac{29}{15}$.

And in this case also, applying to the function $\boldsymbol{f}$ the transformation symmetric about the line $\mathbf{y = -2x + 6}$, we get a new function. So $\boldsymbol{g}$ is a new function.

And getting the expression of the function $\boldsymbol{g(x)}$, we want to solve for $\boldsymbol{y}$ the equation of the curve of $\boldsymbol{g}$. So solving it, we can get first,

$\mathbf{9x^2 + 16y^2 + 24xy - 94x - 167y + 326 = 16y^2 - (167 - 24x)y + 9x^2 - 94x + 326 = 0}$.

Meanwhile,

$$16y^2 - (167-24x)y = 16\{y^2 - \frac{167-24x}{16}y + \left(\frac{167-24x}{32}\right)^2 - \left(\frac{167-24x}{32}\right)^2\}$$

$$= 16\left(y - \frac{167-24x}{32}\right)^2 - \frac{(167-24x)^2}{64}.$$

So we get $16\left(y - \frac{167-24x}{32}\right)^2 - \frac{(167-24x)^2}{64} + 9x^2 - 94x + 326 = 0$.

Thus, we get $16\left(y - \frac{167-24x}{32}\right)^2 = \frac{(167-24x)^2}{64} - 9x^2 + 94x - 326$

$$= \frac{1}{64}(167^2 - 2\cdot167\cdot24x + 24^2x^2 - 64\cdot9x^2 + 64\cdot94x - 326\cdot64)$$

$$= \frac{1}{64}(6016x - 8016x + 27889 - 20864) = \frac{1}{64}(-2000x + 7025) = \frac{25}{64}(281 - 80x).$$

So we get $16\left(y - \frac{167-24x}{32}\right)^2 = \left(\frac{5}{32}\right)^2(281 - 80x)$

$$\Rightarrow y - \frac{167-24x}{32} = \pm\frac{5}{32}\sqrt{281-80x} \Rightarrow y = \frac{167-24x}{32} \pm \frac{5}{32}\sqrt{281-80x},$$

which is however, a set of two equations. So which one?

We have $x \le \frac{17}{5}$. So we get

$$24x \le \frac{17\cdot 24}{5} \Rightarrow -24x \ge -\frac{408}{5} \Rightarrow 167 - 24x \ge 167 - \frac{408}{5} = \frac{427}{5}$$

$$\Rightarrow \frac{167-24x}{32} \ge \frac{427}{5\cdot 32} = \frac{427}{160} = 2.66875.$$

Thus, we get $\frac{167-24x}{32} \ge 2.66875$. And we know that $\frac{5}{32}\sqrt{281-80x}$ cannot be negative anyway.

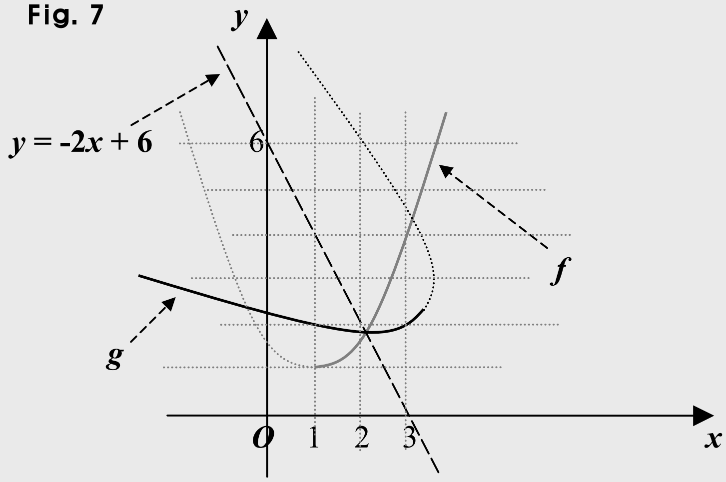
So if we choose $y = \frac{167-24x}{32} + \frac{5}{32}\sqrt{281-80x}$, we have to get $y \ge 2.66875$.

We know however, the range is $y \ge \frac{29}{15} = 1.933...$ So choosing it, we get a discrepancy, where for instance, y cannot be 2, which is in the range. So we cannot choose it, and have to choose the other.

So we get $y = \frac{167-24x}{32} - \frac{5}{32}\sqrt{281-80x} = \frac{1}{32}(167 - 24x - 5\sqrt{281-80x})$.

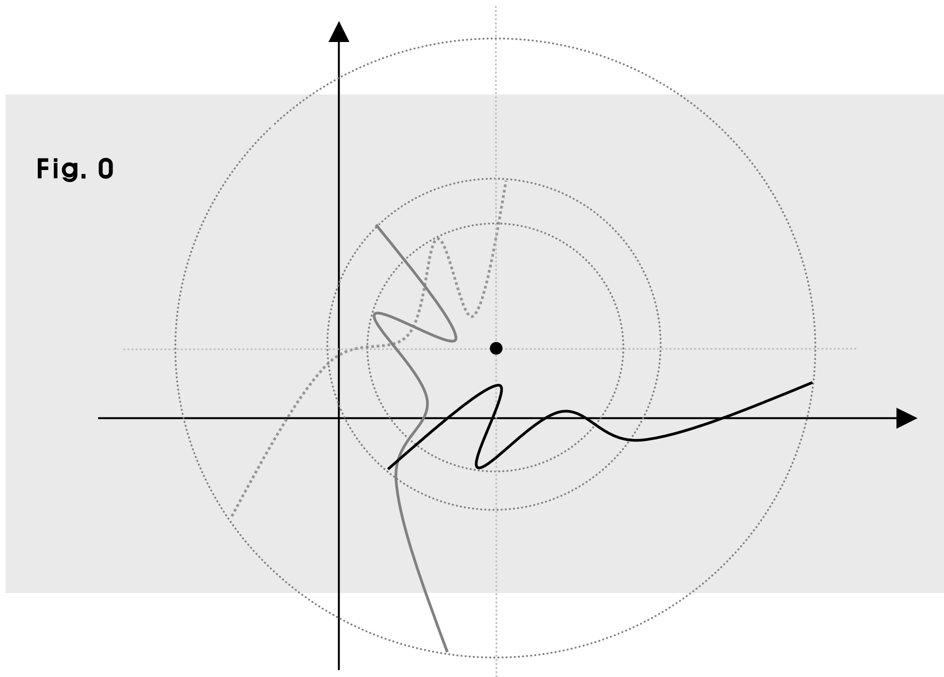
Thus, the new function g is $y = g(x) = \frac{1}{32}(167 - 24x - 5\sqrt{281-80x})$ for $x \le \frac{17}{5}$.

Fig. 7



3.1. Transformations by turning 1

Fig. 0



Applying to a curve a transformation by turning, we turn the curve, and get a new curve, which is the curve turned, of course. We don't just turn the curve though. Turning a curve using a transformation in this book, we turn it around a point, which is called therefore, the axis of turning.

So in a transformation by turning, we specify the point the original curve turns around, and the new curve is the original curve turned around the point.
What then about the amount of turning?

We can specify the amount of turning by an angle. And thus, in a transformation by turning, we specify the angle, together with the point called the axis of turning, and the new curve is the original curve turned around the point in the amount of the angle.

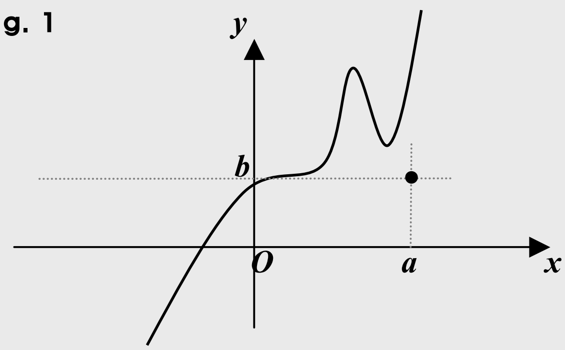
How then is the new curve made?

In a transformation by turning, every point in the original curve turns around the same point in the same amount of angle, and then, all the points turned form the new curve.

So each point moves along its own circular path, that is, a circle.
In such a transformation therefore, we can see a lot of concentric circles, each of which is a path each point in the original curve moves along, and is centered at the same point stated above, called therefore, the axis of turning. And that's the way a curve turns in a transformation by turning.

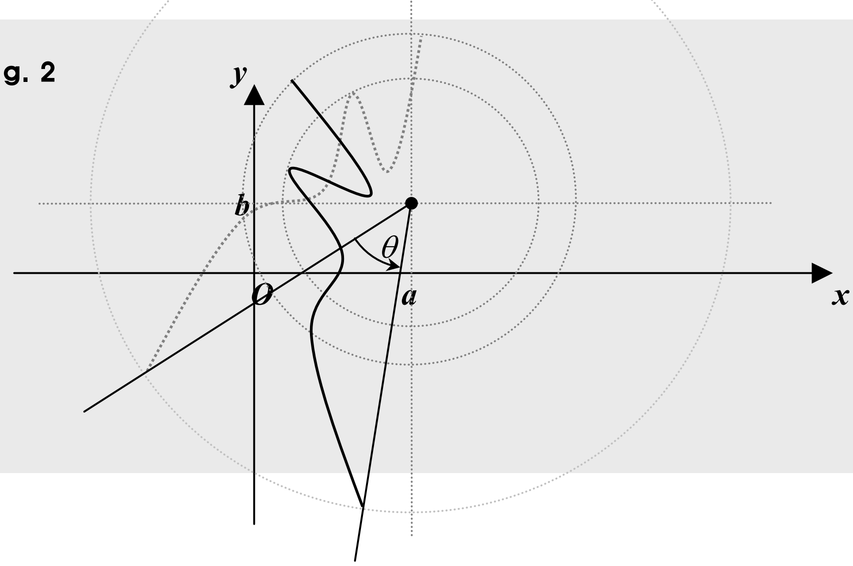
So suppose now that the original curve is the curve of a function $\boldsymbol{y = f(x)}$, and is put in a graph the way below, and the curve of $\boldsymbol{f}$ turns around a point $\boldsymbol{(a, b)}$ in an angle θ.

Fig. 1



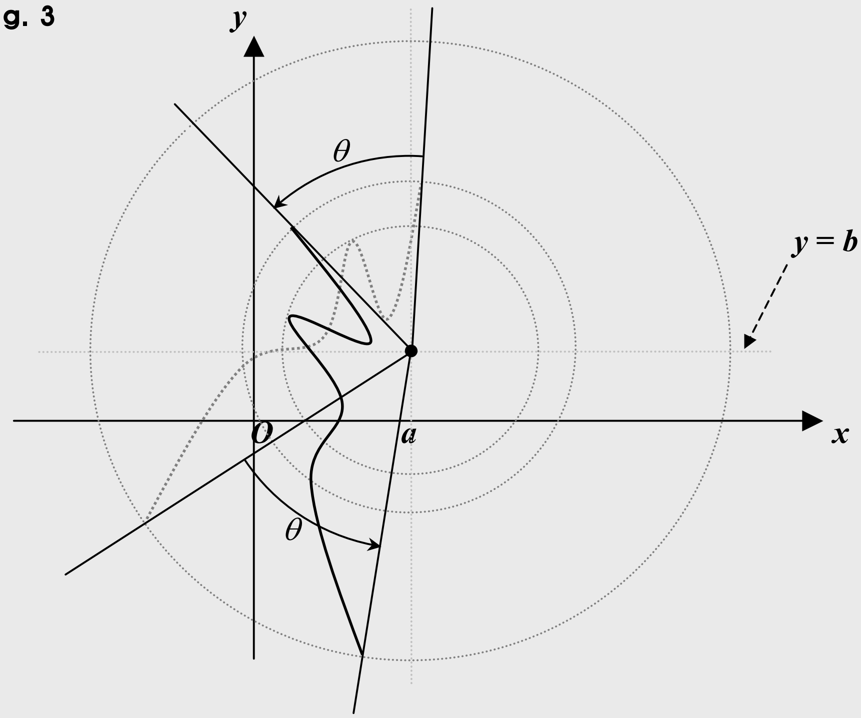
Then, we apply to $\boldsymbol{f}$ the transformation by turning around the point $\boldsymbol{(a, b)}$ in the amount of angle θ. Then, we get a new curve, which is the curve of $\boldsymbol{f}$ turned around the point in the amount of the angle, and thus, is the curve in black in the graph below.

Fig. 2



Note that each and every point in the curve of ***f*** turns around the same point **(*a*, *b*)** in the same amount of angle θ.

Fig. 3



How then can we get the new curve, that is, the new equation?

A transformation definition shows specifically **how** every point gets changed in the original curve. So it specifies ***the way*** the original curve changes, that is, ***the way*** the new curve gets made. Thus, we want to get the definition first.

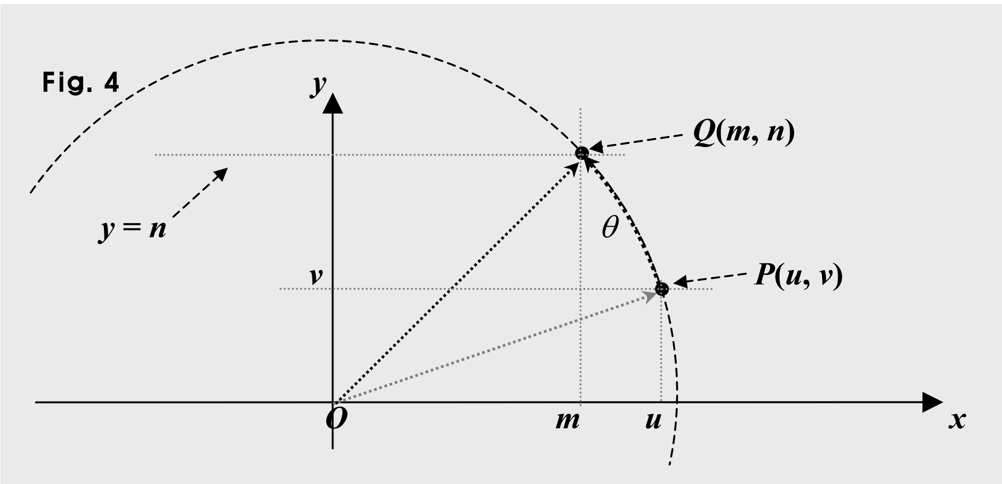
Applying a transformation by turning around a point **(*a*, *b*)** in an amount of angle θ, we make each and every point in the original curve turn around the point **(*a*, *b*)** in angle θ.

Then, the original curve is said to turn around **(*a*, *b*)** in angle θ.

And we can apply a transformation to a point. So let's begin with moving a point in the ***x*-*y*** plane by the transformation stated above.

Suppose now, given a point ***P*(*u*, *v*)**, we want to find the point ***Q*(*m*, *n*)** assuming for simplicity, the axis of turning is the origin, that is, the point ***P*** turns around origin. In other words, we want to find the point we get after turning the point ***P*** around the origin **(0, 0)** in the amount of angle θ. That is, using the fact that **(*m*, *n*)** is the point we get after turning **(*u*, *v*)** the way above, we want to put each of the coordinates of **(*m*, *n*)** in terms of the coordinates of **(*u*, *v*)**.

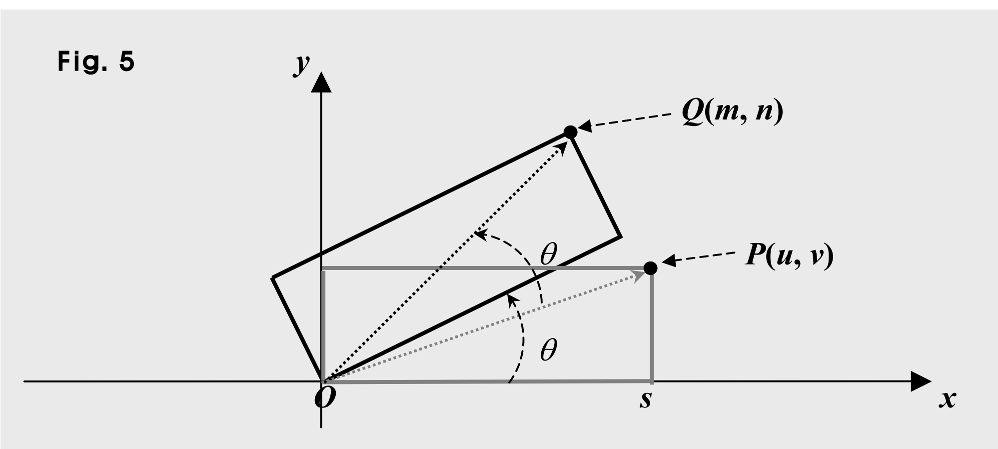
Fig. 4



The point ***P*** has turned around the origin in the amount of angle θ.
So putting each of the coordinates of **(*m*, *n*)** in terms of the coordinates of **(*u*, *v*)**, we get

$\boldsymbol{m = u\cos\theta - v\sin\theta}$, and $\boldsymbol{n = u\sin\theta + v\cos\theta}$.   How?

Fig. 5



To begin with, in the rectangle in gray, the width is ***u***, and the height is ***v***. And the diagonal is the line segment connecting the origin and the point ***P***.

So next, we can see that the rectangle in gray turns around the origin exactly the way the point ***P*** turns around the origin. That is, the rectangle in black is the gray rectangle turned around the origin in the amount of angle θ. In other words, the two rectangles are identical to each other. So in the black rectangle, the width is ***u***, and the height is ***v***, too.

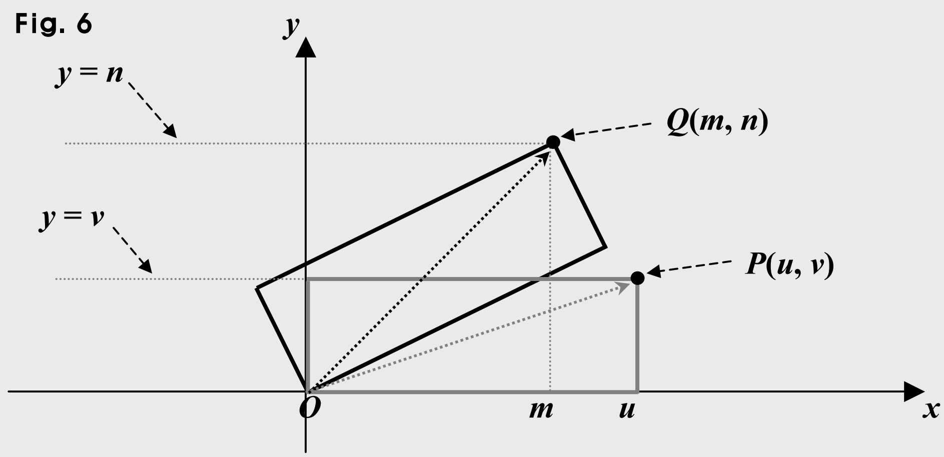
So let's now take a closer look at the two rectangles.

