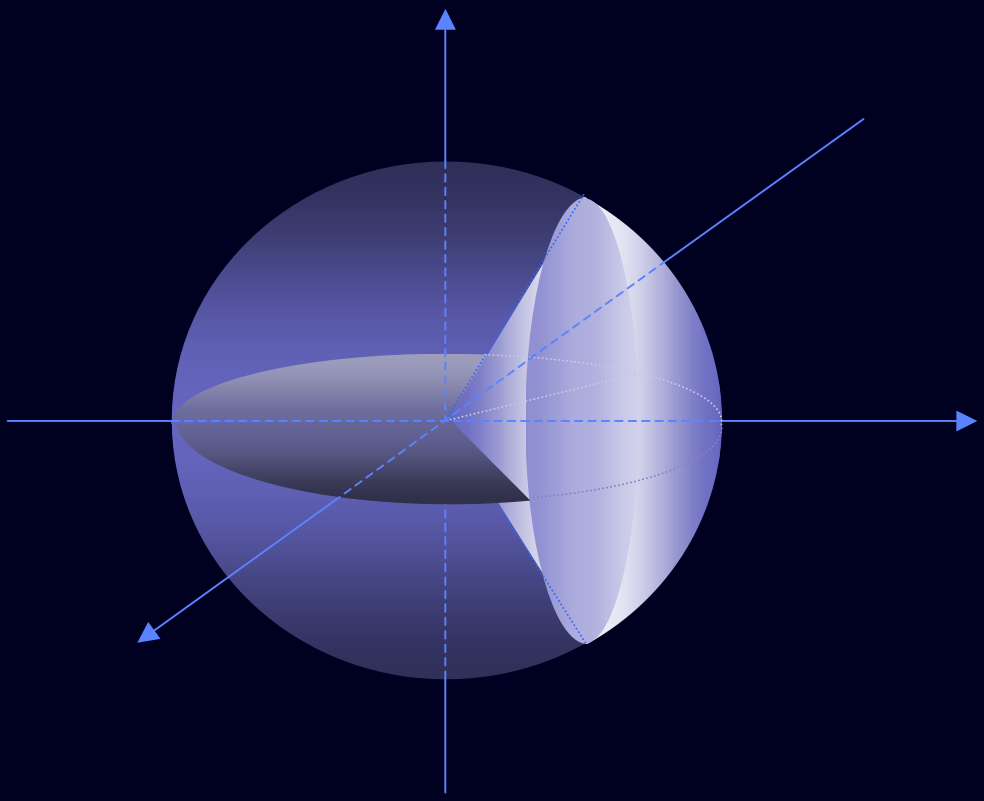
다항식의 인수분해

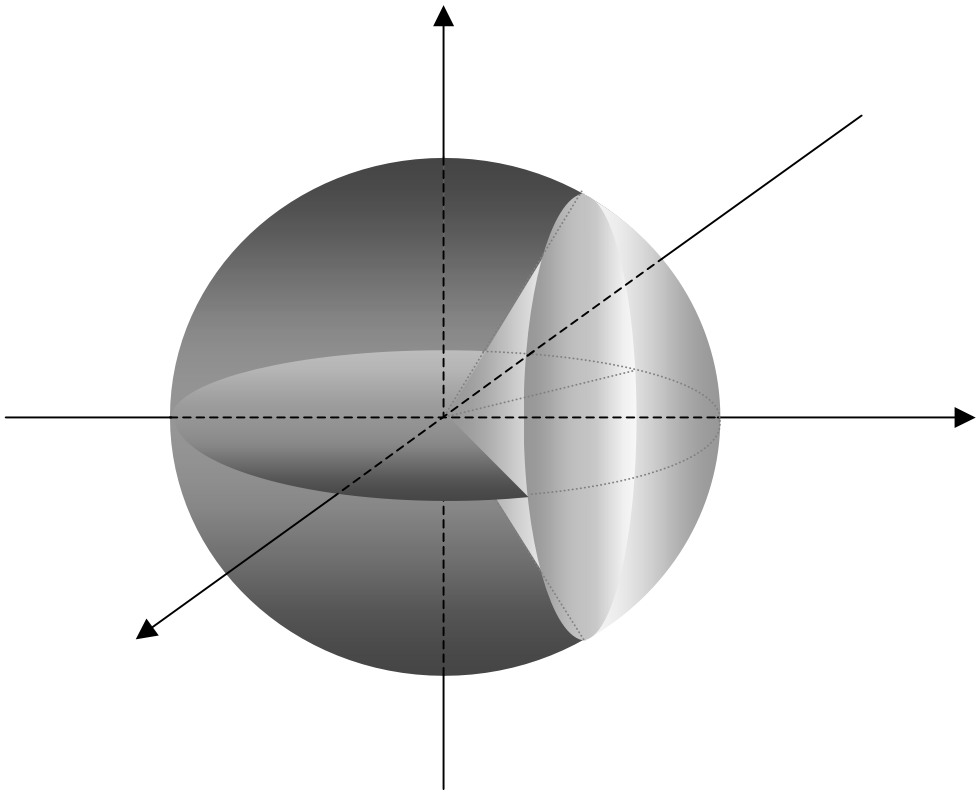
Polynomial Factorizations



김성렬

Seong R. Kim

Polynomial Factorizations



Seong R. KIM

Dear students,

Students need the best teacher, so you need examples, because examples are the best teacher. This is thus, a book of examples. All the examples here are fully worked, and explain **how** the basic but essential tools in math are made, together with **what** they are, **how** they work, and **how** to work with them. Such tools include numbers, formulas, identities, equations, laws, etc.

Doing math, we work with ideas and run ideas, because every thing in math is an idea. What idea though?

It’s a concept. So we need to understand it if we want to use it. A number is an idea, for instance, and the same is true for a line or circle, too. And putting ideas together, we can build another, which becomes the base or an element of another, and each is connected. And that’s the way your math grows.

So you get to build a circuit, and sometimes, need to fill the gap or repair the circuit so that you can get the sense of it. Your calculation thus, runs properly, and gets to the solution.

Getting the concept doing good examples, and gaining the fluency doing right examples, you can build the circuit strong.

This book is however, nothing but a bunch of examples until you get it powered. How then to get it powered, and make it run and work for you?

Read it closely at your pace, and then, do each example yourself. It is quite important to note that you do it in **your** own writing. Just watching someone doing it, you just only feel that you can do it, too, but can hardly do it when you need it. You do it, you get it. And you've got to do it over with different examples. It's a cliché, of course, but is always true that knowing is one thing and doing is another.

I've been helping students get conceptual understanding and fluency so that they can grow, take care of, and run their own math. The area covers the basics in algebra and geometry, and includes equations for unknowns and those for curves, functions and their graphs, and other math tools, which are the basic elements in calculus, which's been the core of my interest from my early age in high school.

Of my students, some are quite poor in math, and thus, are afraid of or even hate math, some require special education because of exceptional intelligence, some are smart enough, some are naïve and diligent, and many are clever but lazy. All the students are badly after though, one thing in common.

A Strong Foundation in Algebra

It is of course, the prime objective of my work, and I'm always happy and eager to help them achieve it. I can only help though. Your math grows as much as you grow and take care of it yourself. So does mine. What math then do students most often do or use in high schools or colleges?

Algebra and geometry. What algebra then?

Elementary algebra, of course. Doing the algebra, you should be able to not only work with, but manipulate, that is, change, alter, and convert math expressions the way you want, so for instance, you can break apart and factor polynomials in many kinds. Also, you can handle properly ratios and rates, trigonometry, factors or divisors, constants and variables, powers and logarithms or exponents, equations and functions, along with their graphs, equalities and inequalities, identities and formulas, laws and rules, etc.

And if you want to handle them properly, you want to know their natures and how they mingle with each other.

So studying those stated above, you want to know **what** they are, **how** they work, and **how** to work with them or **what** to do with them. What then about the geometry?

Basically, the geometry is about shapes, and has much to do with positions and angles in algebra. Why algebra?

You often need to compute lengths, areas, or volumes in many situations. The shapes begin with points, lines, triangles, and circles, and move on to rectangles, squares, parallelograms or rhombuses, trapezoids, tetragons, other polygons as pentagons, polyhedrons as hexahedrons, etc.

Doing the geometry, too, we need to do the algebra stated above such as trigonometry. So it is analytic geometry, often called coordinate geometry, too. And doing it, we can specify positions using coordinates. So in that geometry, basically, we work with graphs. Putting a math idea in a graph, we can not only effectively think about it, but actually see it, too, and therefore, can not just easily, but efficiently work with it, too. What idea then is it?

The idea begins with a point, a line, a parabola, a circle, an ellipse, and a hyperbola, called a conic section or basic curve, and then, moves on to other curves, planes, surfaces, volumes, and other objects in various dimensional spaces, together with vectors. And using an angle, we can specify and work with an amount of turning or change in direction.

So learning, using, or applying those ideas or tools in math, we get to solve problems. And this book can help. It can help you build and run your own math engine in your brain, and thus, can help achieve your solid math skill.

It is however, not a magic book giving you a math skill of high caliber overnight. And it's just a book of examples, and can have many mistakes. I'm a human, too. There's no mojo making a math genius, and math is science, and is full of facts and ideas. And it is after all, not me, not your teacher, and not any book but you who actually make you put together some of those facts and ideas, and understand it.

Putting facts and ideas together, understanding it, and taking care of what you have learned doing examples, you grow your math. And this book is ready to help and can help you help yourself.

Solving a problem, we put it many different ways. For instance, while expanding or reducing it, or modifying or converting it, we keep searching for the solution, keep approaching the solution, and eventually, can get there.

It is often the case a problem itself is the solution. We can put a problem in many different ways, and eventually, can end up with the solution. How come then is the solution no other than the problem?

For instance, the solution to 3232 ÷ 101 is 32. And we can put it this way:

$3232 \div 101 = \frac{3232}{101} = \frac{32 \text{ x } 101}{101} = \frac{32}{1} = 32 \Rightarrow$ 3232 ÷ 101 = 32.

And we can get this, too: 32 = 3232 ÷ 101. How?

$32 = \frac{32}{1} = \frac{32 \text{ x } 101}{101} = \frac{3232}{101} =$ 3232/101 = 3232 ÷ 101.

Too easy?

For another instance, the solution to $\boldsymbol{ax^2 + bx + c = 0}$ is

$\boldsymbol{x} = \frac{-\boldsymbol{b} \pm \sqrt{\boldsymbol{b}^2 - 4\boldsymbol{ac}}}{2\boldsymbol{a}}$, which is called the quadratic formula.

How come then is the solution no other than the problem?

We can put it this way:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow 2ax = -b \pm \sqrt{b^2 - 4ac}$$

$$\Rightarrow 2ax + b = \pm\sqrt{b^2 - 4ac} \Rightarrow (2ax + b)^2 = b^2 - 4ac$$

$$\Rightarrow 4a^2x^2 + 4abx + b^2 = b^2 - 4ac \Rightarrow 4a^2x^2 + 4abx = -4ac$$

$$\Rightarrow ax^2 + bx = -c \Rightarrow ax^2 + bx + c = 0.$$

And we can get this, too:

$$ax^2 + bx + c = 0 \quad \Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ How?

$$ax^2 + bx + c = a(x^2 + \frac{b}{a}x) + c = a(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}) + c$$

$$= a(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}) - \frac{b^2}{4a} + c = a(x + \frac{b}{2a})^2 - \frac{b^2 - 4ac}{4a} = 0$$

$$\Rightarrow a(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a} \Rightarrow (x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\Rightarrow x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \Rightarrow x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

So when solving a problem, we expand or reduce it, modify or convert it, and keep searching for and approaching the solution. Then eventually, we can get there. What then do we do when doing the set of processes above?

We do algebra. You need a strong foundation in algebra. So if a problem is well defined, that is, if it makes sense, we should be able to get it solved the way as follows.

A problem $\Rightarrow$ **…** $\Rightarrow$ **…** $\Rightarrow$ **the solution**,

and thus, **the problem** $\Rightarrow$ **the solution**.

So solving a problem, we put it many different ways so that we can get to the solution. And that's the way math runs.

Many more examples are available for free, and you can get them downloaded at **www.runmath.com**. So visit the website, get them downloaded, and do the examples. The website is getting updated weekly or monthly.

May your math run very well.

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0. Monomials & Polynomials

Doing math, we use a mathematical expression, often called briefly, a math expression. And more briefly, we usually just call it an expression.

And of those expressions, the most often used are monomials and polynomials, together with numbers, of course. Numbers are math expressions showing values.
What then, is a monomial?

Calling briefly a monomial expression, we just call it a monomial. And the simplest is a power such as x^5, a^3, b^2, and y^4, where the exponent is a positive integer.

Thus, technically, x is a power, and is a monomial, because $x = x^1$, which is a power of x and the exponent is 1, which is a positive integer, of course. And since the exponent is a positive integer, we do *not* take as monomials such expressions as these: x^{-2}, x^{-5}, $\sqrt{x} = x^{\frac{1}{2}}$, $x\sqrt{x} = x^{\frac{3}{2}}$, $\frac{1}{x} = x^{-1}$, $\frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$, $\frac{1}{x^2\sqrt{x}} = x^{-\frac{5}{2}}$, etc.

Next, a monomial can be a product of a number and a power such as $3x$, $2a^3$, and $\frac{b^2}{2}$.
And a monomial can be a product of powers or a product of a number and powers, so monomials can be these: $xy = x^1y^1$, $3xy$, $2xyz$, $-5xy^2$, $2x^2a^3$, $4ab^3y^4$, $3x^4y^2a^3b^4$, or such.

However, we don't take as monomials these: $\frac{xy}{2a}$, $-\frac{2x^2y^3}{3ab}$, $\frac{2x^2y^3}{3a^2b}$, $5xy\sqrt{a}$, or $\sqrt{2}axy$, which is called a term. A term can be a monomial though. Those monomials stated above can be terms, but not all terms are monomials. For instance, 9 is a term, so a number is a term, and $\sqrt{x}$ is a term, too, but is not a monomial.

Next, adding together two monomials, or adding a monomial to a number, we get an expression called a *binomial* expression, often just called a binomial.

Note that subtracting an object means adding the negative of the object.

So for instance, we get $x - y = x + (-y)$, which is therefore, a binomial, too.