To begin with, a circle is in a plane, and is a set of points, the distance from each of which to a particular point is the same. And we call the particular point the center, of course. So?

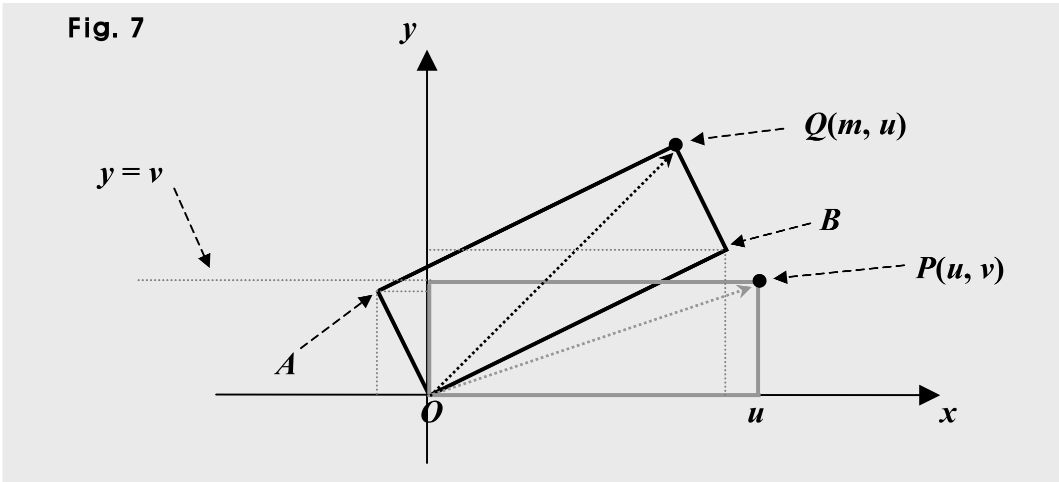
We know that the two points ***P*(*u*, *v*)** and ***Q*(*m*, *n*)** are in the same circle, centered at the origin. So next, their distances from the origin are the same. Using thus, the distance formula, we get $\boldsymbol{m^2 + n^2 = u^2 + v^2}$.

So the easiest way to look at the two rectangles is that we take the two points ***P*** and ***Q*** for two position vectors, often just called vectors, for short. And we can put a vector in a sum of two vectors. One is called the horizontal component, and the other is called the vertical component. So using the vector idea above, and taking the ***x***-coordinate as the horizontal component, and the ***y***-coordinate as the vertical component, we can put two points ***P*** and ***Q*** the way blow.

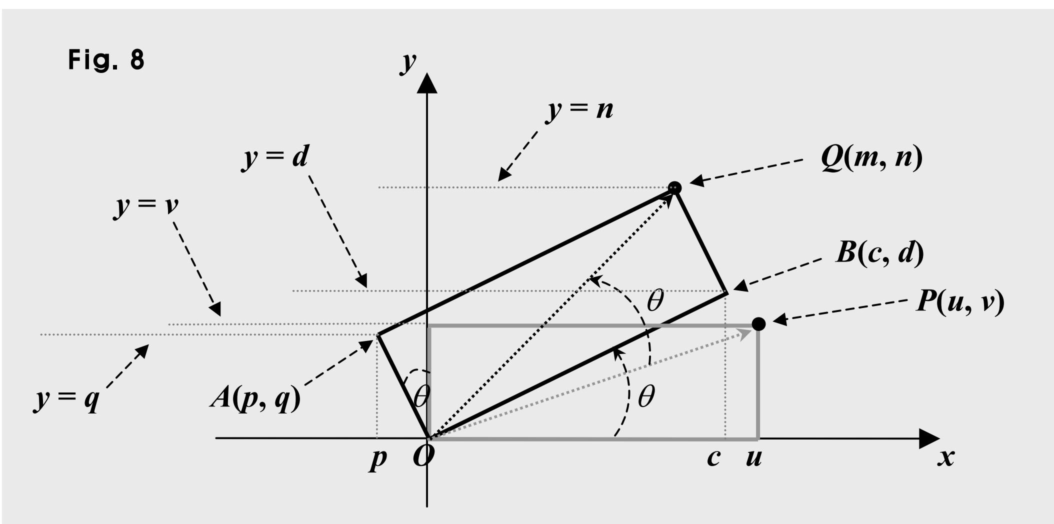
Fig. 6



Then, the horizontal component of the vector ***P*** is ***u***, and the vertical component is ***v***. And the horizontal component of the vector ***Q*** is ***m***, and the vertical component is ***n***. Also, we can put the vector ***Q*** the way below, too.



So we can take the vector ***Q*** as the sum of the two vectors ***A*** and ***B***.
And of course, each of the two vectors ***A*** and ***B*** can have its horizontal and vertical components, too. So we can put the two vectors ***A*** and ***B*** the way below.



Then, to begin with, the vector (point) ***B*** is composed of its horizontal component ***c***, and its vertical component ***d***. And the magnitude of ***B*** is the magnitude of ***u***, which is the horizontal component of ***P***. So we get $\boldsymbol{c = u \cos\theta}$, and $\boldsymbol{d = u \sin\theta}$.

And next, the vector ***A*** is composed of its horizontal component ***p***, and its vertical component ***q***. And the magnitude of ***A*** is the magnitude of ***v***, which is the vertical component of ***P***. So we get $\boldsymbol{p = -v \sin\theta}$, and $\boldsymbol{q = v \cos\theta}$.

And next, we know that the sum of the two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ is the vector $\boldsymbol{Q}$.
What then do we get adding together all the components $\boldsymbol{c}$, $\boldsymbol{d}$, $\boldsymbol{p}$, and $\boldsymbol{q}$?

We get the vector $\boldsymbol{Q}$, of course. And among the four vectors above, $\boldsymbol{c}$ and $\boldsymbol{p}$ are horizontal components.
What then do we get taking the sum of the horizontal components?

We get the horizontal component of the vector $\boldsymbol{Q}$, that is, we get the vector $\boldsymbol{m}$, which is the $\boldsymbol{x}$-coordinate of the point $\boldsymbol{Q}$, of course. In other words, we get

$\boldsymbol{m = c + p = u \cos\theta + (-v \sin\theta) = u \cos\theta - v \sin\theta}$.

And next, what do we get taking the sum of the two vertical components $\boldsymbol{d}$ and $\boldsymbol{v}$ above?

We get the vertical component of the vector $\boldsymbol{Q}$, that is, we get the vector $\boldsymbol{n}$, which is the $\boldsymbol{y}$-coordinate of the point $\boldsymbol{Q}$, of course. In other words, we get

$\boldsymbol{n = d + q = u \sin\theta + v \cos\theta}$.

Therefore, we get $\boldsymbol{m = u \cos\theta - v \sin\theta}$, and $\boldsymbol{n = u \sin\theta + v \cos\theta}$.

What then is the correlation between the original and the new arbitrary points?

The arbitrary point in a curve represents all the points in the curve. So assuming $\boldsymbol{(u, v)}$ is in the original curve, $\boldsymbol{(x, y)}$ is the original arbitrary point, $\boldsymbol{(s, t)}$ is the new arbitrary point, and $\boldsymbol{(m, n)}$ is in the new curve, we get the correlation as follows.

$\boldsymbol{(s, t) = (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta)}$, because $\boldsymbol{(x, y)}$ behaves the way $\boldsymbol{(u, v)}$ does, and vice versa, and the same is true for $\boldsymbol{(s, t)}$ and $\boldsymbol{(m, n)}$, too.

Thus, assuming $\boldsymbol{T}$ is the transformation by turning around the origin in the amount of angle θ, we can put $\boldsymbol{T}$ the way as follows.

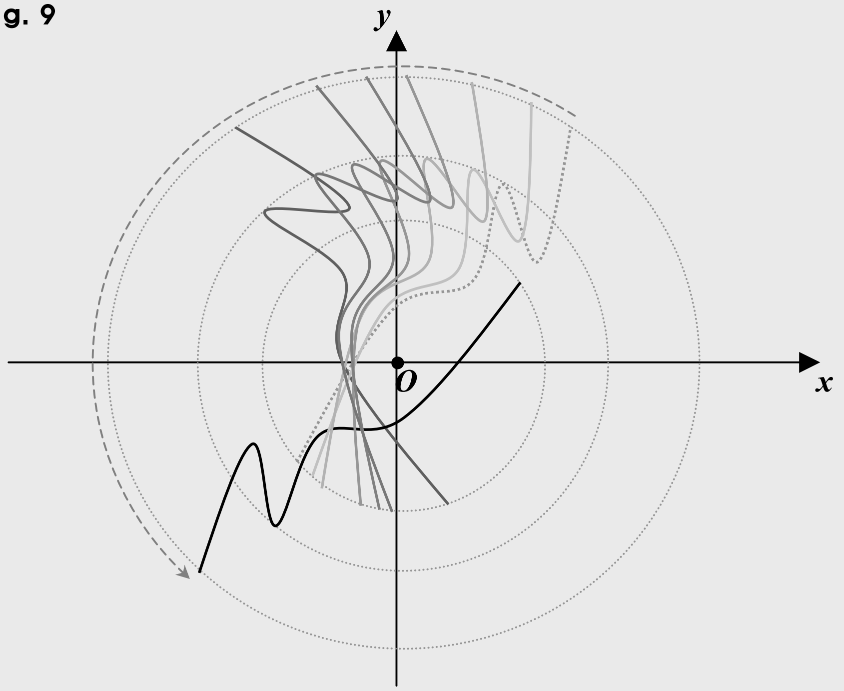
$\boldsymbol{T: (x, y) \longrightarrow (s, t) = (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta)}$.

And briefly, we can put it this way, too: $\boldsymbol{T: (x, y) \longrightarrow (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta)}$.

So we can call the expression above the definition of a transformation by turning around the origin in the amount of angle θ. And the core of the transformation is this: $(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$, which is the new arbitrary point, so we set $(s, t) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$. And it shows how each and every point in the original curve gets changed, that is, the way every point in the new curve gets made.

And applying T to the function f, and changing the angle θ, we can put the transformations the way below. (Note that the axis of turning is the origin.)

Fig. 9

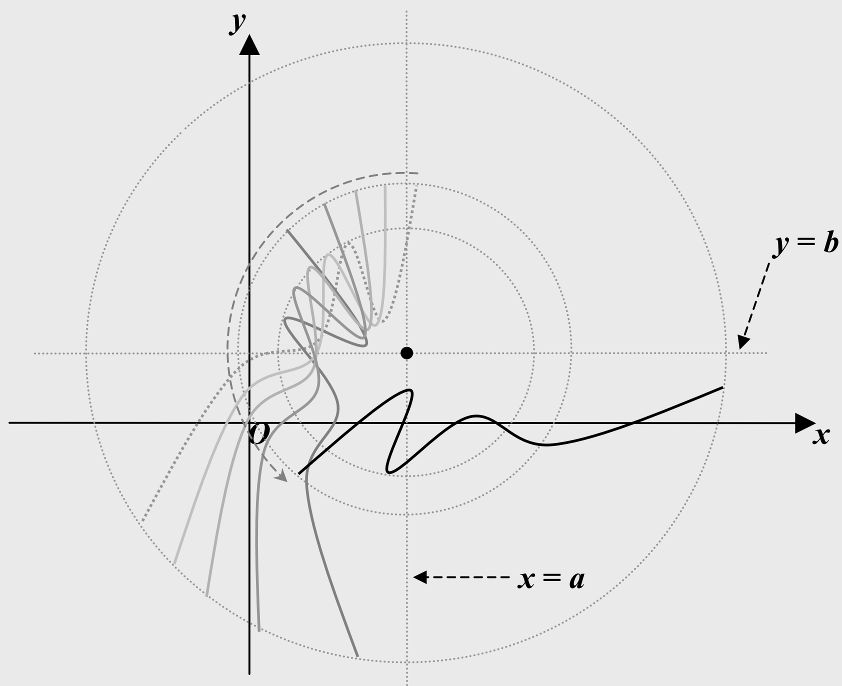


And of course, the way the original curve turns depends on the position of the axis of turning. That is, using a different point as the axis of turning, we get the original curve to turn differently. In other words, using as the axis of turning a point other than the origin, we get to use a different transformation, and will have to find another new arbitrary point other than the one as follows. $(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$.

Also, of course, the way the original curve turns depends on the position of the original curve, too. That is to say that putting in a different location in the same graph the original curve itself, we make it turn differently.

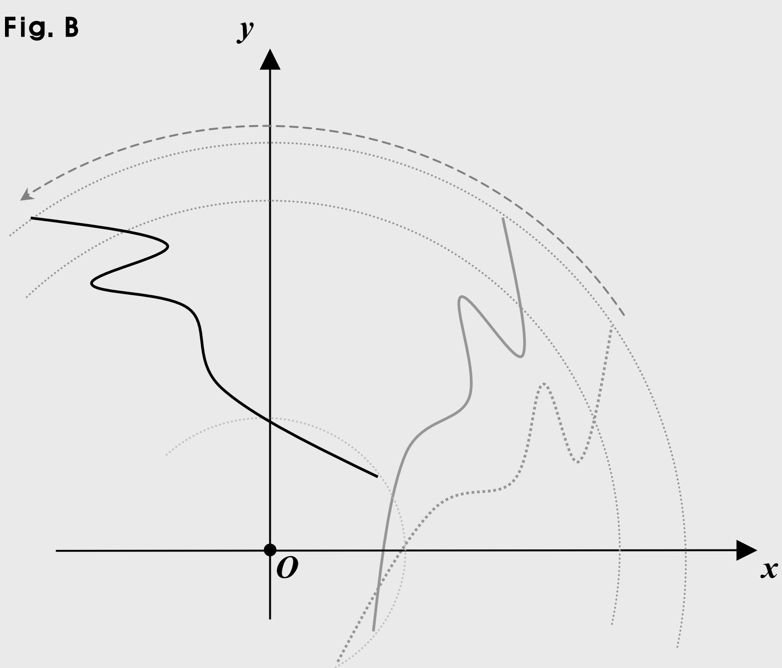
So putting the axis of turning at a point **(*a*, *b*)**, we get this:

Fig. A



And putting the original curve in a different location in the graph, we get this:

Fig. B



Let's now get the new curve, that is, the new equation.

To begin with, assuming (s, t) is the new arbitrary point, and of course, (x, y) is the original arbitrary point, we can set $(s, t) = (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta)$.

Then, we get $s = x \cos\theta - y \sin\theta$……(1), and $t = x \sin\theta + y \cos\theta$……(2)

Next, we want to get the equation connecting s and t. And we can get it mixing (s, t) into $y = f(x)$. We can get them mixed solving for x and y the system of the two equations above, and then, putting the solution into the original equation. Then, the mixture is the equation connecting s and t, and is the new equation.

So let's now solve for x and y the system of the two equations above.

Eliminating y first, we multiply (1) by $\cos\theta$, and multiply (2) by $\sin\theta$, and then, add the two products together.

So first, multiplying (1) by $\cos\theta$, we get $s \cos\theta = x \cos^2\theta - y \cos\theta \sin\theta$.

Multiplying next, (2) by $\sin\theta$, we get $t \sin\theta = x \sin^2\theta + y \cos\theta \sin\theta$.

Adding both together side by side, we get $s \cos\theta + t \sin\theta = x \cos^2\theta + x \sin^2\theta$.

And we have a trig-identity $\cos^2\theta + \sin^2\theta = 1$, so we get $s \cos\theta + t \sin\theta = x$.

Eliminating x this time, we multiply (1) by $\sin\theta$, and multiply (2) by $\cos\theta$, and then, subtract the first product from the second.

So first, multiplying (1) by $\sin\theta$, we get $s \sin\theta = x \sin\theta \cos\theta - y \sin^2\theta$.

Multiplying next, (2) by $\cos\theta$, we get $t \cos\theta = x \sin\theta \cos\theta + y \cos^2\theta$.

And doing the subtraction side by side, we get $t \cos\theta - s \sin\theta = y \cos^2\theta + y \sin^2\theta$.

Then, again, we have $\cos^2\theta + x \sin^2\theta = 1$, so we get $t \cos\theta - s \sin\theta = y$.

So the solution to the system is $x = s \cos\theta + t \sin\theta$, and $y = t \cos\theta - s \sin\theta$.

And we can now mix (s, t) into $y = f(x)$. Mixing them, we put the solution above into the original equation.   How?

In $y = f(x)$, we replace x with ($s \cos\theta + t \sin\theta$), and y with ($t \cos\theta - s \sin\theta$).

So doing the substitutions, we get $y = f(x) \Rightarrow t \cos\theta - s \sin\theta = f(s \cos\theta + t \sin\theta)$, which is the connective equation between s and t, and is the new equation.

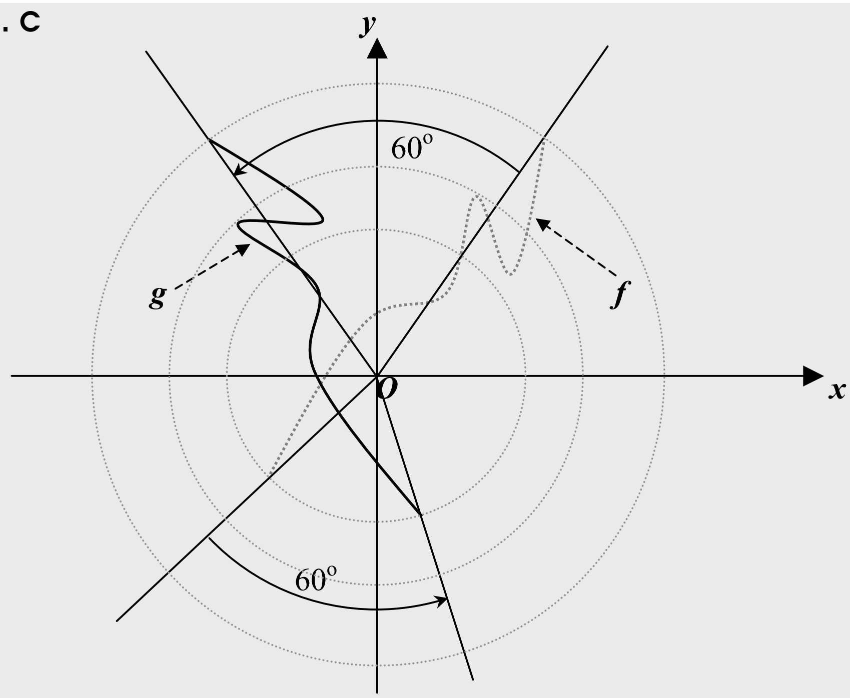
And we know that s is the variable x in the new equation, and t is the variable y.
So next, replacing s with x, and t with y, we get $y \cos\theta - x \sin\theta = f(x \cos\theta + y \sin\theta)$, which is the new equation. Assuming for instance, $\theta = 60^{\circ}$, we get

$x \cos\theta + y \sin\theta = x \cos 60^{\circ} + y \sin 60^{\circ} = \frac{1}{2}x + \frac{\sqrt{3}}{2}y = \frac{x + \sqrt{3}y}{2}$, and

$y \cos\theta - x \sin\theta = y \cos 60^{\circ} - x \sin 60^{\circ} = \frac{1}{2}y - \frac{\sqrt{3}}{2}x = \frac{y - \sqrt{3}x}{2}$.