Making thus, some more binomials, we can get these:

$x + y$, $x - 1$, $2x + y$, $2xy - ay$, $x^2 - 2$, $y^2 - xy$, $xy^2 + a$, $2axy^2 - 3ab$, etc.

What then about these: $\frac{1+x}{2}$, $\frac{x+y}{3}$, $\frac{ab+x}{5}$, and $\frac{ab+xy}{3}$?

They are binomials, too.

That's because $\frac{1+x}{2}=\frac{1}{2}+\frac{x}{2}$, $\frac{x+y}{3}=\frac{x}{3}+\frac{y}{3}$, $\frac{ab-x}{5}=\frac{ab}{5}-\frac{x}{5}$, and $\frac{ab+xy}{3}=\frac{ab}{3}+\frac{xy}{3}$

We do not take however, as binomials these: $x+\frac{1}{x}$, $\frac{1+x}{xy}$, $\frac{ab+x}{2y}$, or $\sqrt{x}+1$.

That's because every exponent has to be a positive integer, and we have these:

$x+\frac{1}{x}=x+x^{-1}$, $\frac{1+x}{xy}=\frac{1}{xy}+\frac{x}{xy}=(xy)^{-1}+y^{-1}$, and $\sqrt{x}+1=x^{\frac{1}{2}}+1$

Next, if adding together three monomials, or if adding a binomial to a number, we get an expression called a *trinomial* expression, often just called a trinomial. So for instance, making some trinomials, we can get these:

$x + 2x + y$, $x - 2xy + 3y$, $2x + y - 1$, $2xy - ay + 1$, $x^2 - 2y^2 + ay$, $y^2 - 1 + x^2$,

$xy^2 + ab + 1$, $2axy^2 - 3ab + 2x^2$, $x^2 + x + 1$, $x^3 - 2x + 5$, etc.

Also, every exponent has to be a positive integer, so for instance,

$x+y+\frac{1}{x}$ and $\frac{1+2x+y}{xy}$ are not trinomials.

What then is a polynomial?

Calling briefly a polynomial expression, we just call it a polynomial.
And in fact, 'poly' means 'two or more', so a polynomial means a sum of two or more monomials. So adding together monomials, we get a polynomial.

Also, adding a monomial to a number, we get a polynomial, too.
So a binomial or a trinomial can be called a polynomial, too.
Thus, the smallest polynomial is a binomial.
So for instance, polynomials can be the expressions as follows.

$x + 2$, $x + y$, $x + y + 1$, $x - 2xy + 3$, $2x^2 + 3xy - x + y$, $x^3y - ay + 3x^2y - 2x + 4$,

$x^2 - 2y^2 + bxy + xy^2 - a + 3$, $2axy^2 - 3aby + bx^2 - 4$, $x^5 + 2x^4 + x^3 - 3x + 5$, etc.

And for another instance, setting $P = x^3 + 2x^2y - a(b - 2x) + \frac{3(y+b)}{4} + 7 + b$, we can say that P is a polynomial. And we can put the polynomial P the way as follows,

$P = x^3 + 2x^2y - ab + 2ax + \frac{3y}{4} + \frac{3b}{4} + 7 + b$.

And every exponent has to be a positive integer, so for instance,

$x - y + 1 + \frac{1}{x}$ and $\frac{1 + 2x + y + y^2}{xy}$ are not polynomials.

Also, we can say that a polynomial is a sum of terms. What is a term though?

A term is a number, a monomial, a product of a number and a monomial, a product of monomials, or a quotient between monomials.
So the polynomial P below is made of 10 terms,

$P = x^3 + 2x^2y - ab + 2ax + \frac{x}{2b} + \frac{3y}{4} + \sqrt{x}y^3 + \frac{2y}{\sqrt{x}} + 7 + b$.

And **7** in the polynomial P is called a *constant* term. Why is 7 a constant term though?

Not only a constant but a number, too, is constant, because it doesn't change its value.
So7 is constant term, and can be called a *numeric* term, too.
For another instance, if c is a constant in an equation $y = 2x^2 + x + c$, we call c a constant term, too.

And if terms are monomials where the variables are the same, and the same exponents are applied to the variables, we call those terms *like*-terms.

So for instance, $\mathbf{2}x$, $\mathbf{-3}x$, and $\mathbf{5}x$ are like-terms.

$\mathbf{2}ax$, $\mathbf{-5}ax$, and $\mathbf{3}ax$ are like-terms.

$\mathbf{3}ax^2$, $\mathbf{2}ax^2$, $\mathbf{5}ax^2$, $\mathbf{-3}ax^2$, and $\frac{2}{3}ax^2$ are like-terms.

$\frac{2}{xy}$, $\frac{5}{xy}$, and $\frac{3}{2xy}$, that is, $\mathbf{2}x^{-1}y^{-1}$, $\mathbf{5}x^{-1}y^{-1}$, $\frac{3}{2}x^{-1}y^{-1}$ are like-terms, not monomials though.

$\frac{2a}{xy}$, $\frac{5a}{xy}$, and $\frac{3a}{2xy}$ are like-terms, but are not monomials.

And $\frac{2ab}{x^2y}$, $-\frac{5ab}{x^2y}$, $\frac{3ab}{2x^2y}$, and $\frac{\sqrt{3}ab}{x^2y}$ are like-terms, too, but of course, are not monomials.

So some like terms are monomials, and some are not.
Is there then anything good about like terms?

We can add them together, and make expressions simpler. So for instance, we can get

$\mathbf{2}x - \mathbf{3}x + \mathbf{5}x = (2 - 3 + 5)x = \mathbf{4}x$.

$\mathbf{2}ax - \mathbf{5}ax + \mathbf{3}ax = (2 - 5 + 3)ax = \mathbf{0}\cdot x = \mathbf{0}$.

$\mathbf{2}ax - \mathbf{5}ax + \mathbf{7}ax = (2 - 5 + 7)ax = \mathbf{4}ax$.

$$\frac{2ab}{x^2y} - \frac{5ab}{x^2y} + \frac{3ab}{2x^2y} + \frac{\sqrt{3}ab}{x^2y} = \frac{2ab - 5ab + 3ab + \sqrt{3}ab}{x^2y} = \frac{(2 - 5 + 3 + \sqrt{3})ab}{x^2y} = \frac{\sqrt{3}ab}{x^2y}$$

1.0. Polynomial Arithmetic 1

Doing math, we do algebra, and do it a lot.
And doing it, we often work with polynomials, along with numbers.
Working with polynomials, we often do arithmetic using them, together with numbers.
And the arithmetic matters.

So what is polynomial arithmetic?

Normally, doing arithmetic, we do it with numbers as integers, fractions, or decimals, etc. And doing it, we do operations in four kinds: additions, subtractions, multiplications, and divisions.

Doing arithmetic with numbers, we do *number arithmetic*, and doing it, we add numbers together, subtract a number from another, multiply a number by another, or divide a number by another.

And the same is true for polynomials, too.
So we do such four operations with polynomials, too.
Doing however, polynomial arithmetic, we don't just do arithmetic.
That is, we don't just do additions, or just do others.

Doing polynomial arithmetic, we often get to some manipulations with numbers, monomials, and polynomials. So for instance, we sometimes, break some numbers and other expressions apart, and put the pieces together so that we can get some meaningful expressions. Why?

It's all because we want to get to the solution we want, and more importantly, we want to get to the solution easier and faster. So let's see now how the arithmetic works..

We may want to begin with some easy and basic operations.
So just worming up for now.

Doing additions first, we can get 0. For instance, adding together $x - 2y$ and $2y - x$, we get $x - 2y + (2y - x) = x - 2y + 2y - x = 0$.

Adding 1 to a polynomial $3x + y$, we get $3x + y + 1$.

Adding $2x$ to $y + x^2$, we get $y + x^2 + 2x$. Adding c to $x + 2x^2$, we get $x + 2x^2 + c$.

Adding $x + 1$ to $x + 2x^2$, then looking at like-terms, we get $2x + 2x^2 + 1$.

Multiplying 2 by $3x + y$, we get $2(3x + y) = 6x + 2y$.

Multiplying $2x$ by $y + x^2$, we get $2x(y + x^2) = 2xy + 2x^3$.

Multiplying c by $x + 2x^2$, we get $c(x + 2x^2) = cx + 2cx^2$.

Multiplying $x + 1$ by $x + 2x^2$,
we get $(x + 1)(x + 2x^2) = x^2 + x + 2x^3 + 2x^2 = 2x^3 + 3x^2 + x$.

Subtracting 1 from $3x + y$, we get $3x + y - 1$.

Subtracting $3x + y$ from $3x + y + 1$,
we get $3x + y + 1 - (3x + y) = 3x + y + 1 - 3x - y = 1$.

Dividing $3x + y$ by 2, we get $\frac{3x + y}{2} = \frac{3}{2}x + \frac{1}{2}y$.

Dividing $6x + 2y$ by $3x + y$, we get $\frac{6x + 2y}{3x + y} = \frac{2(3x + y)}{3x + y} = 2$.

Dividing $x + x^2$ by $x + 1$, we get $\frac{x^2 + x}{x + 1} = \frac{x(x + 1)}{x + 1} = x$.

Dividing $cx^2 + cy$ by $x^2 + y$, we get $\frac{cx^2 + cy}{x^2 + y} = \frac{c(x^2 + y)}{x^2 + y} = c$.

Dividing $x^2 + y$ by x^3, we get $\frac{x^2 + y}{x^3} = \frac{x^2}{x^3} + \frac{y}{x^3} = \frac{1}{x} + \frac{y}{x^3}$.

1.1. Polynomial Arithmetic 2

There are several laws shared by polynomial arithmetic and number arithmetic. They are basic laws in arithmetic, and are probably well-known to you.

Let's now however, go over the laws, simply because they are so important. The more basic, the more important. To begin with, we have a basic law as follows

If we multiply (or divide) by the same both the numerator and the denominator, in a fraction, of course, the value of the fraction does not change.

So assuming $\boldsymbol{B}$ and $\boldsymbol{C} \neq \mathbf{0}$, we can get

$\boldsymbol{A} \div \boldsymbol{B} = (\boldsymbol{AC}) \div (\boldsymbol{BC})$. In other words, $\dfrac{\boldsymbol{A}}{\boldsymbol{B}} = \dfrac{\boldsymbol{AC}}{\boldsymbol{BC}}$.

$\boldsymbol{A} \div \boldsymbol{B} = (\boldsymbol{A} \div \boldsymbol{C}) \div (\boldsymbol{B} \div \boldsymbol{C})$. That is, $\dfrac{\boldsymbol{A}}{\boldsymbol{B}} = \dfrac{\dfrac{\boldsymbol{A}}{\boldsymbol{C}}}{\dfrac{\boldsymbol{B}}{\boldsymbol{C}}}$

In sum, $\boldsymbol{A} \div \boldsymbol{B} = (\boldsymbol{AC}) \div (\boldsymbol{BC}) = (\boldsymbol{A} \div \boldsymbol{C}) \div (\boldsymbol{B} \div \boldsymbol{C})$, that is, $\dfrac{\boldsymbol{A}}{\boldsymbol{B}} = \dfrac{\boldsymbol{AC}}{\boldsymbol{BC}} = \dfrac{\dfrac{\boldsymbol{A}}{\boldsymbol{C}}}{\dfrac{\boldsymbol{B}}{\boldsymbol{C}}}$

Note that $\boldsymbol{B}$ and $\boldsymbol{C}$ above cannot be 0, simply because no denominator can be 0, which is in fact, another fundamental law in arithmetic. So keep in mind no division by 0.

And taking some instances with numbers, we can have

$\mathbf{15 \div 30 = (15 \cdot 3) \div (30 \cdot 3)}$. In other words, $\frac{\mathbf{15}}{\mathbf{30}} = \frac{\mathbf{15 \cdot 3}}{\mathbf{30 \cdot 3}}$.

$\mathbf{15 \div 30 = (15 \div 3) \div (30 \div 3)}$. That is, $\frac{\mathbf{15}}{\mathbf{30}} = \frac{\frac{15}{3}}{\frac{30}{3}}$, simply because we get $\frac{\frac{15}{3} \cdot 3}{\frac{30}{3} \cdot 3} = \frac{\mathbf{15}}{\mathbf{30}}$.

In sum, $\mathbf{15 \div 30 = (15 \cdot 3) \div (30 \cdot 3) = (15 \div 3) \div (30 \div 3)}$, that is, $\frac{\mathbf{15}}{\mathbf{30}} = \frac{\mathbf{15 \cdot 3}}{\mathbf{30 \cdot 3}} = \frac{\frac{15}{3}}{\frac{30}{3}}$

Let's next, move on to a similar basic law.
If we multiply (or divide) by the same both sides of an equality, the equality maintains.
Suppose for instance, $\boldsymbol{A} = \boldsymbol{B}$.
Then, we can get $\boldsymbol{AC} = \boldsymbol{BC}$, and $\frac{\boldsymbol{A}}{\boldsymbol{D}} = \frac{\boldsymbol{B}}{\boldsymbol{D}}$, where $\boldsymbol{D} \neq \mathbf{0}$, of course.

Next, let's move on to another basic law.
If we add the same to both sides of an equality, the equality maintains.
And the same is true for subtractions, too. So if we subtract the same from both sides of an equality, the equality maintains.

Suppose for instance, $\boldsymbol{A} = \boldsymbol{B}$. Then, we can get

$\boldsymbol{A} + \boldsymbol{C} = \boldsymbol{B} + \boldsymbol{C}$, and $\boldsymbol{A} - \boldsymbol{C} = \boldsymbol{B} - \boldsymbol{C}$.

In sum, we have $\boldsymbol{A} \pm \boldsymbol{C} = \boldsymbol{B} \pm \boldsymbol{C}$. What then is good about the basic laws above?