So we get $\frac{y - \sqrt{3}x}{2} = f\left(\frac{x + \sqrt{3}y}{2}\right)$, which is the new curve, and thus, is the transformation of f by T, where the axis of turning is the origin, and the amount of turning, that is, the angle θ is 60°. And calling the new curve is g, and putting both f and g in a graph, we can put them the way below.

Fig. C



And assuming for instance, the original function is $\boldsymbol{y = h(x) = -2x + 3}$, $\boldsymbol{\theta = 60^o}$, and the axis of turning is the origin, we get $\frac{y-\sqrt{3}x}{2} = -2\cdot\frac{x+\sqrt{3}y}{2} + 3$.

So we get $\frac{y-\sqrt{3}x}{2} = -(x+\sqrt{3}\,y) + 3 \Rightarrow y-\sqrt{3}\,x = -2(x+\sqrt{3}\,y) + 6$

$\Rightarrow y + 2\sqrt{3}\,y = -2x + \sqrt{3}x + 6 \Rightarrow y(1+2\sqrt{3}\,) = x(\sqrt{3}-2) + 6$

$$\Rightarrow y = \frac{(\sqrt{3}-2)x+6}{1+2\sqrt{3}} = \frac{(\sqrt{3}-2)(2\sqrt{3}-1)x+6(2\sqrt{3}-1)}{(2\sqrt{3}+1)(2\sqrt{3}-1)} = \frac{(8-5\sqrt{3})x+6(2\sqrt{3}-1)}{11}$$

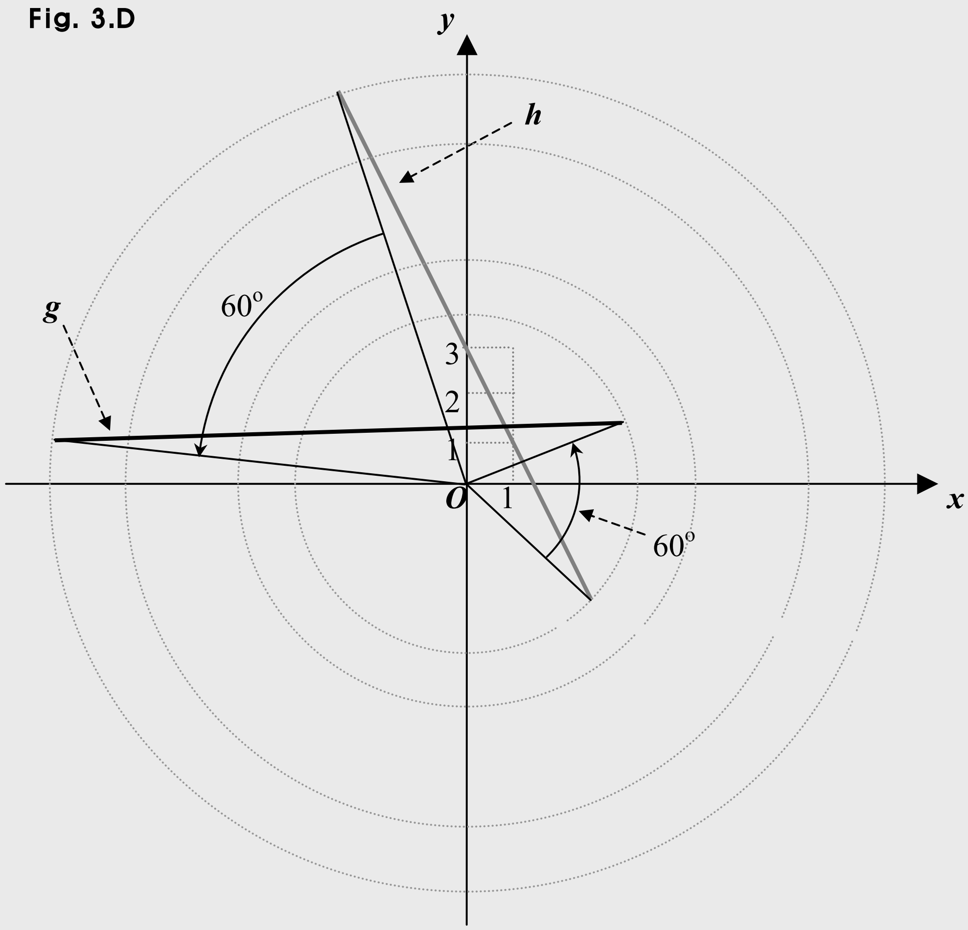
$$\Rightarrow y = \frac{(8-5\sqrt{3})}{11}x + \frac{6(2\sqrt{3}-1)}{11},$$

which is the new curve, which can be a curve of a function. Thus, assuming the new function is $\boldsymbol{g}$, we can put $\boldsymbol{g}$ the way as follows.

$$y = g(x) = \frac{(8-5\sqrt{3})}{11}x + \frac{6(2\sqrt{3}-1)}{11} \approx -0.06x + 1.344.$$

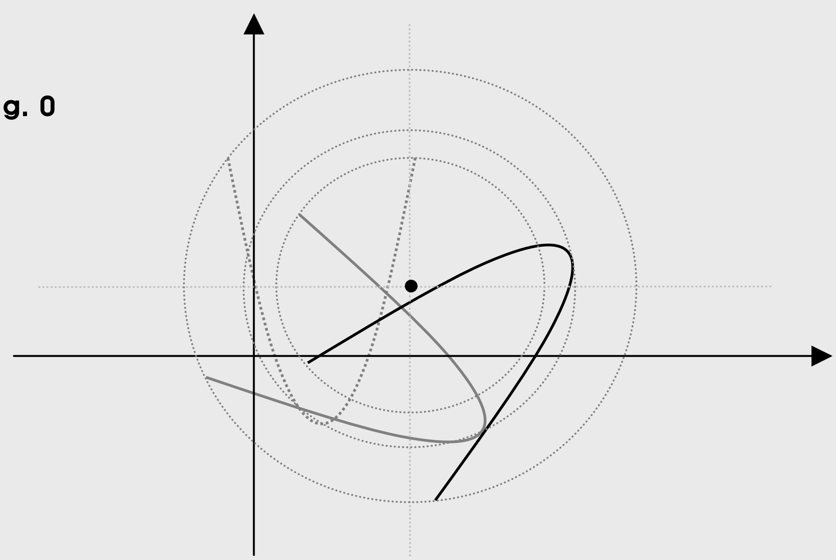
And putting the original and the new in a graph, we get the graph as follows.

Fig. 3.D



3.2. Transformations by turning 2

Fig. 0



Applying to a curve a transformation by turning, we turn the curve, and get a new curve, and the new curve is the original curve turned around a point in the amount of an angle.

So in a transformation by turning around the origin in an angle, the new curve is the original curve turned around the origin in the amount of the angle.

And assuming $\boldsymbol{T}$ is the transformation by turning around the origin in the amount of angle θ, we can put $\boldsymbol{T}$ the way as follows.

$$\boldsymbol{T}: (x, y) \longrightarrow (s, t) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta).$$

And briefly, we can put it this way, too: $\boldsymbol{T}: (x, y) \longrightarrow (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$.

So we can call the expression above the definition of a transformation by <u>turning around the origin in the amount of angle θ</u>. And $(x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta)$ is the new arbitrary point, and shows how each and every point in the original curve gets changed, and thus, explains the way the new curve gets made.

And applying T to a function f, we take the transformation of the function f by T.
And if g is the transformation of the function f by T, we can put it this way: $g = T \bullet f$.

So let's now for instance, find $g = T \bullet f$ where $y = f(x) = (x - 1)^2 + 1$.

Then, first, we can set $(s, t) = (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta)$, where (s, t) is the new point, and (x, y) is the original point.

Let's this time though, take (x_g, y_g) as the new point, and (x_f, y_f) as the original point.

Then, the original equation is $y_f = f(x_f)$, and the new is $y_g = g(x_g)$ or $g(x_g, y_g) = 0$.

And we can set $(x_g, y_g) = (x_f \cos\theta - y_f \sin\theta, x_f \sin\theta + y_f \cos\theta)$.

Then, we get $x_g = x_f \cos\theta - y_f \sin\theta$......(1), and $y_g = x_f \sin\theta + y_f \cos\theta$......(2)

Next, getting the new equation, we want to get the equation connecting x_g and y_g.

We can get the connective equation mixing (x_g, y_g) into the original equation, $y_f = f(x_f)$.
How then can we get the mixture?

Solving first, for x_f and y_f, the system of (1) and (2) above, and then, putting the solution into the original equation $y_f = f(x_f)$, we get the mixture. Then, the mixture is the equation connecting x_g and y_g, that is, the new equation $y_g = g(x_g)$ or $g(x_g, y_g) = 0$.

So solving first, the system for x_f and y_f, we get the solution as follows.
$x_f = x_g \cos\theta + y_g \sin\theta$, and $y_f = y_g \cos\theta - x_g \sin\theta$.

So we can now mix the new point into the original curve. And getting the mixture, we put the solution above, that is, the x_f and y_f above into the original equation $y_f = f(x_f)$.

Then, we get $y_f = f(x_f) \Rightarrow y_g \cos\theta - x_g \sin\theta = f(x_g \cos\theta + y_g \sin\theta)$, which is the equation connecting x_g and y_g, and is the mixture, which is the new equation.

Now, we have $\boldsymbol{y = f(x) = (x-1)^2 + 1}$. So the equation connecting $\boldsymbol{x_g}$ and $\boldsymbol{y_g}$ is

$$y_g \cos\theta - x_g \sin\theta = f(x_g \cos\theta + y_g \sin\theta) = (x_g \cos\theta + y_g \sin\theta - 1)^2 + 1 \;\ldots\ldots(1)$$

$\Rightarrow \boldsymbol{y_g \cos\theta - x_g \sin\theta = (x_g \cos\theta + y_g \sin\theta - 1)^2 + 1}$, which is the new equation.

So the new equation $\boldsymbol{g}$ is $\boldsymbol{g(x_g, y_g) = (x_g \cos\theta + y_g \sin\theta - 1)^2 + x_g \sin\theta - y_g \cos\theta + 1 = 0}$.

And we know the curve of $\boldsymbol{g}$ is in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane. So we can now just remove the subscripts to the variables. Then, we get

$$g(x, y) = (x \cos\theta + y \sin\theta - 1)^2 + x \sin\theta - y \cos\theta + 1 = 0.$$

And of course, we can remove the subscripts right after the substitutions, too.
That is, we can get the new equation the way below, too. From (1) above, we can get

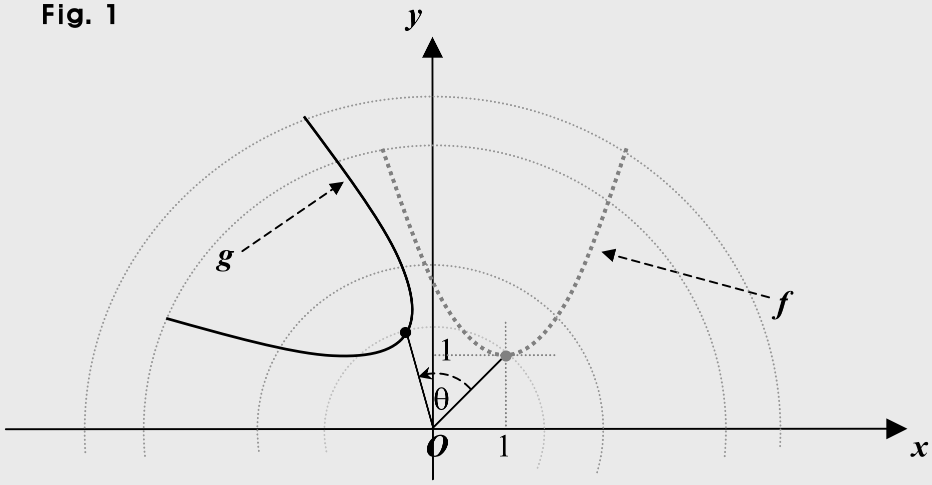
$$y \cos\theta - x \sin\theta = f(x \cos\theta + y \sin\theta) = (x \cos\theta + y \sin\theta - 1)^2 + 1$$
$$\Rightarrow y \cos\theta - x \sin\theta = (x \cos\theta + y \sin\theta - 1)^2 + 1.$$

Then, we get $\boldsymbol{g(x, y) = (x \cos\theta + y \sin\theta - 1)^2 + x \sin\theta - y \cos\theta + 1 = 0}$.

And assuming for instance, the angle $\boldsymbol{\theta = 60^o}$, we get $\boldsymbol{\cos\theta = \frac{1}{2}}$, and $\boldsymbol{\sin\theta = \frac{\sqrt{3}}{2}}$,

so we get $\boldsymbol{g(x, y) = \frac{1}{4}\{x^2 + 2\sqrt{3}xy + 3y^2 - (2 + 4\sqrt{3})y - (4 - 2\sqrt{3})x + 8 = 0}$.

Fig. 1



And we can see in the graph above, the new curve is not a curve of a function.
Let's next, for another instance, find $\boldsymbol{h = T \bullet f}$ where $\boldsymbol{y = f(x) = 2(x - 1)^2 - 1}$.
Then, assuming this time, $\boldsymbol{(s, t)}$ is the new point, we can begin with setting
$\boldsymbol{(s, t) = (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta)}$, where $\boldsymbol{(x, y)}$ is the original point.

Then, the original equation is $\boldsymbol{y = f(x)}$, and the new is $\boldsymbol{y = h(x)}$ or $\boldsymbol{h(x, y) = 0}$.

And we get $\boldsymbol{s = x \cos\theta - y \sin\theta}$......(1), and $\boldsymbol{t = x \sin\theta + y \cos\theta}$......(2)

Next, getting the new equation, we want to get the equation connecting $\boldsymbol{s}$ and $\boldsymbol{t}$.

We can get the connective equation mixing $\boldsymbol{(s, t)}$ into the original equation, $\boldsymbol{y = f(x)}$.
How then can we get the mixture?

Solving first, for $\boldsymbol{x}$ and $\boldsymbol{y}$, the system of (1) and (2) above, and then, putting the solution into the original equation $\boldsymbol{y = f(x)}$, we get the mixture. Then, the mixture is the equation connecting $\boldsymbol{s}$ and $\boldsymbol{t}$, that is, the new equation $\boldsymbol{x = h(x)}$ or $\boldsymbol{h(x, y) = 0}$.

So solving first, the system for $\boldsymbol{x}$ and $\boldsymbol{y}$, we get the solution as follows.
$\boldsymbol{x = s \cos\theta + t \sin\theta}$, and $\boldsymbol{y = t \cos\theta - s \sin\theta}$.

So we can now mix the new point into the original curve. And getting the mixture, we put the solution above, that is, the $\boldsymbol{x}$ and $\boldsymbol{y}$ above into the original equation $\boldsymbol{y = f(x)}$.

Then, we get $\boldsymbol{y = f(x) \Rightarrow t \cos\theta - s \sin\theta = f(s \cos\theta + t \sin\theta)}$, which is the equation connecting $\boldsymbol{s}$ and $\boldsymbol{t}$, and is the mixture, which is the new equation.

Now, we have $\boldsymbol{y = f(x) = 2(x - 1)^2 - 1}$. So the equation connecting $\boldsymbol{s}$ and $\boldsymbol{t}$ is

$$\boldsymbol{t \cos\theta - s \sin\theta = f(s \cos\theta + t \sin\theta) = (s \cos\theta + t \sin\theta - 1)^2 + 1}$$

$\boldsymbol{\Rightarrow t \cos\theta - s \sin\theta = (s \cos\theta + t \sin\theta - 1)^2 + 1}$, which is the new equation.

So the new equation $\boldsymbol{h}$ is $\boldsymbol{h(s, t) = (s \cos\theta + t \sin\theta - 1)^2 + s \sin\theta - t \cos\theta + 1 = 0}$.

And we know $\boldsymbol{s}$ is $\boldsymbol{x}$ in the new equation, and $\boldsymbol{t}$ is $\boldsymbol{y}$ in the new. So doing the replacements, we get $\boldsymbol{h(x, y) = (x \cos\theta + y \sin\theta - 1)^2 + x \sin\theta - y \cos\theta + 1 = 0}$.

And of course, we can do the replacements right after the substitutions, too.
That is, we can get the new equation the way below, too.

$$\boldsymbol{t\cos\theta - s\sin\theta = f(s\cos\theta + t\sin\theta) = (s\cos\theta + t\sin\theta - 1)2 + 1}$$

$$\boldsymbol{\Rightarrow y\cos\theta - x\sin\theta = (x\cos\theta + y\sin\theta - 1)^2 + 1}.$$

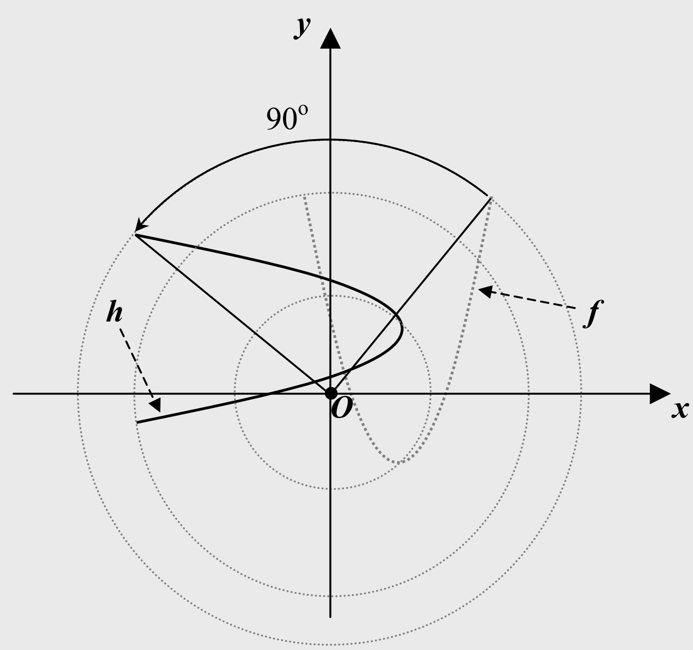
Then, we get $\boldsymbol{g(x, y) = (x\cos\theta + y\sin\theta - 1)^2 + x\sin\theta - y\cos\theta + 1 = 0}$.

And assuming for instance, the angle $\boldsymbol{\theta = 90^o}$, we get $\boldsymbol{\cos\theta = 0}$, and $\boldsymbol{\sin\theta = 1}$, so we get

$$\boldsymbol{g(x, y) = (x\cos\theta + y\sin\theta - 1)^2 + x\sin\theta - y\cos\theta + 1 = 0}$$

$$\boldsymbol{\Rightarrow g(x, y) = (y - 1)^2 + x + 1 = 0} \text{ or } \boldsymbol{x = -(y - 1)^2 - 1}.$$

Fig. 2



3.3. Transformations by turning 3

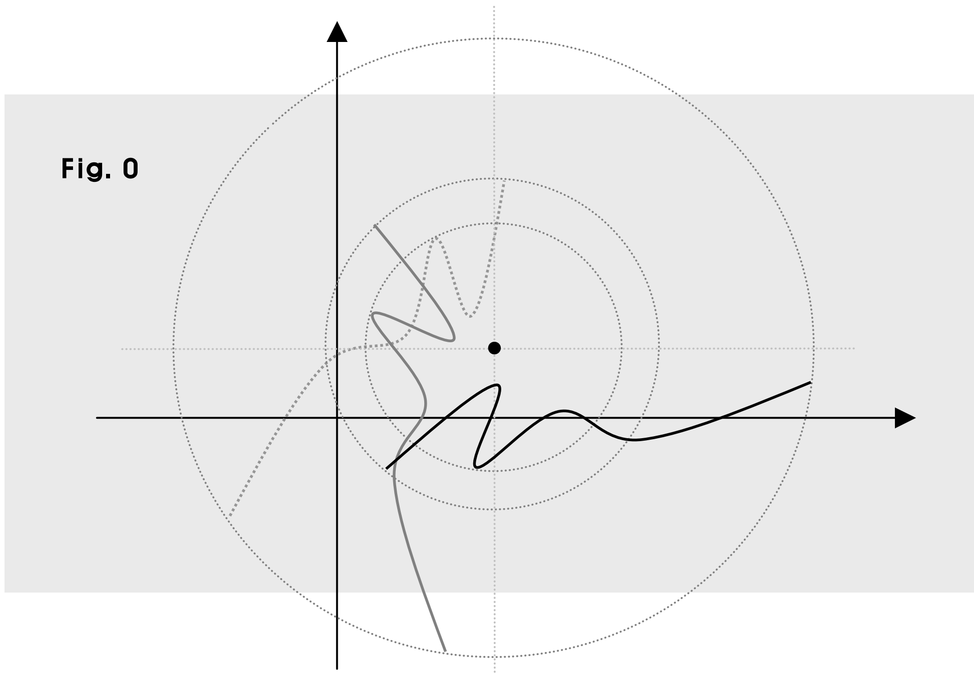


Fig. 0

Assuming $\boldsymbol{T}$ is the transformation by turning around the origin in the amount of angle θ, we can put $\boldsymbol{T}$ the way as follows.

$\boldsymbol{T}$: $(x, y) \longrightarrow (s, t) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$.

And briefly, we can put it this way, too: $\boldsymbol{T}$: $(x, y) \longrightarrow (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$.

What then about the case where the point is not the origin?
That is, the axis of turning is a point (a, b), and not just the origin only?

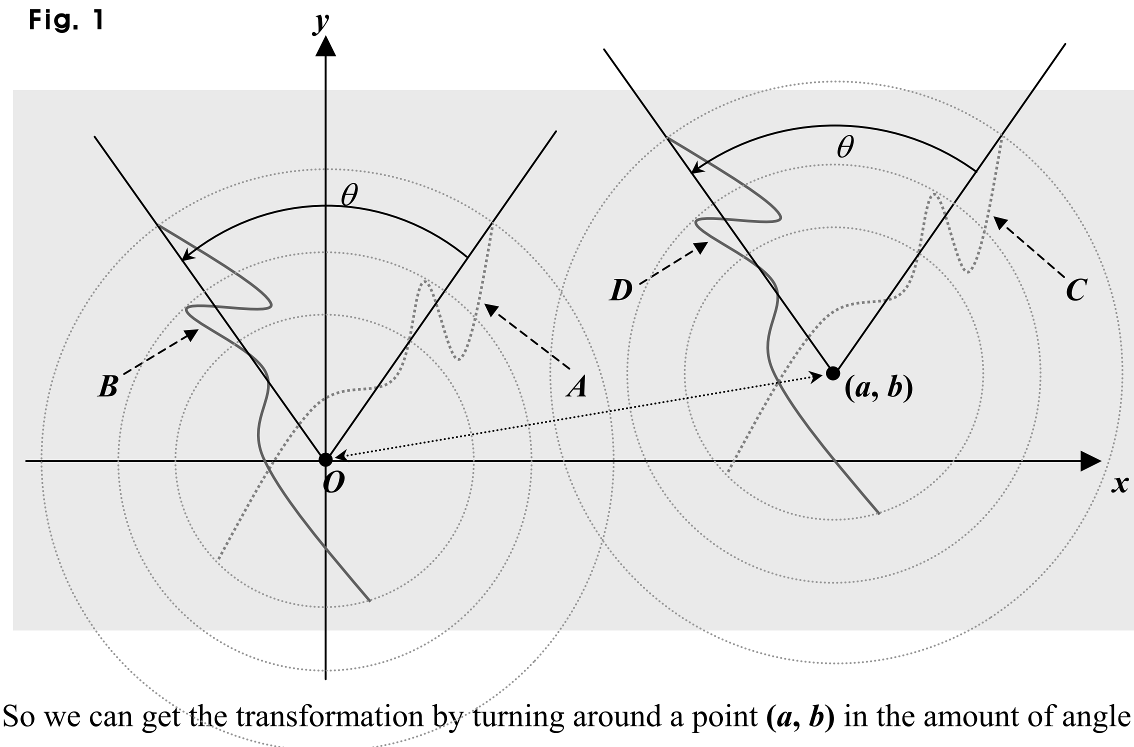
Suppose first, we get a curve $\boldsymbol{B}$ turning a curve $\boldsymbol{A}$ around the origin in an amount of angle θ. Then, of course, $\boldsymbol{B}$ is the new curve, and $\boldsymbol{A}$ is the original.

Suppose next, we translate at the same time, the two curves ***A*** and ***B*** in the amount of ***a*** along the ***x***-axis, and in the amount of ***b*** along the ***y***-axis.

Then, we get two new curves, of course. So suppose now, ***C*** is the new curve we get translating ***A*** the way above, and ***D*** is the new curve we get translating ***B*** that way. What then do we get if we turn the curve ***C*** around a point **(*a*, *b*)** in the amount of angle θ?

We get the curve ***D***. So it looks as if the axis turning got translated twice opposite ways while the curve is turning.

Fig. 1



So we can get the transformation by turning around a point **(*a*, *b*)** in the amount of angle θ the way as follows.

First, translate a curve in the amount of ***–a*** along the ***x***-axis, and in the amount of ***–b*** along the ***y***-axis, and then, turn the curve around the origin in the amount of angle θ.

Next, translate the curve in the amount of ***a*** along the ***x***-axis, and in the amount of ***b*** along the ***y***-axis.

Then, we get the original curve turned around the point **(*a*, *b*)** in the amount of angle θ.

So by applying the two translations and the transformation T to a curve, we can apply to the curve the transformation by turning around a point (a, b) in the amount of angle θ.

So assuming T_1 and T_2 are the transformations, we can put them the way below.

T_1: $(x, y) \longrightarrow (s, t) = (x - a, y - b)$, and T_1: $(x, y) \longrightarrow (s, t) = (x + a, y + b)$.

Applying thus, to an original curve the transformations T_1, T, and T_2 in sequence, we get the new curve, the original curve turned around the point (a, b) in the amount of angle θ.

So assuming R is the transformation by turning the way above, if applying R, we apply the three transformations in sequence. How then can we get the transformation R?

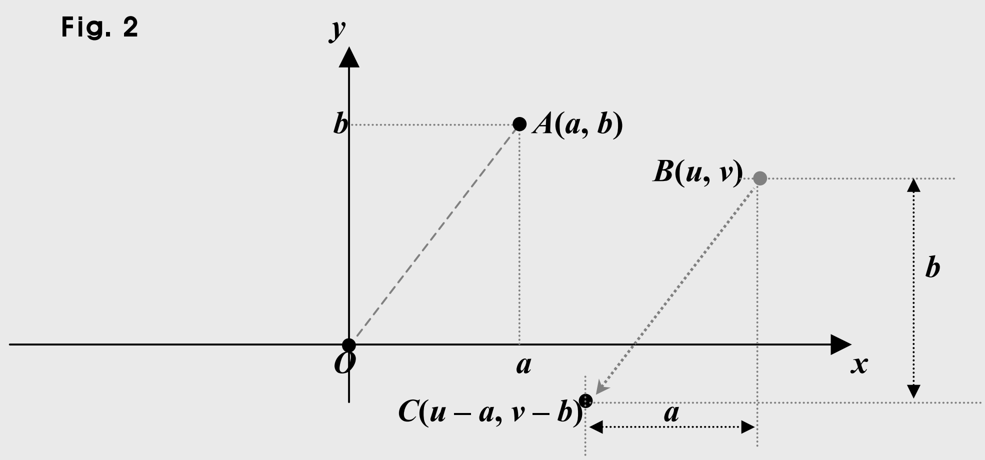
Suppose now, given a point (u, v), we want to find the point (m, n) applying to the point (u, v) the transformation R. Then, getting straight to the conclusion first, we get.

$m = (u - a)\cos\theta - (v - b)\sin\theta + a$, and $n = (u - a)\sin\theta + (v - b)\cos\theta + b$.

And let's see now how we can get the two equations above.

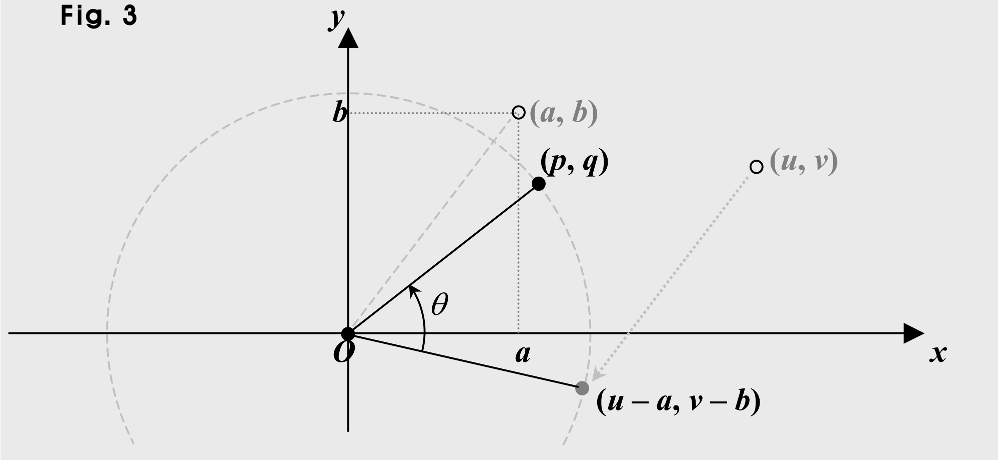
To begin with, we move the point (u, v) in the amount of $-a$ horizontally, and in the amount of $-b$ vertically. That is, the point (u, v) moves to $(u - a, v - b)$ as shown below.

Fig. 2



Then, assuming O is the origin, we get $\overline{AO} = \overline{BC}$, and $\overline{AO} \parallel \overline{BC}$ ($\overline{AO}$ is parallel to $\overline{BC}$). Next, turn the point $(u - a, v - b)$ around the origin in the amount of angle θ as below.

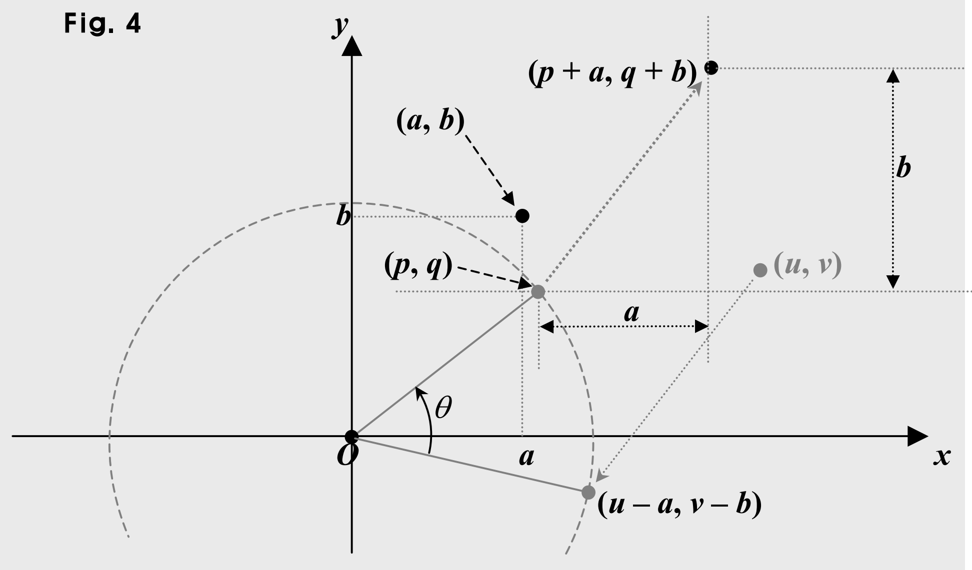
Fig. 3



Then, we get a new point, which is ***(p, q)*** in the graph above. And next, we want to translate ***(p, q)*** exactly the opposite of the way we translated ***(u, v)*** in the first place.

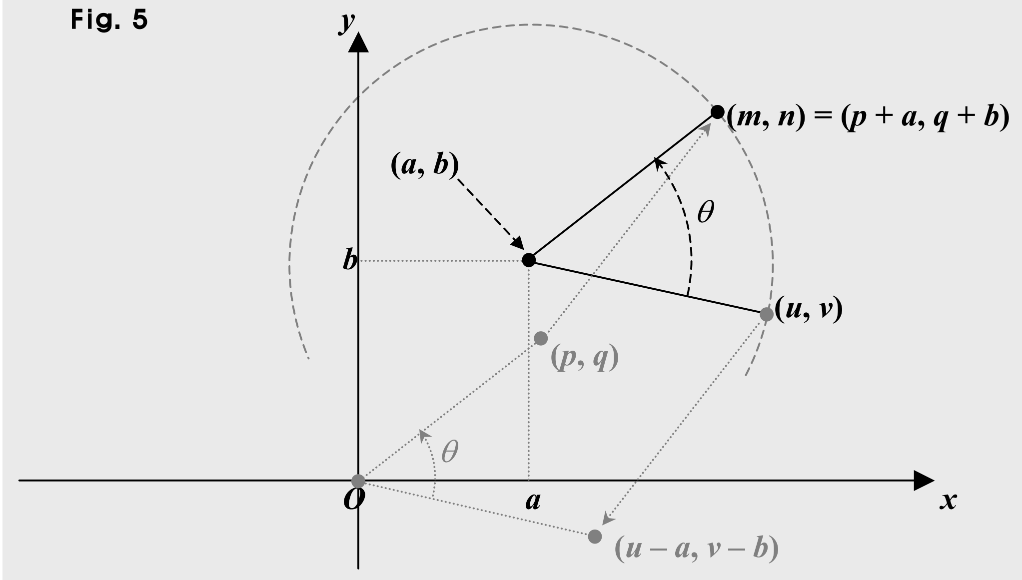
That is, we move the point ***(p, q)*** in the amount of ***a*** horizontally, and in the amount of ***b*** vertically.
Then, we get a new point, which is ***(p + a, q + b)*** as shown in the graph below.

Fig. 4



Then, the new point ***(p + a, q + b)*** is the very point we want to get applying to the point ***(u, v)*** the transformation where the axis of turning is ***(a, b)***, and the amount of turning is the angle θ, and thus, is the point ***(m, n)***.

Fig. 5



So we can notice that the transformation $\boldsymbol{R}$ can be said to be composed of three transformations. Two of them are translations, and the other is a transformation by turning. And assuming one of the translations is $\boldsymbol{T_1}$, we can put it the way below.

$\boldsymbol{T_1}$: $(x, y) \longrightarrow (s, t) = (x - a, y - b)$.

Assuming next, $\boldsymbol{T_2}$ is the other translation, we can put it the way below.

$\boldsymbol{T_2}$: $(x, y) \longrightarrow (s, t) = (x + a, y + b)$.

And assuming $\boldsymbol{T}$ is the transformation where the axis of turning is the origin $(0, 0)$, and the amount of turning is an angle θ, we can put it the way below.

$\boldsymbol{T}$: $(x, y) \longrightarrow (s, t) = (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta)$.

And in sequence, we do the three transformations above the way below.
We begin with $\boldsymbol{T_1}$, then $\boldsymbol{T}$, and then $\boldsymbol{T_2}$.

(And in fact, we can combine transformations to get one transformation where all the transformations combined can be performed. We call such a transformation a composite transformation, the details of which will be covered in the next section.)

So applying first, the transformation $\boldsymbol{T_1}$ to the point (u, v), and assuming the new point is (c, d), we get $(u, v) \longrightarrow (c, d) = (u - a, v - b)$.

And next, applying the transformation $\boldsymbol{T}$ to the point $(\boldsymbol{c}, \boldsymbol{d})$, that is, $(\boldsymbol{u} - \boldsymbol{a}, \boldsymbol{v} - \boldsymbol{b})$, and assuming the new point is $(\boldsymbol{p}, \boldsymbol{q})$, we get

$(\boldsymbol{u} - \boldsymbol{a}, \boldsymbol{v} - \boldsymbol{b}) \longrightarrow (\boldsymbol{p}, \boldsymbol{q}) = \{(\boldsymbol{u} - \boldsymbol{a})\mathbf{cos}\boldsymbol{\theta} - (\boldsymbol{v} - \boldsymbol{b})\mathbf{sin}\boldsymbol{\theta}, (\boldsymbol{u} - \boldsymbol{a})\mathbf{sin}\boldsymbol{\theta} + (\boldsymbol{v} - \boldsymbol{b})\mathbf{cos}\boldsymbol{\theta}\}$.

And next, applying the transformation $\boldsymbol{T_2}$ to the point $(\boldsymbol{p}, \boldsymbol{q})$, and assuming the new point is $(\boldsymbol{m}, \boldsymbol{n})$, we get $(\boldsymbol{p}, \boldsymbol{q}) \longrightarrow (\boldsymbol{m}, \boldsymbol{n}) = (\boldsymbol{p} + \boldsymbol{a}, \boldsymbol{q} + \boldsymbol{b})$.