We often use the laws, or rather, we have to go by them when solving equations.
Solving for instance, two equations $\mathbf{2}\boldsymbol{x} - \mathbf{5} = \mathbf{6}$, and $\mathbf{0.5}\boldsymbol{x} + \mathbf{3} = \mathbf{4}$,
we can get $\mathbf{2}\boldsymbol{x} - \mathbf{5} = \mathbf{3} \Rightarrow \mathbf{2}\boldsymbol{x} - \mathbf{5} + \mathbf{5} = \mathbf{3} + \mathbf{5} \Rightarrow \mathbf{2}\boldsymbol{x} = \mathbf{8} \Rightarrow \frac{2\boldsymbol{x}}{2} = \frac{8}{2} = \mathbf{4} \Rightarrow \boldsymbol{x} = \mathbf{4}$.
And next, we get

$\mathbf{0.5}\boldsymbol{x} + \mathbf{3} = \mathbf{4} \Rightarrow \mathbf{0.5}\boldsymbol{x} + \mathbf{3} - \mathbf{3} = \mathbf{4} - \mathbf{3} \Rightarrow \mathbf{0.5}\boldsymbol{x} = \mathbf{1} \Rightarrow \mathbf{0.5}\boldsymbol{x} \cdot \mathbf{2} = \mathbf{1} \cdot \mathbf{2} = \mathbf{2} \Rightarrow \boldsymbol{x} = \mathbf{2}$.

So doing the same to both sides at each step, we eventually get the solution.
Normally though, in case of adding the same to both sides, we do it this way, of course:

$\mathbf{2}\boldsymbol{x} \mathbf{- 5 = 3} \Rightarrow \mathbf{2}\boldsymbol{x} \mathbf{= 3 + 5}$.

So it looks as if we moved a term to the other side changing the sign.
What's actually happened is however, we've added the same to both sides.

Next, we have another set of laws often used in arithmetic, and usually call them ***three basic laws***, which however, do *not always* apply in subtractions and divisions. The three laws *always apply* in additions and multiplications only. And the three are as follows.

One is ***communicative***, another is ***associative***, and the other is ***distributive***.

Commutative law: $\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$, and $\boldsymbol{AB} = \boldsymbol{BA}$.

So for instance, we can have

$\mathbf{1 + 2 = 2 + 1 = 3}$, $\mathbf{1 + (-2) = (-2) + 1 = -1}$, and $\mathbf{(-1) + (-2) = (-2) + (-1) = 3}$.

$\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}$, $\boldsymbol{x} + (-\boldsymbol{y}) = (-\boldsymbol{y}) + \boldsymbol{x} = \boldsymbol{x} - \boldsymbol{y}$, and $(-\boldsymbol{x}) + (-\boldsymbol{y}) = (-\boldsymbol{y}) + (-\boldsymbol{x}) = -\boldsymbol{x} - \boldsymbol{y} = -(\boldsymbol{x} + \boldsymbol{y})$.

$\mathbf{3 \cdot 4 = 4 \cdot 3 = 12}$, $\mathbf{3 \cdot (-4) = (-4) \cdot 3 = -12}$, and $\mathbf{(-3) \cdot (-4) = (-4) \cdot (-3) = 12}$.

$\boldsymbol{xy} = \boldsymbol{yx}$, $\boldsymbol{x}(-\boldsymbol{y}) = (-\boldsymbol{y})\boldsymbol{x} = -\boldsymbol{xy}$, and $(-\boldsymbol{x})(-\boldsymbol{y}) = (-\boldsymbol{y})(-\boldsymbol{x}) = \boldsymbol{xy}$.

Associative law: $\boldsymbol{A} + \boldsymbol{B} + \boldsymbol{C} = (\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C} = \boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C})$,

and $\boldsymbol{ABC} = (\boldsymbol{AB})\boldsymbol{C} = \boldsymbol{A}(\boldsymbol{BC})$

So when we do additions only, the order does not matter.
And the same is true for doing multiplications only, too.

So for instance,

1 + 2 + 3 = (1 + 2) + 3 = 1 + (2 + 3) = 6

-1 + (-2) + (-3) = {-1 + (-2)} + (-3) = -1 + {-2 + (-3)} = -6

1 + (-2) + (-3) = {1 + (-2)} + (-3) = 1 + {-2 + (-3)} = -4

-1 + 2 + (-3) = (-1 + 2) + (-3) = -1 + {2 + (-3)} = -2

1 + 2 + (-3) = (1 + 2) + (-3) = 1 + {2 + (-3)} = 0

1 + (-2) + 3 = {1 + (-2)} + 3 = 1 + (-2 + 3) = 2

$x + y + z = (x + y) + z) = x + (y + z)$

$-x + (-y) + (-z) = \{-x + (-y)\} + (-z) = -x + \{-y + (-z)\} = -x - y - z = -(x + y + z)$

$x + (-y) + (-z) = \{x + (-y)\} + (-z) = x + \{-y + (-z)\} = x - y - z$

$-x + y + (-z) = (-x + y) + (-z) = -x + \{y + (-z)\} = -x + y - z = y - x - z$

$x + y + (-z) = (x + y) + (-z) = x + \{y + (-z)\} = x + y - z$

$x + (-y) + z = \{x + (-y)\} + z = x + (-y + z) = x - y + z$

2·3·4 = (2·3)4 = 2(3·4) = 24 **-2·(-3)·(-4) = {-2·(-3)}(-4) = -2{-3·(-4)} = -24**

2·(-3)·(-4) = {2·(-3)}(-4) = 2{-3·(-4)} = 24 **-2·3·(-4) = (-2·3)(-4) = -2{3·(-4)} = 24**

-2·3·4 = (-2·3)4 = -2(3·4) = -24 **2·(-3)·4 = {2·(-3)}4 = 2(-3·4) = -24**

$x \cdot y \cdot z = (x \cdot y)z = x(y \cdot z) = xyz$ $-x \cdot (-y) \cdot (-z) = \{-x \cdot (-y)\}(-z) = -x\{-y \cdot (-z)\} = -xyz$

$x \cdot (-y) \cdot (-z) = \{x \cdot (-y)\}(-z) = x\{-y \cdot (-z)\} = xyz$ $x \cdot y \cdot (-z) = (x \cdot y)(-z) = x\{y \cdot (-z)\} = -xyz$

$x\cdot(-y)\cdot z = \{x\cdot(-y)\}z = x(-y\cdot z) = -xyz$ $\quad -x\cdot y\cdot(-z) = (-x\cdot y)(-z) = -x\{y\cdot(-z)\} = xyz$

$\{(x^2 - yz)x\}(y + 1) = (x^3 - xyz)(y + 1) = x^3y + x^3 - xy^2z - xyz$, and

$(x^2 - yz)\{x(y + 1)\} = (x^2 - yz)(xy + x) = x^3y + x^3 - xy^2z - xyz$.

So we get $(x^2 - yz)x(y + 1) = \{(x^2 - yz)x\}(y + 1) = (x^2 - yz)\{x(y + 1)\}$.

Distributive law: $A(B + C) = AB + AC$, and $A(B - C) = AB - AC$

So for instance,

$2(3 + 4) = 2\cdot3 + 2\cdot4 = 6 + 8 = 14$ $\quad -2\{-3 + (-4)\} = -2\cdot(-3) + (-2)\cdot(-4) = 6 + 8 = 14$

$2\{-3 + (-4)\} = 2\cdot(-3) + 2\cdot(-4) = -6 + (-8) = -6 - 8 = -14$

$-2\{3 + (-4)\} = -2\cdot3 + (-2)\cdot(-4) = -6 + 8 = 2$

$-2\{(-3) + 4\} = -2\cdot(-3) + (-2)\cdot4 = 6 + (-8) = 6 - 8 = -2$

$x(y + z) = xy + xz$, and $-x\{-y + (-z)\} = xy + xz$.