And we know $\boldsymbol{p} = (\boldsymbol{u} - \boldsymbol{a})\mathbf{cos}\boldsymbol{\theta} - (\boldsymbol{v} - \boldsymbol{b})\mathbf{sin}\boldsymbol{\theta}$, and $\boldsymbol{q} = (\boldsymbol{u} - \boldsymbol{a})\mathbf{sin}\boldsymbol{\theta} + (\boldsymbol{v} - \boldsymbol{b})\mathbf{cos}\boldsymbol{\theta}$.

So we get $\boldsymbol{m} = (\boldsymbol{u} - \boldsymbol{a})\mathbf{cos}\boldsymbol{\theta} - (\boldsymbol{v} - \boldsymbol{b})\mathbf{sin}\boldsymbol{\theta} + \boldsymbol{a}$, and $\boldsymbol{n} = (\boldsymbol{u} - \boldsymbol{a})\mathbf{sin}\boldsymbol{\theta} + (\boldsymbol{v} - \boldsymbol{b})\mathbf{cos}\boldsymbol{\theta} + \boldsymbol{b}$.

And thus, applying to $(\boldsymbol{u}, \boldsymbol{v})$ all the transformations $\boldsymbol{A}$, $\boldsymbol{T}$, and $\boldsymbol{B}$ in sequence, we get

$$(\boldsymbol{u}, \boldsymbol{v}) \longrightarrow (\boldsymbol{m}, \boldsymbol{n}) = \{(\boldsymbol{u} - \boldsymbol{a})\mathbf{cos}\boldsymbol{\theta} - (\boldsymbol{v} - \boldsymbol{b})\mathbf{sin}\boldsymbol{\theta} + \boldsymbol{a}, (\boldsymbol{u} - \boldsymbol{a})\mathbf{sin}\boldsymbol{\theta} + (\boldsymbol{v} - \boldsymbol{b})\mathbf{cos}\boldsymbol{\theta} + \boldsymbol{b}\}.$$

What then is the correlation between the original arbitrary point and the new arbitrary point in the transformation described above?

The arbitrary point in a curve represents all the points in the curve. So assuming $(\boldsymbol{u}, \boldsymbol{v})$ is in the original curve, $(\boldsymbol{x}, \boldsymbol{y})$ is the original arbitrary point, $(\boldsymbol{s}, \boldsymbol{t})$ is the new arbitrary point, and $(\boldsymbol{m}, \boldsymbol{n})$ is in the new curve, we get the correlation as follows.

$$(\boldsymbol{s}, \boldsymbol{t}) = \{(\boldsymbol{x} - \boldsymbol{a})\mathbf{cos}\boldsymbol{\theta} - (\boldsymbol{y} - \boldsymbol{b})\mathbf{sin}\boldsymbol{\theta} + \boldsymbol{a}, (\boldsymbol{x} - \boldsymbol{a})\mathbf{sin}\boldsymbol{\theta} + (\boldsymbol{y} - \boldsymbol{b})\mathbf{cos}\boldsymbol{\theta} + \boldsymbol{b}\}.$$

And thus, assuming $\boldsymbol{R}$ is the transformation by turning around a point $(\boldsymbol{a}, \boldsymbol{b})$ in the amount of angle θ, we can put $\boldsymbol{R}$ the way as follows.

$$\boldsymbol{R}: (\boldsymbol{x}, \boldsymbol{y}) \longrightarrow (\boldsymbol{s}, \boldsymbol{t}) = \{(\boldsymbol{x} - \boldsymbol{a})\mathbf{cos}\boldsymbol{\theta} - (\boldsymbol{y} - \boldsymbol{b})\mathbf{sin}\boldsymbol{\theta} + \boldsymbol{a}, (\boldsymbol{x} - \boldsymbol{a})\mathbf{sin}\boldsymbol{\theta} + (\boldsymbol{y} - \boldsymbol{b})\mathbf{cos}\boldsymbol{\theta} + \boldsymbol{b}\}.$$

And briefly, we can put it this way, too:

$$\boldsymbol{R}: (\boldsymbol{x}, \boldsymbol{y}) \longrightarrow \{(\boldsymbol{x} - \boldsymbol{a})\mathbf{cos}\boldsymbol{\theta} - (\boldsymbol{y} - \boldsymbol{b})\mathbf{sin}\boldsymbol{\theta} + \boldsymbol{a}, (\boldsymbol{x} - \boldsymbol{a})\mathbf{sin}\boldsymbol{\theta} + (\boldsymbol{y} - \boldsymbol{b})\mathbf{cos}\boldsymbol{\theta} + \boldsymbol{b}\}.$$

So we can call it <u>the definition of the transformation by turning around a point $(\boldsymbol{a}, \boldsymbol{b})$ in the amount of angle θ</u>. And the new arbitrary point is

$\{(\boldsymbol{x} - \boldsymbol{a})\mathbf{cos}\boldsymbol{\theta} - (\boldsymbol{y} - \boldsymbol{b})\mathbf{sin}\boldsymbol{\theta} + \boldsymbol{a}, (\boldsymbol{x} - \boldsymbol{a})\mathbf{sin}\boldsymbol{\theta} + (\boldsymbol{y} - \boldsymbol{b})\mathbf{cos}\boldsymbol{\theta} + \boldsymbol{b}\}$, which shows thus, the mechanism where every point in the original curve gets changed, that is, the mechanism where every point in the new curve gets made. How then can we get the new equation?

Assuming (s, t) is the new arbitrary point, and of course, (x, y) is the original arbitrary point, we can set $(s, t) = \{(x - a)\cos\theta - (y - b)\sin\theta + a, (x - a)\sin\theta + (y - b)\cos\theta + b\}$.

And getting the equation connecting s and t, we get the new equation. We can get it mixing (s, t) into $y = f(x)$. And we can get them mixed using the setting above.

And using it, we get first, a system of two equations from it, and solve the system for x and y, and then, put the solution into the original equation $y = f(x)$. Then, we get the mixture, which is the connective equation between s and t.

So solving it now, we want to solve for x and y the system of the two equations below. $s = (x - a)\cos\theta - (y - b)\sin\theta + a$, and $t = (x - a)\sin\theta + (y - b)\cos\theta + b$, which is not quite simple. So for simplicity, setting $C = \cos\theta$, and $S = \sin\theta$, we can put it the way below.

$s - a = (x - a)C - (y - b)S$……(1), and $t - b = (x - a)S + (y - b)C$……(2)

Now, eliminating y first, we multiply (1) by C, and multiply (2) by S, and then, add the two products together.

So first, multiplying (1) by C, we get $(s - a)C = (x - a)C^2 - (y - b)SC$.

Multiplying next, (2) by S, we get $(t - b)S = (x - a)S^2 + (y - b)CS$.

And adding both together side by side, we get $(s - a)C + (t - b)S = (x - a)C^2 + (x - a)S^2$.

And using the trig identity $\cos^2\theta + \sin^2\theta = 1$, we get $(s - a)C + (t - b)S = x - a$.

And next, eliminating x this time, we multiply (1) by S, and multiply (2) by C, and then, subtract the first product from the second.

So first, multiplying (1) by S, we get $(s - a)S = (x - a)CS - (y - b)S^2$.

Multiplying next, (2) by C, we get $(t - b)C = (x - a)SC + (y - b)C^2$.

And doing the subtraction side by side, we get $(t - b)C - (s - a)S = (y - b)C^2 + (y - b)S^2$.

And using the trig identity $\cos^2\theta + x\sin^2\theta = 1$ again, we get $(t - b)C - (s - a)S = y - b$.

So putting threads together, we put the solution to the system the way as follows.

$x = (s - a)\cos\theta + (t - b)\sin\theta + a$, and $y = (t - b)\cos\theta - (s - a)\sin\theta + b$.

Thus, we can now mix the new arbitrary point into the original curve, that is, we can mix (s, t) into $y = f(x)$. And getting the mixture, we put the solution above into $y = f(x)$.

And putting the solution into $y = f(x)$, we do substitutions, so we get

$y = f(x) \Rightarrow (t - b)\cos\theta - (s - a)\sin\theta + b = f((s - a)\cos\theta + (t - b)\sin\theta + a)$.

Then, the equation above is the equation connecting s and t. And we know s is x in the new equation, and t is y in the new. So replacing s with x, and t with y, we get

$(y - b)\cos\theta - (x - a)\sin\theta + b = f\{(x - a)\cos\theta + (y - b)\sin\theta + a\}$.

Then, the equation above is the new equation, which is the transformation of f by R, and thus, we can call it the formula for the transformation where the axis of turning is (a, b) and the amount of turning is the angle θ.

Assuming for instance, the original function f is $y = f(x) = (x - 1)^2 + 1$, the axis of turning (a, b) is $(1, 2)$, and the angle θ is 60^o, we get $\cos\theta = \frac{1}{2}$, and $\sin\theta = \frac{\sqrt{3}}{2}$, so we get

$$(x - a)\cos\theta + (y - b)\sin\theta + a = \frac{1}{2}(x - 1) + \frac{\sqrt{3}}{2}(y - 2) + 1 = \frac{x + \sqrt{3}y + 1 - 2\sqrt{3}}{2},$$ and

$$(y - b)\cos\theta - (x - a)\sin\theta + b = \frac{1}{2}(y - 2) - \frac{\sqrt{3}}{2}(x - 1) + 2 = \frac{y - \sqrt{3}x + 2 + \sqrt{3}}{2}.$$

So we get $\frac{y - \sqrt{3}x + 2 + \sqrt{3}}{2} = \left(\frac{x + \sqrt{3}y + 1 - 2\sqrt{3}}{2} - 1\right)^2 + 1$, which is the new equation.

That is to say that turning around the point (1, 2), the curve of $y = f(x) = (x - 1)^2 + 1$ in the amount of 60^o, we get the curve of the new equation above.

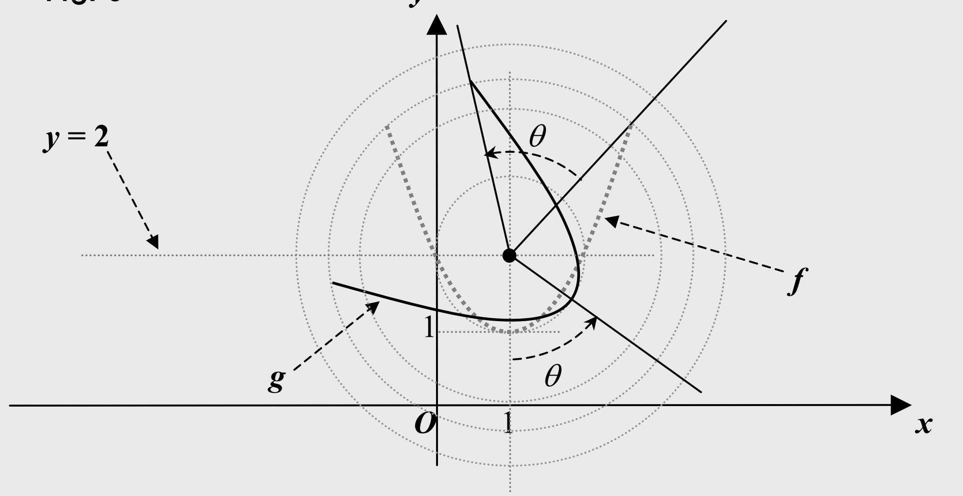
In short, assuming g is the new equation, we just call g the transformation of f by R.

And using the transformation operator, we can just put it the way below.

$g = f \bullet R$, where $(a, b) = (1, 2)$, and $\theta = 60^{o}$.

And putting f and g both in a graph, we can put them the way below.

Fig. 6



And for another instance, let's find $g = R \bullet f$ where $f(x, y) = \dfrac{y^2}{1 - x} + 2 = 0$.

Then, assuming (x, y) is the original point, and (s, t) is the new point, we can set first,

$(s, t) = \{(x - a)\cos\theta - (y - b)\sin\theta + a, (x - a)\sin\theta + (y - b)\cos\theta + b\}$.

Assuming this time, (x_g, y_g) is the new point, and (x_f, y_f) is the original, we can set first,

$(x_g, y_g) = \{(x_f - a)\cos\theta - (y_f - b)\sin\theta + a, (x_f - a)\sin\theta + (y_f - b)\cos\theta + b\}$.

Then, the right hand side is the new arbitrary point, and shows how every point in the original curve gets changed, that is, how every point in the new curve gets made.

And getting the equation connecting x_g and y_g, we get the new equation. We can get it mixing (x_g, y_g) into $f(x_f, y_f) = 0$. And we can get them mixed using the setting above.

And using it, we get first, a system of two equations from it, and solve the system for x_f and y_f, and then, put the solution into the original equation $f(x_f, y_f) = 0$. Then, we get the mixture, which is the connective equation between x_g and y_g.

So using it now, we want to solve first, for x_f and y_f the system of equations below.
$x_g = (x_f - a)\cos\theta - (y_f - b)\sin\theta + a$, and $y_g = (x_f - a)\sin\theta + (y_f - b)\cos\theta + b$.

And solving it, we get
$x_f = (x_g - a)\cos\theta + (y_g - b)\sin\theta + a$, and $y_f = (y_g - b)\cos\theta - (x_g - a)\sin\theta + b$.

So we can now mix the new point into the original curve, that is, we can mix (x_g, y_g) into $f(x_f, y_f) = 0$. And getting the mixture, we put the solution above into $f(x_f, y_f) = 0$.

And putting the solution into $f(x_f, y_f) = 0$, we do substitutions. Then, we get

$$f(x_f, y_f) = 0 \Rightarrow f\{(x_g - a)\cos\theta + (y_g - b)\sin\theta + a,\ (y_g - b)\cos\theta - (x_g - a)\sin\theta + b\} = 0,$$

which is the equation connecting x_g and y_g, that is, the new equation g. So we can put it the way below.

$$g(x_g, y_g) = f\{(x_g - a)\cos\theta + (y_g - b)\sin\theta + a,\ (y_g - b)\cos\theta - (x_g - a)\sin\theta + b\} = 0.$$

And normally, we put it the way below.

$$g(x, y) = f\{(x - a)\cos\theta + (y - b)\sin\theta + a,\ (y - b)\cos\theta - (x - a)\sin\theta + b\} = 0.$$

And we have $f(x, y) = \dfrac{y^2}{1 - x} + 2 = 0$.

So we get $f(x_g, y_g) = \dfrac{\{(y_g - b)\cos\theta - (x_g - a)\sin\theta + b\}^2}{1 - \{(x_g - a)\cos\theta + (y_g - b)\sin\theta + a\}} + 2 = 0$.

Thus, we get $g(x, y) = \dfrac{\{(y - b)\cos\theta - (x - a)\sin\theta + b\}^2}{1 - \{(x - a)\cos\theta + (y - b)\sin\theta + a\}} + 2 = 0$.

And assuming for instance, the axis of turning is $(1, 2)$, and the angle $\theta = 60^\circ$, we get $\cos\theta = \dfrac{1}{2}$, and $\sin\theta = \dfrac{\sqrt{3}}{2}$, so we get

$$g(x, y) = \frac{\{\frac{1}{2}(y-2) - \frac{\sqrt{3}}{2}(x-1) + 2\}^2}{1 - \{\frac{1}{2}(x-1) + \frac{\sqrt{3}}{2}(y-2) + 1\}} + 2 = \frac{\{\frac{1}{2}(y-2) - \frac{\sqrt{3}}{2}(x-1) + 2\}^2}{\frac{1}{2}(1-x) + \frac{\sqrt{3}}{2}(2-y)} + 2 = 0$$

And assuming $\{\frac{1}{2}(1-x)+\frac{\sqrt{3}}{2}(2-y)\} \neq 0$, we can put the equation above the way below.

$$\frac{\{\frac{1}{2}(y-2)-\frac{\sqrt{3}}{2}(x-1)+2\}^2}{\frac{1}{2}(1-x)+\frac{\sqrt{3}}{2}(2-y)}+2=0 \Rightarrow \frac{\{\frac{1}{2}(y-2)-\frac{\sqrt{3}}{2}(x-1)+2\}^2}{\frac{1}{2}(1-x)+\frac{\sqrt{3}}{2}(2-y)}=-2$$

$$\Rightarrow \{\tfrac{1}{2}(y-2)-\tfrac{\sqrt{3}}{2}(x-1)+2\}^2 = -2\{\tfrac{1}{2}(1-x)+\tfrac{\sqrt{3}}{2}(2-y)\} = (x-1)+\sqrt{3}(y-2).$$

Thus, we get $\{\frac{1}{2}(y-2)-\frac{\sqrt{3}}{2}(x-1)+2\}^2 = x+\sqrt{3}y-1-2\sqrt{3}$.

Meanwhile, $\frac{1}{2}(y-2)-\frac{\sqrt{3}}{2}(x-1)+2 = \dfrac{y-\sqrt{3}x+2+\sqrt{3}}{2}$. So we get

$$\left(\frac{y-\sqrt{3}x+2+\sqrt{3}}{2}\right)^2 = x+\sqrt{3}y-1-2\sqrt{3}$$

$$\Rightarrow (y-\sqrt{3}x+2+\sqrt{3})^2 = 4(x+\sqrt{3}y-1-2\sqrt{3}).$$

Meanwhile,
$$(y-\sqrt{3}x+2+\sqrt{3})^2 = y^2+3x^2+(2+\sqrt{3})^2-2\sqrt{3}xy-2\sqrt{3}(2+\sqrt{3})x+2(2+\sqrt{3})y.$$

Thus, we get,

$$y^2+3x^2+(2+\sqrt{3})^2-\ 2\sqrt{3}xy-2\sqrt{3}(2+\sqrt{3})x+2(2+\sqrt{3})y = 4(x+\sqrt{3}y-1-2\sqrt{3})$$

$$\Rightarrow y^2+3x^2-2\sqrt{3}xy-(4\sqrt{3}+6+4)x+(4+2\sqrt{3}-4\sqrt{3})y+(2+\sqrt{3})^2+4(1+2\sqrt{3})$$

$$= y^2+3x^2-2\sqrt{3}xy-(4\sqrt{3}+10)x+(4-2\sqrt{3})y+11+12\sqrt{3}=0.$$

So assuming $\{\frac{1}{2}(1-x)+\frac{\sqrt{3}}{2}(2-y)\} \neq 0$, that is, $x+\sqrt{3}y-1-2\sqrt{3} \neq 0$, we get

$$3x^2-2\sqrt{3}xy+y^2-(4\sqrt{3}+10)x+(4-2\sqrt{3})y+11+12\sqrt{3}=0.$$

And we can put g the way below.

$g(x, y) = 3x^2 - 2\sqrt{3}\,xy + y^2 - (4\sqrt{3} + 10)x + (4 - 2\sqrt{3}\,)y + 11 + 12\sqrt{3} = 0$, where $x + \sqrt{3}\,y - 1 - 2\sqrt{3} \neq 0$. What kind of curve then is the curve of g?

It is a parabola, and we can check to see if it is a parabola using the discriminant.

A parabola is one of basic curves called conic sections, which include lines, parabolas, circles, ellipses, and hyperbolas. And we can put in general, a conic section in such an equation as follows. $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where A, B, C, D, E, and F are constant.
Then, if we get $B^2 - 4AC = 0$, the equation above indicates a parabola. And the expression $B^2 - 4AC$ is called the discriminant. What if the discriminant is not 0?

If $B^2 - 4AC < 0$, it indicates an ellipse, and if $B^2 - 4AC > 0$, it indicates a hyperbola. How do we know if it indicates a circle though?

If we get $A = C$, and $B = 0$, it indicates a circle. (And if not sure of conic sections, refer to **CONICS**.) What curve then is the curve of f?

It is a parabola, too, of course, since if we get a parabola turning a curve, the curve is a parabola. Let's now though, check to see if the curve of f is a parabola, too.

We have $f(x, y) = \dfrac{y^2}{1-x} + 2 = 0$. So assuming $x \neq 1$, we can get

$\dfrac{y^2}{1-x} + 2 = 0 \Rightarrow \dfrac{y^2}{1-x} = -2 \Rightarrow y^2 = 2(x - 1) \Rightarrow x = \dfrac{1}{2}y^2 + 1$, which is a parabola.
Note however, the parabola above does not include a point $(1, 0)$.
That is because the actual equation of f is not $x = \dfrac{1}{2}y^2 + 1$ but $\dfrac{y^2}{1-x} + 2 = 0$, and also, if $x = 1$, the denominator $(1 - x)$ is 0, which is not allowed. So the curve of f is a parabola, but does not meet the line $x = 1$, and does not meet the line $y = 0$, either.

So the curve of f does not include the point (1, 0). What then about the curve of g?

The curve of g is a parabola, too, but is not continuous at a point. What point then is it?

It does not include the point where the line $\boldsymbol{x+\sqrt{3}\,y-1-2\sqrt{3}=0}$ meets the parabola $\boldsymbol{3x^2-2\sqrt{3}\,xy+y^2-(4\sqrt{3}+10)x+(4-2\sqrt{3}\,)y+11+12\sqrt{3}=0}$. Why not?

That's because the actual equation of $\boldsymbol{g}$ is not the one below.
$\boldsymbol{3x^2-2\sqrt{3}\,xy+y^2-(4\sqrt{3}+10)x+(4-2\sqrt{3}\,)y+11+12\sqrt{3}=0}$ but this:

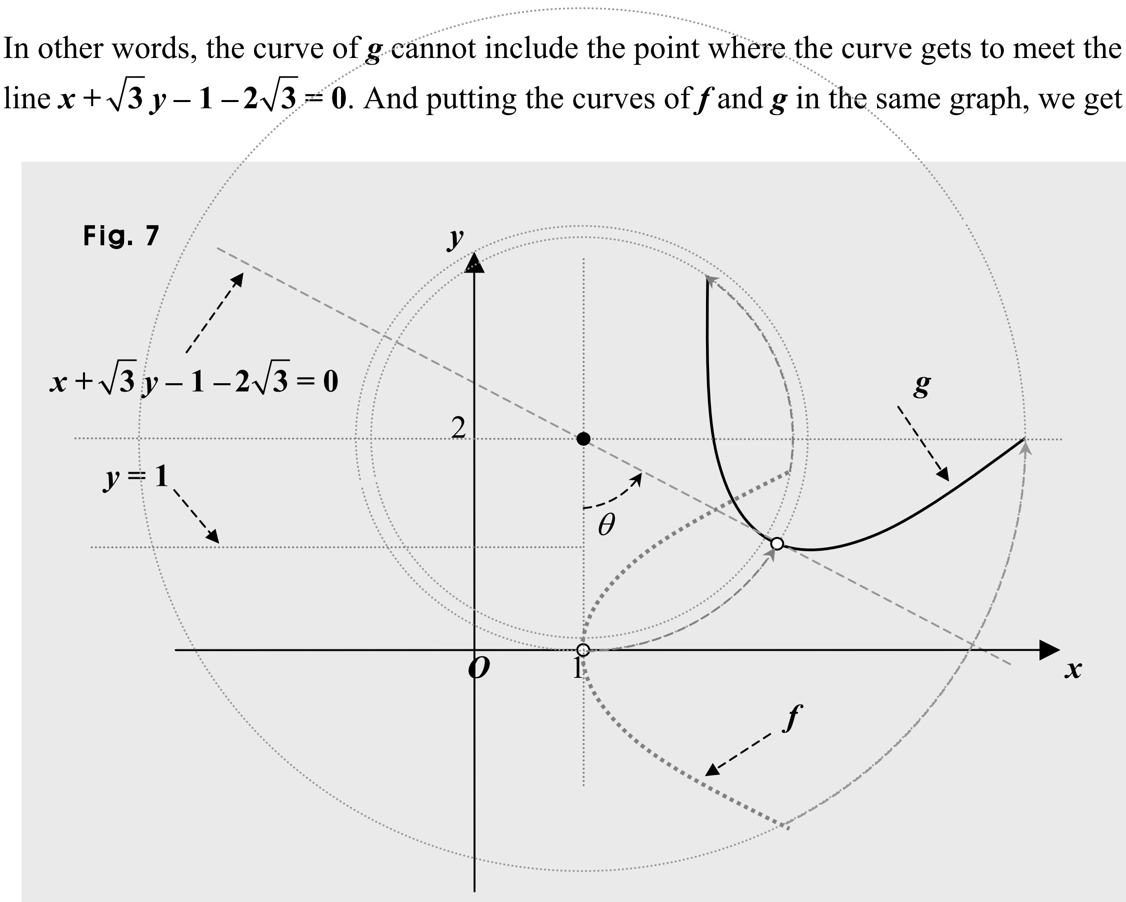
$$\frac{\{\frac{1}{2}(y-2)-\frac{\sqrt{3}}{2}(x-1)+2\}^2}{\frac{1}{2}(1-x)+\frac{\sqrt{3}}{2}(2-y)}+2=0.$$

And if $\boldsymbol{\frac{1}{2}(1-x)+\frac{\sqrt{3}}{2}(2-y)=0}$, that is, if $\boldsymbol{x+\sqrt{3}\,y-1-2\sqrt{3}=0}$, the denominator is 0.

So the curve of $\boldsymbol{g}$ is a parabola, but does not meet the line $\boldsymbol{x+\sqrt{3}\,y-1-2\sqrt{3}=0}$.

In other words, the curve of $\boldsymbol{g}$ cannot include the point where the curve gets to meet the line $\boldsymbol{x+\sqrt{3}\,y-1-2\sqrt{3}=0}$. And putting the curves of $\boldsymbol{f}$ and $\boldsymbol{g}$ in the same graph, we get

Fig. 7



4. Composite Transformations

Putting functions together, what do we get?

We can get a function. So putting functions together, we can make a function. And such a function is called a composite function. And the same is true for transformations, too. So putting together transformations, we can make a transformation, too, called therefore, a composite transformation.

We have already covered in fact, such transformations in some sections covered earlier. For instance, getting a transformation where a curve turns around a point other than the origin, we make a composite transformation combining the transformation by turning around the origin and two parallel transformations.

And as mentioned in the first section, a transformation is no other than a function.
So the way we make a composite transformation is just about the same as the way we make a composite function. So combining transformations, we put them in a sequence, and combine two at a time in sequence. (If not quite sure of composite functions, refer to the section in the book, **Basic Functions**.)

And the sequence matters.
Combining two transformations, we don't just combine them. We put one into the other. So putting together two transformations, we want to make sure which one of the two gets put into the other.

And putting one transformation into another, we apply the other to the one.
So for instance, putting a transformation $\boldsymbol{B}$ into a transformation $\boldsymbol{A}$, we apply $\boldsymbol{A}$ to $\boldsymbol{B}$.

And making a composite transformation, we do composite operations on transformations. And doing such an operation, we use the dot • as the operator, and the sequence matters. So for instance, combining two transformations $\boldsymbol{A}$ and $\boldsymbol{B}$ applying $\boldsymbol{A}$ to $\boldsymbol{B}$, and assuming $\boldsymbol{C}$ is the composite transformation, we can set $\boldsymbol{C} = \boldsymbol{A} \bullet \boldsymbol{B}$.

So assuming C is the composite transformation we get applying A to B, we set $C = A \bullet B$. And the sequence matters. So applying this time, B to A assuming D is the composite transformation, we set $D = B \bullet A$, and normally, we get $C \neq D$. That is, $A \bullet B \neq B \bullet A$.

Let's see now how we put together transformations, that is, how a composite transformation gets made, and how it works.

Assuming now, A is the transformation symmetric about the line $y = x$, and B is a parallel transformation, we can put them the way below.

A: $(x, y) \longrightarrow (y, x)$.
B: $(x, y) \longrightarrow (x + a, y + b)$, where a and b are constant.

Then, applying A to B, we get $A \bullet B$, and assuming $C = A \bullet B$, we put C the way below.
C: $(x, y) \longrightarrow (y + b, x + a)$. Also, we have $A \bullet B$: $(x, y) \longrightarrow (y + b, x + a)$.

And we can put the idea above the way below, too.

Applying B first, to a curve, and then, applying A to the result, we apply $A \bullet B$ to the curve.
In other words, if applying A to the transformation of a curve by B, we apply $A \bullet B$ to the curve, that is, we apply C to the curve.
So applying to a curve in sequence, B first, and A next, we apply $A \bullet B$ to the curve, that is, we apply C to the curve.
If for instance, $B \bullet f = g$, and $A \bullet g = h$, we get $h = A \bullet B \bullet f$, and of course, $h = C \bullet f$, too.
How?

Assuming first, B is applied to a curve called F, and (s, t) is the new point, we get $(s, t) = (x + a, y + b)$, and (x, y) is the original point, of course.

Then, we get $s = x + a$, and $t = y + b$.

Suppose now, G is the new curve.
Then, applying A to the curve G, we get another new point.
What then is the original point in this case?
We know that the curve G has (s, t), which is $(x + a, y + b)$, and G is the original curve

now. So the original point now is (s, t), that is, $(x + a, y + b)$.

Now, we know A: $(x, y) \longrightarrow (y, x)$.

So in A, the coordinates of the original point swap their positions.

So applying A to G now, assuming (u, v) is the other new point stated above, we get $(s, t) \longrightarrow (u, v) = (t, s)$. And we know $s = x + a$, and $t = y + b$.

So we get $(u, v) = (y + b, x + a)$. Suppose now, H is the new curve.
Then, we can say that applying A to G, we get H, which has (u, v), that is, $(y + b, x + a)$.

So in sum, applying first B to F, we get G, which has (s, t), that is, $(x + a, y + b)$, and applying next, A to G, we get H, which has (u, v), that is, $(y + b, x + a)$.

In short, to the curve F, applying B first, and A next, we get $(y + b, x + a)$.

And applying B first, and A next, we apply $A \bullet B$, and we have set $C = A \bullet B$.
So applying C now, assuming (m, n) is the new point, we get $(m, n) = (y + b, x + a)$.

Thus, we can put C this way: C: $(x, y) \longrightarrow (m, n) = (y + b, x + a)$.

In short, we put it this way: C: $(x, y) \longrightarrow (y + b, x + a)$.

And a transformation is no other than a function, and we can apply it to a function, so we can apply it to a transformation, too.

Applying a transformation to a function, we apply it to a point, and the point is the arbitrary point in the curve of the function. So applying a transformation to another transformation, we apply it to a point, too. What point then is it?

It is the new arbitrary point the other transformation makes.
So we can put a composite transformation the way blow.

Transforming a transformation, we get a composite transformation. And the composite transformation is the transformation transformed.

And applying a transformation to another transformation, we apply it to the new point the other transformation makes. Then, we get the new point the composite transformation makes.
Using thus, the fact above, we can get the composite transformation $\boldsymbol{C}$ the way below.

Assuming $\boldsymbol{C}$ is the composite transformation we get applying $\boldsymbol{A}$ to $\boldsymbol{B}$, we set $\boldsymbol{C = A \bullet B}$.

So assuming $\boldsymbol{A}$ is applied to $\boldsymbol{B}$, $\boldsymbol{(p, q)}$ is the new point $\boldsymbol{B}$ makes, and $\boldsymbol{(s, t)}$ is the new point $\boldsymbol{A}$ makes, we get $\boldsymbol{A \bullet B}$: $\boldsymbol{(p, q) \longrightarrow (s, t) = (q, p)}$, because $\boldsymbol{A}$: $\boldsymbol{(x, y) \longrightarrow (y, x)}$.

So we can put $\boldsymbol{C}$ the way as follows. $\boldsymbol{C}$: $\boldsymbol{(p, q) \longrightarrow (s, t) = (q, p)}$.

And we have $\boldsymbol{B}$: $\boldsymbol{(x, y) \longrightarrow (p, q) = (x + a, y + b)}$, and $\boldsymbol{(x, y)}$ is the original point.

So we get $\boldsymbol{p = x + a}$, and $\boldsymbol{q = y + b}$. Thus, we get $\boldsymbol{(s, t) = (q, p) = (y + b, x + a)}$.

And we know $\boldsymbol{(x, y)}$ is the very first original point. So we can put all the processes in $\boldsymbol{C}$ this way: $\boldsymbol{(x, y) \xrightarrow{B} (p, q) = (x + a, y + b) \xrightarrow{A} (s, t) = (q, p) = (y + b, x + a)}$.

In short, we can put $\boldsymbol{C}$ this way: $\boldsymbol{C}$: $\boldsymbol{(x, y) \longrightarrow (y + b, x + a)}$.

> What if this time, $\boldsymbol{A}$ gets applied first, and $\boldsymbol{B}$ gets applied next?

We know applying $\boldsymbol{B}$ first, and $\boldsymbol{A}$ next, we apply $\boldsymbol{A \bullet B}$. So applying $\boldsymbol{A}$ first, and $\boldsymbol{B}$ next, we apply $\boldsymbol{B \bullet A}$. And we have

$\boldsymbol{A}$: $\boldsymbol{(x, y) \longrightarrow (y, x)}$.
$\boldsymbol{B}$: $\boldsymbol{(x, y) \longrightarrow (x + a, y + b)}$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are constant.
So assuming $\boldsymbol{D = B \bullet A}$, we can put $\boldsymbol{D}$ this way: $\boldsymbol{D}$: $\boldsymbol{(x, y) \longrightarrow (y + a, x + b)}$. How?

Assuming this time, $\boldsymbol{A}$ is applied first to the curve $\boldsymbol{F}$, and $\boldsymbol{(s, t)}$ is the new point, we get $\boldsymbol{(s, t) = (y, x)}$, and $\boldsymbol{(x, y)}$ is the original point, of course.

And suppose now, $\boldsymbol{K}$ is the new curve.
Then, applying $\boldsymbol{B}$ to the curve $\boldsymbol{K}$, we get another new point.
And we know that the curve $\boldsymbol{K}$ has $\boldsymbol{(s, t)}$, which is $\boldsymbol{(y, x)}$, and $\boldsymbol{K}$ is the original curve now.

So the original point now is (s, t), that is, (y, x).

And we know B: $(x, y) \longrightarrow (x + a, y + b)$.

So applying B to K now, assuming (u, v) is the other new point stated above, we get $(s, t) \longrightarrow (u, v) = (s + a, t + b)$. And we know $s = y$, and $t = x$.

So we get $(u, v) = (y + a, x + b)$. Suppose now, L is the new curve.
Then, we can say that applying B to K, we get L, which has (u, v), that is, $(y + a, x + b)$.

So in sum, applying first A to F, we get K, which has (s, t), that is, (y, x), and applying next, B to K, we get L, which has (u, v), that is, $(y + a, x + b)$.

In short, to the curve F, applying A first, and B next, we get $(y + a, x + b)$.

And applying A first, and B next, we apply $B \bullet A$, and we have set $D = B \bullet A$.

So applying D now, assuming (m, n) is the new point, we get $(m, n) = (y + a, x + b)$.

Thus, we can put D this way: D: $(x, y) \longrightarrow (m, n) = (y + a, x + b)$.
And quickly, we put it this way: D: $(x, y) \longrightarrow (y + a, x + b)$.

And we can get D the way below, too.

Assuming D is the composite transformation we get applying B to A, we set $D = B \bullet A$.

And we know B: $(x, y) \longrightarrow (x + a, y + b)$.

So assuming now, B is applied to A, (p, q) is the new point A makes, and (s, t) is the new point B makes, we get $B \bullet A$: $(p, q) \longrightarrow (s, t) = (p + a, q + b)$.

So we can put D the way as follows. D: $(p, q) \longrightarrow (s, t) = (p + a, q + b)$.
And we have A: $(x, y) \longrightarrow (p, q) = (y, x)$, and (x, y) is the original point.

So we get $p = y$, and $q = x$. Thus, we get $(s, t) = (p + a, q + b) = (y + a, x + b)$.

And we know (x, y) is the very first original point. So we can put all the processes in D this way: $(x, y) \xrightarrow{A} (p, q) = (y, x) \xrightarrow{B} (s, t) = (p + a, q + b) = (y + a, x + b)$.

In short, we can put $\boldsymbol{D}$ this way: $\boldsymbol{D}$: $(\boldsymbol{x}, \boldsymbol{y}) \longrightarrow (\boldsymbol{y} + \boldsymbol{a}, \boldsymbol{x} + \boldsymbol{b})$.

And we can put $\boldsymbol{D} = \boldsymbol{B} \bullet \boldsymbol{A}$ the way below, too.

$\boldsymbol{D}$: $(\boldsymbol{x}, \boldsymbol{y}) \xrightarrow{\boldsymbol{A}} (\boldsymbol{y}, \boldsymbol{x}) \xrightarrow{\boldsymbol{B}} (\boldsymbol{y} + \boldsymbol{a}, \boldsymbol{x} + \boldsymbol{b})$.

And by the same token, we can put $\boldsymbol{C} = \boldsymbol{A} \bullet \boldsymbol{B}$ the way below, too.

$\boldsymbol{C}$: $(\boldsymbol{x}, \boldsymbol{y}) \xrightarrow{\boldsymbol{B}} (\boldsymbol{x} + \boldsymbol{a}, \boldsymbol{y} + \boldsymbol{b}) \xrightarrow{\boldsymbol{A}} (\boldsymbol{y} + \boldsymbol{b}, \boldsymbol{x} + \boldsymbol{a})$.

(Note that we can use any letter as a variable or constant, as long as the use is consistent, and causes no ambiguity.)