$x\{-y + (-z)\} = -xy + (-xz) = -xy - xz = -x(y + z)$

$-x\{y + (-z)\} = -xy + xz = x(z - y)$

$x\{y + (-z)\} = xy + (-xz) = xy - xz = x(y - z)$

$2(3 - 4) = 2\cdot3 - 2\cdot4 = 6 - 8 = -2$ $\quad -2\{-3 - (-4)\} = -2\cdot(-3) - (-2)\cdot(-4) = 6 - 8 = -2$

$2\{-3 - (-4)\} = 2\cdot(-3) - 2\cdot(-4) = -6 - (-8) = -6 + 8 = 2$

$-2\{3-(-4)\} = -2\cdot 3-(-2)\cdot(-4) = -6-8 = -14$

$-2\{(-3)-4\} = -2\cdot(-3)-(-2)\cdot 4 = 6-(-8) = 6+8 = 14$

$x(y-z) = xy-xz$, and $-x\{-y-(-z)\} = xy-xz$.

$x\{-y-(-z)\} = -xy-(-xz) = -xy+xz = -x(-y+z) = x(y-z)$

$-x\{y-(-z)\} = -xy-xz = -x(y+z)$

$x\{y-(-z)\} = xy-(-xz) = xy+xz = x(y+z)$

Besides, we have $A\cdot B\div C = (A\cdot B)\div C = A\cdot(B\div C)$.

That's because, $A\cdot B\div C = A\cdot B\cdot\frac{1}{C} = (A\cdot B)\cdot\frac{1}{C} = A\cdot(B\cdot\frac{1}{C}) = \frac{AB}{C}$.

In fact, we have $A\cdot B\div C = (A\cdot B)\div C = \frac{AB}{C}$, and $A\cdot(B\div C) = A\cdot\frac{B}{C} = \frac{AB}{C}$.

For instance, $3\cdot 4\div 2 = 3\cdot 4\cdot\frac{1}{2} = (3\cdot 4)\cdot\frac{1}{2} = 3\cdot(4\cdot\frac{1}{2}) = 6$, and

$-3\cdot(-4)\div(-2) = -3\cdot(-4)\cdot(-\frac{1}{2}) = \{-3\cdot(-4)\}\cdot(-\frac{1}{2}) = -3\cdot\{-4\cdot(-\frac{1}{2})\} = -6$.

In cases of subtractions and divisions however, the three basic laws do ***not always*** apply. In divisions and subtractions, operands do not commute or associate. And a division doesn't distribute itself. And let's see now, some examples.

To begin with, we have $A\div B\neq B\div A$, that is, $\frac{A}{B}\neq\frac{B}{A}$, and $A-B\neq B-A$.

For instance, we have $15\div 3\neq 3\div 15$, that is, $\frac{15}{3}\neq\frac{3}{15}$, and $15-3\neq 3-15$.

Next, we have $(A\div B)\div C\neq A\div(B\div C)$, so operands do not associate in divisions.

That's because $(A \div B) \div C = \dfrac{\dfrac{A}{B}}{C} = \dfrac{A}{B} \cdot \dfrac{1}{C} = \dfrac{A}{BC}$, but we get

$$A \div (B \div C) = \frac{A}{\dfrac{B}{C}} = A \cdot \frac{1}{\dfrac{B}{C}} = A \cdot \frac{C}{B} = \frac{AC}{B}$$

So we have $A \div B \div C = (A \div B) \div C \neq A \div (B \div C)$.

For instance, we have $36 \div 6 \div 3 = (36 \div 6) \div 3 \neq 36 \div (6 \div 3)$.

Also, we have $(A \div B) \cdot C \neq A \div (B \cdot C)$.

That's because $A \div B \cdot C = (A \div B) \cdot C = \dfrac{A}{B} \cdot C = \dfrac{AC}{B}$, but $A \div B \cdot C \neq A \div (B \cdot C) = \dfrac{A}{BC}$.

So for instance, we have $(36 \div 6) \cdot 3 \neq 36 \div (6 \cdot 3)$.

Next, we have $A - B - C = (A - B) - C$, but $A - B - C \neq A - (B - C) = A - B + C$.

Thus, we have $A - B - C = (A - B) - C \neq A - (B - C)$, so operands do not associate in subtractions, either.

For instance, we have $5 - 3 - 1 = 2 - 1 = 1$, and $(5 - 3) - 1 = 2 - 1 = 1$,

so we get $5 - 3 - 1 = (5 - 3) - 1$,

but $5 - 3 - 1 \neq 5 - (3 - 1) = 5 - 3 + 1 = 2 + 1$, or $5 - 3 - 1 \neq 5 - (3 - 1) = 5 - 2$

Thus, we get $5 - 3 - 1 = (5 - 3) - 1 \neq 5 - (3 - 1)$.

Next, we have $A \div (B + C) \neq (A \div B) + (A \div C)$, that is, $\dfrac{A}{B+C} \neq \dfrac{A}{B} + \dfrac{A}{C}$.

So for instance, we have $24 \div (2 + 4) \neq (24 \div 2) + (24 \div 4)$, that is, $\dfrac{24}{2+4} \neq \dfrac{24}{2} + \dfrac{24}{4}$.

Also, we have $\boldsymbol{A \div (B - C) \neq (A \div B) - (A \div C)}$, that is, $\boldsymbol{\frac{A}{B-C} \neq \frac{A}{B} - \frac{A}{C}}$.

For instance, we have $\boldsymbol{24 \div (4 - 2) \neq (24 \div 4) - (24 \div 2)}$, that is, $\boldsymbol{\frac{24}{4-2} \neq \frac{24}{4} - \frac{24}{2}}$.

So a division does not distribute itself.
The four operations in arithmetic are all connected, though. How?

Doing a subtraction, we are adding the negative. $5 - 3 = 5 + (-3)$.

Doing a division, we are multiplying the reciprocal. $\frac{6}{2} = 6 \cdot \frac{1}{2}$

And adding together many of the same things, we do a multiplication. $4 + 4 + 4 = 3 \cdot 4$.

1.2. Polynomial Arithmetic 3

The four operations in arithmetic are all connected. How?

Arithmetic starts with an addition, which is a reverse operation of a subtraction, which is in fact, an addition of the number negative.
And adding together same numbers, we do a multiplication, which is a reverse operation of a division, which is in fact, the multiplication of the reciprocal.

Let's now, have a look at some more examples on operations in polynomial arithmetic.