What then do we get applying $\boldsymbol{D} \bullet \boldsymbol{A}$?

We have
$\boldsymbol{A}$: $(\boldsymbol{x}, \boldsymbol{y}) \longrightarrow (\boldsymbol{y}, \boldsymbol{x})$.
$\boldsymbol{D}$: $(\boldsymbol{x}, \boldsymbol{y}) \longrightarrow (\boldsymbol{y} + \boldsymbol{a}, \boldsymbol{x} + \boldsymbol{b})$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are constant.

So assuming the new point is $(\boldsymbol{s}, \boldsymbol{t})$, we get $(\boldsymbol{s}, \boldsymbol{t}) = (\boldsymbol{x} + \boldsymbol{a}, \boldsymbol{y} + \boldsymbol{b})$. How?

Assuming $\boldsymbol{E}$ is the composite transformation we get applying $\boldsymbol{D}$ to $\boldsymbol{A}$, we set $\boldsymbol{E} = \boldsymbol{D} \bullet \boldsymbol{A}$.

So assuming now, $\boldsymbol{D}$ is applied to $\boldsymbol{A}$, $(\boldsymbol{p}, \boldsymbol{q})$ is the new point $\boldsymbol{A}$ makes, and $(\boldsymbol{s}, \boldsymbol{t})$ is the new point $\boldsymbol{D}$ makes, we get $\boldsymbol{D} \bullet \boldsymbol{A}$: $(\boldsymbol{p}, \boldsymbol{q}) \longrightarrow (\boldsymbol{s}, \boldsymbol{t}) = (\boldsymbol{q} + \boldsymbol{a}, \boldsymbol{p} + \boldsymbol{b})$.

So we can put $\boldsymbol{E}$ the way as follows. $\boldsymbol{E}$: $(\boldsymbol{p}, \boldsymbol{q}) \longrightarrow (\boldsymbol{s}, \boldsymbol{t}) = (\boldsymbol{q} + \boldsymbol{a}, \boldsymbol{p} + \boldsymbol{b})$.

And we have $\boldsymbol{A}$: $(\boldsymbol{x}, \boldsymbol{y}) \longrightarrow (\boldsymbol{p}, \boldsymbol{q}) = (\boldsymbol{y}, \boldsymbol{x})$, and $(\boldsymbol{x}, \boldsymbol{y})$ is the original point.
So we get $\boldsymbol{p} = \boldsymbol{y}$, and $\boldsymbol{q} = \boldsymbol{x}$. Thus, we get $(\boldsymbol{s}, \boldsymbol{t}) = (\boldsymbol{q} + \boldsymbol{a}, \boldsymbol{p} + \boldsymbol{b}) = (\boldsymbol{x} + \boldsymbol{a}, \boldsymbol{y} + \boldsymbol{b})$.

And we know $(\boldsymbol{x}, \boldsymbol{y})$ is the very first original point. So we can put all the processes in $\boldsymbol{E}$ this way: $(\boldsymbol{x}, \boldsymbol{y}) \xrightarrow{\boldsymbol{A}} (\boldsymbol{p}, \boldsymbol{q}) = (\boldsymbol{y}, \boldsymbol{x}) \xrightarrow{\boldsymbol{D}} (\boldsymbol{s}, \boldsymbol{t}) = (\boldsymbol{q} + \boldsymbol{a}, \boldsymbol{p} + \boldsymbol{b}) = (\boldsymbol{x} + \boldsymbol{a}, \boldsymbol{y} + \boldsymbol{b})$.

In short, we can put $\boldsymbol{E}$ this way: $\boldsymbol{E}$: $(\boldsymbol{x}, \boldsymbol{y}) \longrightarrow (\boldsymbol{x} + \boldsymbol{a}, \boldsymbol{y} + \boldsymbol{b})$.

And let's this time, take a look at the transformation above the way below.

We know applying $D \bullet A$, we apply first A, and D next.

So assuming again, A is applied first to the original curve F, and (s, t) is the new point, we get $(s, t) = (y, x)$, and (x, y) is the original point, of course.

Thus, we get $s = y$, and $t = x$.
And suppose again, H is the new curve, and D is now applied to the curve H.

Then, the original point now is (s, t), which is (y, x).
And we know D: $(x, y) \longrightarrow (y + a, x + b)$.

So applying D to H assuming (u, v) is the new point, we get
$(s, t) \longrightarrow (u, v) = (t + a, s + b)$. And we know $s = y$, and $t = x$.

So we get $(u, v) = (x + a, y + b)$.

Thus, applying A first, and D next, we get $(x + a, y + b)$.

And we know applying A first, and D next, we apply $D \bullet A$. So setting $E = D \bullet A$, and applying E to F assuming (m, n) is the new point, we get $(m, n) = (x + a, y + b)$.

So we can put E this way: E: $(x, y) \longrightarrow (x + a, y + b)$.

We have however, B: $(x, y) \longrightarrow (x + a, y + b)$.
So E is in fact, the same as B.
So we get $B = E = D \bullet A \Rightarrow B = D \bullet A$. And we know $D = B \bullet A$.

So we get $B = D \bullet A = B \bullet A \bullet A \Rightarrow B = B \bullet A \bullet A$.

Thus, we can notice that we get $B = (B \bullet A) \bullet A = B \bullet (A \bullet A)$, which means that composite-operations on transformations are associative <u>in this particular case</u>.
Let's this time, take <u>another particular case</u>, so let's now check to see if we get
$A \bullet B \bullet C = A \bullet (B \bullet C) = (A \bullet B) \bullet C$.

We have,
$\boldsymbol{A}$: $(\boldsymbol{x}, \boldsymbol{y}) \longrightarrow (\boldsymbol{y}, \boldsymbol{x})$.
$\boldsymbol{B}$: $(\boldsymbol{x}, \boldsymbol{y}) \longrightarrow (\boldsymbol{x} + \boldsymbol{a}, \boldsymbol{y} + \boldsymbol{b})$.
$\boldsymbol{C}$: $(\boldsymbol{x}, \boldsymbol{y}) \longrightarrow (\boldsymbol{y} + \boldsymbol{b}, \boldsymbol{x} + \boldsymbol{a})$.

To begin with, applying $\boldsymbol{A \bullet (B \bullet C)}$, we apply $\boldsymbol{(B \bullet C)}$ first, and $\boldsymbol{A}$ next.
And beginning with $\boldsymbol{B \bullet C}$, we apply $\boldsymbol{C}$ first, and $\boldsymbol{B}$ next.

So applying $\boldsymbol{C}$ first, assuming $(\boldsymbol{x}, \boldsymbol{y})$ is the original point, and $(\boldsymbol{s}, \boldsymbol{t})$ is the new point, we get $(\boldsymbol{s}, \boldsymbol{t}) = (\boldsymbol{y} + \boldsymbol{b}, \boldsymbol{x} + \boldsymbol{a})$. That is, we get $\boldsymbol{s} = \boldsymbol{y} + \boldsymbol{b}$, and $\boldsymbol{t} = \boldsymbol{x} + \boldsymbol{a}$.

Next, applying $\boldsymbol{B}$ to the curve where the arbitrary point is $(\boldsymbol{s}, \boldsymbol{t})$, and assuming $(\boldsymbol{u}, \boldsymbol{v})$ is the new arbitrary point, we get $(\boldsymbol{s}, \boldsymbol{t}) \longrightarrow (\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{s} + \boldsymbol{a}, \boldsymbol{t} + \boldsymbol{b})$.

And we know $\boldsymbol{s} = \boldsymbol{y} + \boldsymbol{b}$, and $\boldsymbol{t} = \boldsymbol{x} + \boldsymbol{a}$. So we get $(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{y} + \boldsymbol{b} + \boldsymbol{a}, \boldsymbol{x} + \boldsymbol{a} + \boldsymbol{b})$.

Thus, we get $\boldsymbol{u} = \boldsymbol{y} + \boldsymbol{a} + \boldsymbol{b}$, and $\boldsymbol{v} = \boldsymbol{x} + \boldsymbol{a} + \boldsymbol{b}$. And know $\boldsymbol{A}$: $(\boldsymbol{x}, \boldsymbol{y}) \longrightarrow (\boldsymbol{y}, \boldsymbol{x})$.

So next, applying $\boldsymbol{A}$ to the curve where the arbitrary point is $(\boldsymbol{u}, \boldsymbol{v})$, and assuming the new arbitrary point is $(\boldsymbol{m}, \boldsymbol{n})$, we get $(\boldsymbol{u}, \boldsymbol{v}) \longrightarrow (\boldsymbol{m}, \boldsymbol{n}) = (\boldsymbol{v}, \boldsymbol{u})$.

And we know $\boldsymbol{u} = \boldsymbol{y} + \boldsymbol{a} + \boldsymbol{b}$, and $\boldsymbol{v} = \boldsymbol{x} + \boldsymbol{a} + \boldsymbol{b}$. So we get $(\boldsymbol{m}, \boldsymbol{n}) = (\boldsymbol{x} + \boldsymbol{a} + \boldsymbol{b}, \boldsymbol{y} + \boldsymbol{a} + \boldsymbol{b})$.

Thus, assuming $\boldsymbol{P} = \boldsymbol{A \bullet (B \bullet C)}$, we can put $\boldsymbol{P}$ the way below.
$\boldsymbol{P}$: $(\boldsymbol{x}, \boldsymbol{y}) \longrightarrow (\boldsymbol{x} + \boldsymbol{a} + \boldsymbol{b}, \boldsymbol{y} + \boldsymbol{a} + \boldsymbol{b})$.

And next, moving on to $\boldsymbol{(A \bullet B) \bullet C}$, we apply $\boldsymbol{C}$ first, and $\boldsymbol{(A \bullet B)}$ next.
Doing so however, we just apply $\boldsymbol{C}$ twice, because we have $\boldsymbol{C} = \boldsymbol{A \bullet B}$.

And we know applying $\boldsymbol{C}$ assuming $(\boldsymbol{x}, \boldsymbol{y})$ is the original point, and $(\boldsymbol{s}, \boldsymbol{t})$ is the new point, we get $(\boldsymbol{s}, \boldsymbol{t}) = (\boldsymbol{y} + \boldsymbol{b}, \boldsymbol{x} + \boldsymbol{a})$. Thus, we get $\boldsymbol{s} = \boldsymbol{y} + \boldsymbol{b}$, and $\boldsymbol{t} = \boldsymbol{x} + \boldsymbol{a}$.

And next, applying again $\boldsymbol{C}$ to the curve where $(\boldsymbol{s}, \boldsymbol{t})$ is the arbitrary point, and assuming $(\boldsymbol{u}, \boldsymbol{v})$ is the new arbitrary point, we get $(\boldsymbol{s}, \boldsymbol{t}) \longrightarrow (\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{t} + \boldsymbol{b}, \boldsymbol{s} + \boldsymbol{a})$.
So we get $\boldsymbol{u} = \boldsymbol{t} + \boldsymbol{b}$, and $\boldsymbol{v} = \boldsymbol{s} + \boldsymbol{a}$. And we know $\boldsymbol{s} = \boldsymbol{y} + \boldsymbol{b}$, and $\boldsymbol{t} = \boldsymbol{x} + \boldsymbol{a}$.
Thus, we get $\boldsymbol{u} = \boldsymbol{x} + \boldsymbol{a} + \boldsymbol{b}$, and $\boldsymbol{v} = \boldsymbol{y} + \boldsymbol{b} + \boldsymbol{a} = \boldsymbol{y} + \boldsymbol{a} + \boldsymbol{b}$.

So assuming $\boldsymbol{Q} = (\boldsymbol{A}\bullet\boldsymbol{B})\bullet\boldsymbol{C}$, we can put $\boldsymbol{Q}$ the way below.

$\boldsymbol{Q}$: $(x, y) \longrightarrow (x + a + b, y + a + b)$.

So we get $\boldsymbol{P} = \boldsymbol{Q}$, that is, $\boldsymbol{A}\bullet(\boldsymbol{B}\bullet\boldsymbol{C}) = (\boldsymbol{A}\bullet\boldsymbol{B})\bullet\boldsymbol{C}$.

It is however, a particular case, too, and not a general case, so it is not the proof that the operations are associative in all cases. What then do we do?

Until we find the proof, we need to apply transformations from the right to the left in sequence, because it is the way composite transformations are made.

That is to say that, for instance, given $\boldsymbol{U}\bullet\boldsymbol{V}\bullet\boldsymbol{W}$, where $\boldsymbol{U}$, $\boldsymbol{V}$, and $\boldsymbol{W}$ are transformations, we need to do the operations this way: $\boldsymbol{U}\bullet(\boldsymbol{V}\bullet\boldsymbol{W})$.
In other words, we do $\boldsymbol{V}\bullet\boldsymbol{W}$ first, and then, do $\boldsymbol{U}\bullet(\boldsymbol{V}\bullet\boldsymbol{W})$.

And using the examples taken above, we can simplify the processes the way below.

We have
$\boldsymbol{A}$: $(x, y) \longrightarrow (y, x)$.
$\boldsymbol{B}$: $(x, y) \longrightarrow (x + a, y + b)$.
$\boldsymbol{C}$: $(x, y) \longrightarrow (y + b, x + a)$.

Then first, assuming (p, q) is the new point $\boldsymbol{B}\bullet\boldsymbol{C}$ makes, we get
$(p, q) = \{(y + b) + a, (x + a) + b\} = (y + a + b, x + a + b)$.

And next, assuming (s, t) is the new point $\boldsymbol{A}\bullet(\boldsymbol{B}\bullet\boldsymbol{C})$ makes, we get
$(s, t) = (x + a + b, y + a + b)$. So assuming $\boldsymbol{D} = \boldsymbol{A}\bullet\boldsymbol{B}\bullet\boldsymbol{C}$, we get

$\boldsymbol{D}$: $(x, y) \longrightarrow (s, t) = (x + a + b, y + a + b)$.
In short, $\boldsymbol{D}$: $(x, y) \longrightarrow (x + a + b, y + a + b)$.

And finding a transformation, we find in fact, the new point the transformation makes.
So finding a composite transformation, we find the new point the composite transformation makes, too.
And when finding the new point a composite transformation makes, we can use a quick method as follows.

In the new point the transformation applied makes,
replace x with the x-coordinate in the new point the other transformation makes.
And replace y with the y-coordinate in the new point the other transformation makes.

For instance, finding the new point $A \bullet B$ makes, in the new point A makes, replace x with the x-coordinate in the new point B makes, and replace y with the y-coordinate in the new point B makes.

So let's now do some practice on finding the new point of a composite transformation.

We have
A: $(x, y) \longrightarrow (y, x)$.
B: $(x, y) \longrightarrow (x + a, y + b)$.
C: $(x, y) \longrightarrow (y + b, x + a)$.

Then, finding the new point $A \bullet B$ makes, in the new point A makes, replace x with the x-coordinate in the new point B makes, and replace y with the y-coordinate in the new point B makes.
That is to say that in (y, x), replace x with $x + a$, and replace y with $y + b$.
Then, we get $(y + b, x + a)$, which is the new point $A \bullet B$ makes

Next, finding the new point $C \bullet A$ makes, in the new point C makes, replace x with the x-coordinate in the new point A makes, and replace y with the y-coordinate in the new point A makes.
That is to say that in $(y + b, x + a)$, replace x with y, and replace y with x.
Then, we get $(x + b, y + a)$, which is the new point $C \bullet A$ makes

And next, finding the new point $B \bullet C$ makes, in the new point B makes, replace x with the x-coordinate in the new point C makes, and replace y with the y-coordinate in the new point C makes.
That is to say that in $(x + a, y + b)$, replace x with $y + b$, and replace y with $x + a$.
Then, we get $\{(y + b) + a, (x + a) + b\} = (y + a + b, x + a + b)$, which is the new point $B \bullet C$ makes.

What if the transformations are as follows?

$\boldsymbol{A}$: $(x, y) \longrightarrow (x + y, x)$.
$\boldsymbol{B}$: $(x, y) \longrightarrow (x - y, x + y + b)$.
$\boldsymbol{C}$: $(x, y) \longrightarrow (b, x - y + a)$.

Then, finding the new point $A \bullet B$ makes, in the new point A makes, replace x with the x-coordinate in the new point B makes, and replace y with the y-coordinate in the new point B makes.

That is to say that in $(x + y, x)$, replace x with $x - y$, and replace y with $x + y + b$.

Then, we get $\{(x - y) + (x + y + b), (x - y)\} = (2x + b, x - y)$, which is the new point $A \bullet B$ makes

Next, finding the new point $C \bullet A$ makes, in the new point C makes, replace x with the x-coordinate in the new point A makes, and replace y with the y-coordinate in the new point A makes.

That is to say that in $(b, x - y + a)$, replace x with $x + y$, and replace y with x.

Then, we get $\{b, (x + y) - x + a\} = (b, y + a)$, which is the new point $C \bullet A$ makes

And finding next, the new point $B \bullet C$ makes, in the new point B makes, replace x with the x-coordinate in the new point C makes, and replace y with the y-coordinate in the new point C makes.

That is to say that in $(x - y, x + y + b)$, replace x with b, and replace y with $x - y + a$.

Then, we get $\{(b - (x - y + a), b + (x - y + a) + b)\} = (y - x + b - a, x - y + a + 2b)$, which is the new point $B \bullet C$ makes.

5. Parametric Transformations

In a parametric transformation, we use a parameter.
What then is a parameter in such a transformation?

It is a variable, and connects variables. And each of the variables connected is in an equation. So eventually, such a parameter connects the variables so that another equation gets made. So a parametric transformation connects equations to form another equation.

And expressing a variable in terms of a parameter, we parameterize the variable. So for instance, forming $y = f(t)$ and $x = g(t)$ where t is a parameter, we parameterize x and y.

Then, x and y can get connected through t. And connecting x and y, we say we do a parametric transformation, and apply it to $y = f(t)$ and $x = g(t)$. And then, we get for instance, $y = h(x)$. That is to say that we get the connective equation between x and y. Thus, we get a new equation made of x and y.

Why parametric transformations though?

Suppose x is a horizontal distance, y is a vertical distance, and both change as time t changes the way as follows. $x = 18t$, and $y = 24t - 5t^2$ for $0 \le t \le 2.4$.

Suppose for instance, a soccer player kicks a soccer ball, and we want to know the position of the ball at time t. Then, assuming the ball is a point in the x-y plane, the x-axis is put on the ground, and the y-axis is perpendicular to the ground, we can take x above as the x-coordinate, and take y above as the y-coordinate of the point that indicates the ball. That is, $(x, y) = (18t, 24t - 5t^2)$, which is the position of the ball at time t.

Then, connecting x and y, we get a curve, along which the ball moves, that is, we get the trajectory the ball makes. Applying thus, a parametric transformation to the two equations $x = 18t$, and $y = 24t - 5t^2$, we get a curve whose equation is made of x and y. And of course, graphing the curve, we can see the trajectory.

So applying a parametric transformation to $\boldsymbol{y = 24t - 5t^2}$, and $\boldsymbol{x = 18t}$ is no other than connecting $\boldsymbol{x}$ and $\boldsymbol{y}$ using $\boldsymbol{t}$ called the parameter.
And connecting them, we can do it the way below.

$$x = 18t \Rightarrow t = \frac{x}{18} \Rightarrow y = \frac{24x}{18} - \frac{5x^2}{18^2} = -\frac{5}{18^2}(x^2 - \frac{24 \cdot 18}{5}x)$$

$$= -\frac{5}{324}\{x^2 - \frac{432}{5}x + \left(\frac{432}{10}\right)^2 - \left(\frac{432}{10}\right)^2\}$$

$$\Rightarrow y = -\frac{5}{324}(x - 43.2)^2 + \frac{5}{324} \cdot \frac{432^2}{100} = -\frac{5}{324}(x - 43.2)^2 + 28.8$$

$\Rightarrow y = -\dfrac{5}{324}(x - 43.2)^2 + 28.8$, which is a parabola with the vertex at (43.2, 28.8).

Thus, the trajectory is a parabola, which is concave-down.

So for instance, at time $\boldsymbol{t = 2}$, we get $\boldsymbol{x = 18t = 36}$.
Thus next, we get $x = 36 \Rightarrow y = -\dfrac{5}{324}(36 - 43.2)^2 + 28.8 = -\dfrac{5 \cdot 7.2^2}{324} + 28.8 = 28$.

So we can see that the ball is at **(36, 28)** after 2 seconds.

And assuming $\boldsymbol{P}$ is the parametric transformation we did above, we can put it this way:
$\boldsymbol{P: t \longrightarrow (x, y) = (18t, 24t - 5t^2)}$.

And briefly, we can put it this way, too: $\boldsymbol{P: t \longrightarrow (18t, 24t - 5t^2)}$.

So given $\boldsymbol{P: t \longrightarrow (18t, 24t - 5t^2)}$, we want to assume that $\boldsymbol{x = 18t}$, and $\boldsymbol{y = 24t - 5t^2}$.

How then can we call $\boldsymbol{(x, y)}$, that is, $\boldsymbol{(18t, 24t - 5t^2)}$?

It is the new arbitrary point, that is, the arbitrary point in the new curve.
What then is the original curve?

We have two. One is $\boldsymbol{x = 18t}$, and the other is $\boldsymbol{y = 24t - 5t^2}$.
And so to speak, mixing the two together, we get $\boldsymbol{x}$ and $\boldsymbol{y}$ connected in one curve, which is the new. Getting the mixture, we mix $\boldsymbol{x = 18t}$ into $\boldsymbol{y = 24t - 5t^2}$ using $\boldsymbol{t}$ as the medium.

So connecting $\boldsymbol{x}$ and $\boldsymbol{y}$ through the medium of $\boldsymbol{t}$, we get $\boldsymbol{y = -\frac{5}{324}(x - 43.2)^2 + 28.8}$.

And more precisely, we get $\boldsymbol{y = -\frac{5}{324}(x - 43.2)^2 + 28.8}$ for $\boldsymbol{0 \le x \le 86.4}$, because we have $\boldsymbol{0 \le t \le 2.4}$.

6. General Transformations

Doing transformations, we change curves. And in this book, we call the curve to be changed is an original curve, and call the curve changed a new curve. So applying a transformation to an original curve, we get a new curve.

How then do we get the original curve changed?

A curve is a set of points. So changing a curve, we change every point in the curve. And then, we get a new curve.

Changing a curve though, we don't actually change one point at a time.
Changing a curve in fact, we change only one point, which represents all the points in the original curve. And we call the one point the arbitrary point in the original curve.

In short, we just call it the original arbitrary point, or more quickly, the original point.

And changing a curve, we apply a transformation to it.
And applying a transformation, we use the definition of it if the definition is given. If not, we need to find it, of course.

And the definition is made of two points.
One is the original arbitrary point in the original curve, and the other is the arbitrary point in the new curve, so the other is just called the new arbitrary point or more quickly, the new point.

And getting the new curve applying a transformation to an original curve, we mix the new point into the original curve.
Then, we get the equation connecting the coordinates of the new point, and the connective equation is the new equation.

And if the definition is not given, that is, if the new point is not given, we need to get the new point changing the original point the way the original curve changes. Then, mixing the new point into the original curve, we get the equation connecting the coordinates of the new point. And then, the equation is the equation of the new curve we want.

And for instance, assuming (x, y) is the original point, and (s, t) is the new point, we get the new point in the transformation definition the way below.

$(s, t) = (x - 1, y + 3)$ in a parallel transformation.

$(s, t) = (x, 4 - y)$ in a transformation symmetric about a line $y = 2$.

$(s, t) = (2 - x, y)$ in a transformation symmetric about a line $x = 1$.

$(s, t) = (2 - x, 4 - y)$ in a transformation symmetric about a point $(1, 2)$.

$(s, t) = (y, x)$ in a transformation symmetric about the line $y = x$.

$(s, t) = (-y, -x)$ in a transformation symmetric about the line $y = -x$.

$(s, t) = (-y, -x)$ in a transformation symmetric about the line $y = -x$.

$(s, t) = \left(\frac{(1-a^2)x + 2ay - 2ab}{1+a^2}, \frac{2ax + (a^2-1)y + 2b}{1+a^2}\right)$ in a transformation symmetric about a line $y = ax + b$.

$(s, t) = (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta)$ in a transformation by turning around the origin in the amount of angle θ.

$(s, t) = \{(x - a)\cos\theta - (y - b)\sin\theta + a, (x - a)\sin\theta + (y - b)\cos\theta + b\}$ in a transformation by turning around a point (a, b) in the amount of angle θ.

$(s, t) = (y + 1, x - 2)$ in a composite transformation.

In a parametric transformation however, we just take (x, y) as the new point, and set, for instance, $(x, y) = (\sin t, \cos t)$, which can also be called the parametric equation of the unit circle centered at the origin.

Transforming a curve, we do many operations to a curve as expanding or stretching, shrinking or compressing, shifting, as well as translating here and there, flipping or reflecting using symmetry, folding using the absolute value signs, and turning.
And some examples are as follows.

Assuming first, $y = f(x) = 2x^2 - 3x + 1$, we can do transformations the way as follows.

Stretching f vertically by 2, we get $y = g(x) = 2f(x) = 2(2x^2 - 3x + 1) = 4x^2 - 6x + 2$

Stretching f horizontally by 2, we get

$$y = g(x) = f(\frac{x}{2}) = 2(\frac{x}{2})^2 - 3(\frac{x}{2}) + 1 = \frac{x^2}{2} - \frac{3x}{2} + 1$$

Compressing f horizontally by 2, we get

$$y = g(x) = f(2x) = 2(2x)^2 - 3(2x) + 1 = 8x^2 - 6x + 1$$

Shifting f to the right by 2, we get

$$y = g(x) = f(x - 2) = 2(x - 2)^2 - 3(x - 2) + 1$$

$$= 2(x^2 - 4x + 4) - 3x + 6 + 1 = 2x^2 - 11x + 9$$

Shifting f to the left by 2, we get

$$y = g(x) = f(x + 2) = 2(x + 2)^2 - 3(x + 2) + 1$$

$$= 2(x^2 + 4x + 4) - 3x - 6 + 1 = 2x^2 + 5x + 3$$

Shifting up f by 2, we get

$$y = g(x) = f(x) + 2 = 2x^2 - 3x + 1 + 2 = 2x^2 - 3x + 3$$

Shifting down $\boldsymbol{f}$ by 2, we get

$$y = g(x) = f(x) - 2 = 2x^2 - 3x + 1 - 2 = 2x^2 - 3x - 1$$

Assuming again, $y = f(x) = 2x^2 - 3x + 1$, and taking the reciprocal of $\boldsymbol{f}$, we can get

$$y = g(x) = \frac{1}{f(x)} = \frac{1}{2x^2 - 3x + 1} = \frac{1}{(2x-1)(x-1)}$$

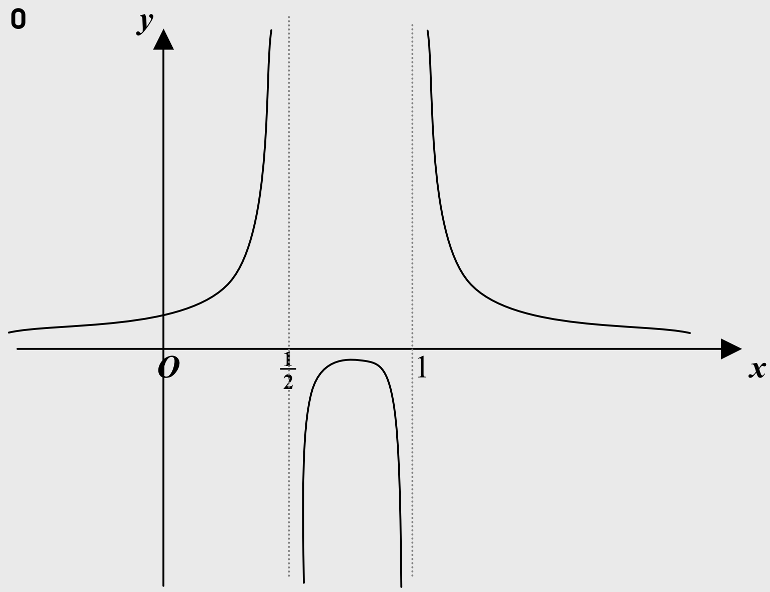
So the function $\boldsymbol{g}$ is not defined for $x = \frac{1}{2}$ or $x = 1$.

That is, if $\boldsymbol{R}$ is a set of all real numbers, the domain of $\boldsymbol{g}$ is $R - \{\frac{1}{2}, 1\}$.

So $\boldsymbol{g}$ has two vertical asymptotes, one is $x = \frac{1}{2}$, and the other is $x = 1$.

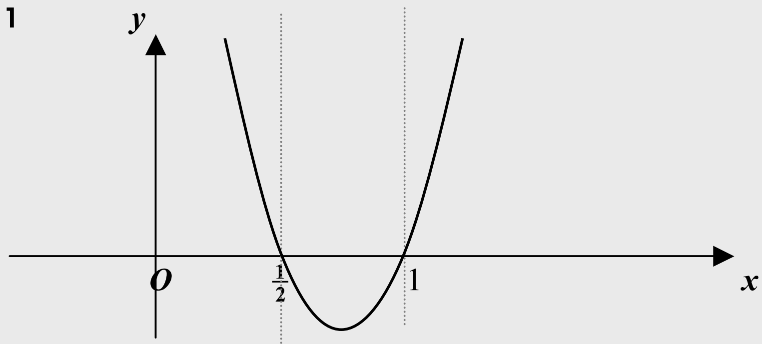
And the horizontal asymptote is $y = 0$, that is, the $\boldsymbol{x}$-axis.
Thus, the curve of $\boldsymbol{g}$ is as follows.

Fig. 0



And the curve of the original function f is as follows.

Fig. 1



Assuming next, $y = f(x) = \sin x$, we can say that the amplitude is 1, frequency is 1, period is 2π, and phase is 0, and can do transformations the way as follows.

Assuming h is a new function, and stretching f vertically by 2,
we get $y = h(x) = 2f(x) = 2\sin x$.
Then the amplitude is 2, the frequency is 1, the period is 2π, and the phase is 0.

Stretching f horizontally by 2, we get $y = h(x) = f(\frac{x}{2}) = \sin(\frac{x}{2})$.

Then, the amplitude is 1, the frequency is $\frac{1}{2}$, the period is 4π, and the phase is 0.

Compressing f horizontally by 2, we get $y = h(x) = f(2x) = \sin(2x)$.
Then the amplitude is 1, the frequency is 2, the period is π, and the phase is 0.

Shifting up f by 2, we get $y = h(x) = f(x) + 2 = \sin x + 2$.
Then the amplitude is 1, the frequency is 1, the period is 2π, and the phase is 0.

Shifting down f by 2, we get $y = h(x) = f(x) - 2 = \sin x - 2$.
Then the amplitude is 1, the frequency is 1, the period is 2π, and the phase is 0.

Shifting f to the right by 2, we get $y = h(x) = f(x - 2) = \sin(x - 2)$.
Then the amplitude is 1, the frequency is 1, the period is 2π, and the phase is -2.

Shifting f to the left by 2, we get $y = h(x) = f(x + 2) = \sin(x + 2)$.
Then the amplitude is 1, the frequency is 1, the period is 2π, and the phase is 2.

And setting $y = F(x) = A \sin B(x \pm C) + D$, we get

the amplitude = A, frequency = B, period = $2\pi/B$, phase = $\pm C$, and vertical shift = D.

Assuming again, $y = f(x) = 2x^2 - 3x + 1$, we can do transformations the way as follows.

Reflecting about the origin, change x with $-x$, and change y with $-y$.

So we get $y = f(x) \Rightarrow -y = f(-x) \Rightarrow y = -f(-x)$. Thus, we get

$y = f(x) = 2x^2 - 3x + 1 \Rightarrow -y = f(-x) = 2(-x)^2 - 3(-x) + 1 = 2x^2 + 3x + 1$

$\Rightarrow y = -f(-x) = -(2x^2 + 3x + 1) = -2x^2 - 3x - 1$

Reflecting about the y-axis, change x with $-x$. So we get $y = f(x) \Rightarrow y = f(-x)$.

Thus, we get
$y = f(x) = 2x^2 - 3x + 1 \Rightarrow y = f(-x) = 2(-x)^2 - 3(-x) + 1 = 2x^2 + 3x + 1$

Reflecting about the x-axis, change y with $-y$. So we get $y = f(x) \Rightarrow -y = f(x)$.

Thus, we get
$y = f(x) = 2x^2 - 3x + 1 \Rightarrow -y = f(x) = 2x^2 - 3x + 1$

$\Rightarrow y = -f(x) = -(2x^2 - 3x + 1) = -2x^2 + 3x - 1$

Reflecting about the line $\boldsymbol{y = x}$, exchange $\boldsymbol{x}$ with $\boldsymbol{y}$. So we get $\boldsymbol{y = f(x) \Rightarrow x = f(y)}$.

Thus, we get $\boldsymbol{y = f(x) = 2x^2 - 3x + 1 \Rightarrow x = f(y) = 2y^2 - 3y + 1}$

And taking the inverse of $\boldsymbol{f}$, we can get

$$\mathbf{2y^2 - 3y - x + 1 = 0 \Rightarrow \quad y = \frac{3 \pm \sqrt{3^2 - 4 \cdot 2(1 - x)}}{4} = \frac{3 \pm \sqrt{9 - 8 + 8x}}{4} = \frac{3 \pm \sqrt{8x + 1}}{4}}$$

So we can get $\boldsymbol{y = g(x) = \frac{3 + \sqrt{8x+1}}{4}}$ for $\boldsymbol{x \geq \frac{3}{4}}$, and $\boldsymbol{y = h(x) = \frac{3 - \sqrt{8x+1}}{4}}$ for $\boldsymbol{x \geq \frac{3}{4}}$.

Assuming next, $\boldsymbol{y = f(x) = \sin x}$, $\boldsymbol{y = g(x) = \cos x}$, and $\boldsymbol{y = h(x) = \tan x}$, we can do transformations the way as follows.

Reflecting about the origin, change $\boldsymbol{x}$ with $\boldsymbol{-x}$, and change $\boldsymbol{y}$ with $\boldsymbol{-y}$.

So we get $\boldsymbol{y = f(x) \Rightarrow -y = f(-x) \Rightarrow y = -f(-x)}$. Thus, we get

$$\boldsymbol{y = f(x) = \sin x \Rightarrow -y = f(-x) = \sin(-x) = -\sin x \Rightarrow y = -f(-x) = -(-\sin x) = \sin x}$$

$$\boldsymbol{y = g(x) = \cos x \Rightarrow -y = g(-x) = \cos(-x) = \cos x \Rightarrow y = -g(-x) = -(\cos x) = -\cos x}$$

$$\boldsymbol{y = h(x) = \tan x \Rightarrow -y = h(-x) = \tan(-x) = -\tan x \Rightarrow y = -h(-x) = -(-\tan x) = \tan x}$$

Reflecting about the $\boldsymbol{y}$-axis, change $\boldsymbol{x}$ with $\boldsymbol{-x}$.

So we get $\boldsymbol{y = f(x) \Rightarrow y = f(-x)}$. Thus, we get

$$\boldsymbol{y = f(x) = \sin x \Rightarrow y = f(-x) = \sin(-x) = -\sin x}$$

$$\boldsymbol{y = g(x) = \cos x \Rightarrow y = g(-x) = \cos(-x) = \cos x}$$

$$\boldsymbol{y = h(x) = \tan x \Rightarrow y = h(-x) = \tan(-x) = -\tan x}$$

Reflecting about the x-axis, change y with $-y$.

So we get $y = f(x) \Rightarrow -y = f(x)$. Thus, we get

$y = f(x) = \sin x \Rightarrow -y = f(x) = \sin x \Rightarrow y = -f(x) = -\sin x$

$y = g(x) = \cos x \Rightarrow -y = g(x) = \cos x \Rightarrow y = -g(x) = -\cos x$

$y = h(x) = \tan x \Rightarrow -y = h(x) = \tan x \Rightarrow y = -h(x) = -\tan x$

Reflecting about the line $y = x$, exchange x with y. So we get $y = f(x) \Rightarrow x = f(y)$

Thus, we get

$y = f(x) = \sin x \Rightarrow x = f(y) = \sin y$

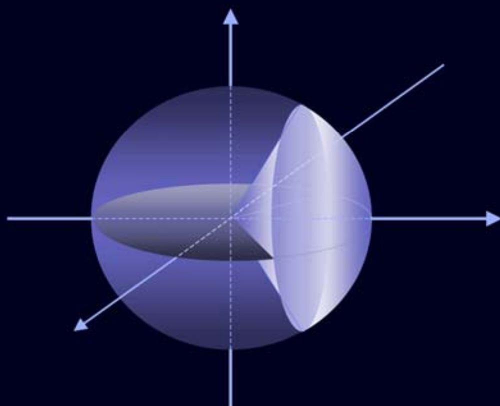
And taking the function of x, we can get $y = \sin^{-1}(x)$ for $-1 \leq x \leq 1$.

$y = g(x) = \cos x \Rightarrow x = g(y) = \cos y$

And taking the function of x, we can get $y = \cos^{-1}(x)$ for $-1 \leq x \leq 1$.

$y = h(x) = \tan x \Rightarrow x = h(y) = \tan y$

And taking the function of x, we can get $y = \tan^{-1}(x)$.



함수의 변환

Function Transformations

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