To begin with, assuming $\boldsymbol{A} = 2x + 3y$, and $\boldsymbol{B} = 5x + 7y$, we can get

$$\boldsymbol{A} + \boldsymbol{B} = 2x + 3y + 5x + 7y = x(2 + 5) + y(3 + 7) = 7x + 10y$$

$$2\boldsymbol{A} - \boldsymbol{B} = 2(2x + 5y) - (5x + 7y) = 4x + 10y - 5x - 7y = x(4 - 5) + y(10 - 7) = -x + 3y$$

$$3\boldsymbol{A} + 2y = 3(2x + 5y) + 2y = 6x + 15y + 2y = 6x + 17y$$

$$\frac{1}{2}(2\boldsymbol{A} - \boldsymbol{B}) = \frac{1}{2}\{2(2x + 3y) - (5x + 7y)\} = \frac{1}{2}(4x + 6y - 5x - 7y) = \frac{1}{2}(-x - y) = -\frac{1}{2}(x + y).$$

Next, assuming $\boldsymbol{A} = 3xy^2 + yz$, $\boldsymbol{B} = 2x + 7xy^2 + z + 9yz$, and $\boldsymbol{C} = 3x + 8y + 9z$,

we can get $\frac{1}{3}\boldsymbol{A} + 2\boldsymbol{B} - 3\boldsymbol{C} = \frac{1}{3}(3xy + yz) + 2(2x + 7xy^2 + z + 9yz) - 3(3x + 8y + 9z)$

$$= xy + \frac{1}{3}yz + 4x + 14xy^2 + 2z + 18yz - 9x - 24y - 27z$$

$$= 14xy^2 + xy + \frac{1 + 3 \cdot 18}{3}yz - 5x - 24y - 25z = 14xy^2 + xy + \frac{55}{3}yz - 5x - 24y - 25z.$$

Assuming next, $A = x + y$, $B = y + z$, and $C = x + z$, we can get

$A + B + C = x + y + y + z + x + z = 2x + 2y + 2z = 2(x + y + z)$

$A - B + C = x + y - y - z + x + z = 2x$

$A + B - C = x + y + y + z - x - z = 2y$

$-A + B + C = -x - y + y + z + x + z = 2z$

$(A - B + C) + (A + B - C) + (-A + B + C) = 2x + 2y + 2z = 2(x + y + z) = A + B + C$

$A + (B + C) = x + y + (y + z + x + z) = x + y + y + z + x + z = 2(x + y + z) = A + B + C$

$(A + B) + C = (x + y + y + z) + x + z = x + y + y + z + x + z = 2(x + y + z) = A + B + C$

$AB = (x + y)(y + z) = (x + y)B = xB + yB = x(y + z) + y(y + z) = xy + xz + y^2 + yz$

$= A(y + z) = Ay + Az = (x + y)y + (x + y)z = xy + y^2 + xz + yz$

$(AB)C = (x + y)(y + z)C = (xy + xz + y^2 + yz)C = (xy + xz + y^2 + yz)(x + z)$

$= x(xy + xz + y^2 + yz) + z(xy + xz + y^2 + yz)$

$= x^2y + x^2z + xy^2 + xyz + xyz + xz^2 + y^2z + yz^2 = (y + z)x^2 + (y^2 + 2yz + z^2)x + y^2z + yz^2.$

$A(BC) = A\{(y + z)(x + z)\} = A(xy + yz + xz + z^2) = (x + y)(xy + yz + xz + z^2)$

$= x(xy + yz + xz + z^2) + y(xy + yz + xz + z^2)$

$= x^2y + xyz + x^2z + xz^2 + xy^2 + y^2z + xyz + yz^2 = x^2y + x^2z + 2xyz + xz^2 + xy^2 + y^2z + yz^2$

$= (y + z)x^2 + (y^2 + 2yz + z^2)x + y^2z + yz^2.$

Assuming next, $A = x - y$, and $B = y - z$, we can get

$A + B = (x - y) + (y - z) = x - y + y - z = x - z$

$B + A = (y - z) + (x - y) = y - z + x - y = x - z$

$AB - BA = (x - y)(y - z) - (y - z)(x - y) = xy - xz - y^2 + yz - (xy - y^2 - xz + yz)$

$= xy - xz - y^2 + yz - xy + y^2 + xz - yz = 0$.

Suppose this time, $A = x + y$, $B = y - z$, and $C = z - x$. Then, we can get

$A(B + C) = (x + y)\{(y - z) + (z - x)\} = (x + y)(y - z + z - x) = (x + y)(y - x)$

$= xy - x^2 + y^2 - xy = y^2 - x^2$, and

$AB + AC = (x + y)(y - z) + (x + y)(z - x) = xy - xz + y^2 - yz + xz - x^2 + yz - xy = y^2 - x^2$.

And let's next, take a look at how we can do divisions with polynomials.

Suppose for instance, $A = 2x^4 + 3x^3 + 5x^2 + 7x$, $B = 8x$, and we want to divide A by B. Then, we can get

$$\frac{A}{B} = \frac{2x^4 + 3x^3 + 5x^2 + 7x}{8x} = \frac{2x^4}{8x} + \frac{3x^3}{8x} + \frac{5x^2}{8x} + \frac{7x}{8x} = \frac{2x^3}{8} + \frac{3x^2}{8} + \frac{5x}{8} + \frac{7}{8}$$

$= \frac{1}{4}x^3 + \frac{3}{8}x^2 + \frac{5}{8}x + \frac{7}{8}$, which can be put this way, too: $\frac{1}{8}(2x^3 + 3x^2 + 5x + 7)$.

However, divisions with polynomials are not normally that simple as above.
So let's now take a look at the division a bit more systematically.
Basically, doing a division, we get the number of the divisors that the dividend can have.
More precisely, we get the maximum number of the divisors that the dividend can have.
What do we mean by the divisor though?

Doing a division, we divide a dividend by a divisor.
Then, we get a quotient and a remainder if any.

If a divisor divides a dividend, the remainder is 0.
If a divisor doesn't divide a dividend, the remainder is not 0.

So for instance, if we just say that we divide ***A*** by ***B***, we call ***B*** the divisor, and call ***A*** the dividend. And we don't know if the remainder is 0 or not until we get the division done. What then do we mean by the quotient?

It was stated above that basically, doing a division, we get the number of the divisors that the dividend can have, and that more precisely, we get the maximum number of the divisors that the dividend can have.

So for instance, dividing ***A*** by ***B***, we get the maximum number of ***B***s that the dividend ***A*** can have. Then, the maximum number is called the quotient, and the remainder can be 0 or nonzero.

And if the remainder exists, that is, the remainder is not 0, ***A*** is not a multiple of ***B***.
If the remainder does not exist, that is, the remainder is 0, ***A*** is a multiple of ***B***.

So for instance, dividing $(\mathbf{3}\boldsymbol{x}^2 + \mathbf{6}\boldsymbol{y})$ by $(\boldsymbol{x}^2 + \mathbf{2}\boldsymbol{y})$, we get 3 as the quotient, since the number of the divisors, that is, the number of $(\boldsymbol{x}^2 + \mathbf{2}\boldsymbol{y})$s that the dividend $(\mathbf{3}\boldsymbol{x}^2 + \mathbf{6}\boldsymbol{y})$ can have is 3. How?

We have $\mathbf{3}\boldsymbol{x}^2 + \mathbf{6}\boldsymbol{y} = (\boldsymbol{x}^2 + \mathbf{2}\boldsymbol{y}) + (\boldsymbol{x}^2 + \mathbf{2}\boldsymbol{y}) + (\boldsymbol{x}^2 + \mathbf{2}\boldsymbol{y})$, so we get $\mathbf{3}\boldsymbol{x}^2 + \mathbf{6}\boldsymbol{y} = \mathbf{3}\cdot(\boldsymbol{x}^2 + \mathbf{2}\boldsymbol{y})$.

And also, dividing $\mathbf{3}\boldsymbol{x}^2 + \mathbf{6}\boldsymbol{y} + \boldsymbol{x} + \mathbf{1}$ by $\boldsymbol{x}^2 + \mathbf{2}\boldsymbol{y}$, we get 3 as the quotient, too, since the number of $(\boldsymbol{x}^2 + \mathbf{2}\boldsymbol{y})$s that the dividend $(\mathbf{3}\boldsymbol{x}^2 + \mathbf{6}\boldsymbol{y} + \boldsymbol{x} + \mathbf{1})$ can have is 3.
And in this case, the remainder is not 0, and is $\boldsymbol{x} + \mathbf{1}$. How?

We have $\mathbf{3}\boldsymbol{x}^2 + \mathbf{6}\boldsymbol{y} + \boldsymbol{x} + \mathbf{1} = (\boldsymbol{x}^2 + \mathbf{2}\boldsymbol{y}) + (\boldsymbol{x}^2 + \mathbf{2}\boldsymbol{y}) + (\boldsymbol{x}^2 + \mathbf{2}\boldsymbol{y}) + \boldsymbol{x} + \mathbf{1}$.
So we get $\mathbf{3}\boldsymbol{x}^2 + \mathbf{6}\boldsymbol{y} + \boldsymbol{x} + \mathbf{1} = \mathbf{3}\cdot(\boldsymbol{x}^2 + \mathbf{2}\boldsymbol{y}) + (\boldsymbol{x} + \mathbf{1})$. And $\boldsymbol{x} + \mathbf{1}$ is the remainder.

So we do multiplications, too, doing divisions.
Doing a division though, we do not just find the maximum number of the divisors that the dividend can have.

Doing a division in fact, we find first, the product of the quotient and the divisor, and then, the sum of the product and the remainder.
So for instance, dividing $\boldsymbol{A}$ by $\boldsymbol{B}$, and assuming $\boldsymbol{Q}$ is the quotient, and $\boldsymbol{R}$ is the remainder, we can put the dividend $\boldsymbol{A}$ the way as follows.

$$\boldsymbol{A = BQ + R}.$$

So for instance, dividing $\mathbf{0.3}\boldsymbol{x}^2 + \mathbf{0.6}\boldsymbol{y} + \boldsymbol{x} + \mathbf{1}$ by $\boldsymbol{x}^2 + \mathbf{2}\boldsymbol{y}$, we get 0.3 as the quotient, and get $\boldsymbol{x} + \mathbf{1}$ as the remainder. Thus, we can put the dividend the way as follows.

$$\mathbf{0.3}\boldsymbol{x}^2 + \mathbf{0.6}\boldsymbol{y} + \boldsymbol{x} + \mathbf{1} = \mathbf{0.3}(\boldsymbol{x}^2 + \mathbf{2}\boldsymbol{y}) + \boldsymbol{x} + \mathbf{1}.$$

So the dividend can be expressed by the sum of the remainder and the product of the quotient and the divisor.
And of course, the quotient does not have to be a number.
In fact, if a polynomial gets divided by a polynomial, the quotient can be not only a number but a monomial or a polynomial, too.

And doing polynomial divisions, we often do a special division called a *synthetic division*, which is a division of a polynomial by a binomial as $\boldsymbol{x} + \mathbf{1}$, and is quite important. So we are going to cover such a division in a separate section.

We can use such a division when doing polynomial factorizations, often called factoring polynomials. We can use it, too, though, when just dividing a polynomial by a binomial. And it is covered in fact, in the next section.

We'll begin with how a synthetic division works, and will cover a general case where we divide a polynomial by a binomial $\boldsymbol{ax - b}$. So in the next section, an example will be as follows.

$\boldsymbol{P = (ax - b)Q + R}$, where $\boldsymbol{a \neq 0}$, and $\boldsymbol{P = c_0x^3 + c_1x^2 + c_2x + c_3}$.

Then, dividing the polynomial $\boldsymbol{P}$ by a binomial $(\boldsymbol{ax - b})$, we get

$\boldsymbol{Q = \frac{1}{a}\{c_0x^2 + (c_1 + c_0\alpha)x + c_2 + c_1\alpha + c_0\alpha^2\}}$, which is the quotient, which is a polynomial, too, of course, and $\boldsymbol{R = c_3 + c_2\alpha + c_1\alpha^2 + c_0\alpha^3}$ where $\boldsymbol{\alpha = \frac{b}{a}}$, and $\boldsymbol{R}$ is the remainder.

And of course, if the remainder $\boldsymbol{R}$ is 0, the binomial $(\boldsymbol{ax - b})$ divides the polynomial $\boldsymbol{P}$, and of course, so does the quotient $\boldsymbol{Q}$. It's simply because if $\boldsymbol{R = 0}$, we get $\boldsymbol{P = (ax - b)Q}$. So $(\boldsymbol{ax - b})$ and $\boldsymbol{Q}$ both are divisors of $\boldsymbol{P}$.

And we'll get to see the details on the division itself in the next section.

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Doing polynomial divisions, we often do a synthetic division, which is a division of a polynomial by a binomial as $x + 1$.

So for instance, dividing $x^3 + 2x^2 - 3x + 2$ by $x - 3$, we can do a synthetic division.

We often use synthetic divisions doing polynomial factorizations. And of course, we use it just dividing a polynomial by a binomial, since it is convenient to do synthetic division. So in this section, we are going to begin with a polynomial as follows.

$$P = (ax - b)Q + R, \text{ where } a \neq 0, \text{ and } P = c_0x^3 + c_1x^2 + c_2x + c_3.$$

Then, dividing the polynomial P by the polynomial $(ax - b)$, we get

$Q = \frac{1}{a}\{c_0x^2 + (c_1 + c_0\alpha)x + c_2 + c_1\alpha + c_0\alpha^2\}$, which is the quotient, which is a polynomial and $R = c_3 + c_2\alpha + c_1\alpha^2 + c_0\alpha^3$ where $\alpha = \frac{b}{a}$, and R is the remainder. How?

Suppose now, $T = b_0x^2 + b_1x + b_2$. Suppose also, $P = (x - \alpha)T + r$.

Then, if we divide P by $x - \alpha$, T is the quotient, and r is the remainder, then we get

$P = c_0x^3 + c_1x^2 + c_2x + c_3 = (x - \alpha)T + r = (x - \alpha)(b_0x^2 + b_1x + b_2) + r$

$= b_0x^3 + (b_1 - b_0\alpha)x^2 + (b_2 - b_1\alpha)x + (r - b_2\alpha)$.

And comparing terms in P, we get $c_0 = b_0$, $c_1 = b_1 - b_0\alpha$, $c_2 = b_2 - b_1\alpha$, and $c_3 = r - b_2\alpha$. So next, we can get

$b_0 = c_0$,

$b_1 = c_1 + b_0\alpha = c_1 + c_0\alpha \Rightarrow b_1 = c_1 + c_0\alpha$,

$b_2 = c_2 + b_1\alpha = c_2 + c_1\alpha + c_0\alpha^2 \Rightarrow b_2 = c_2 + c_1\alpha + c_0\alpha^2$, and

$r = c_3 + b_2\alpha = c_3 + c_2\alpha + c_1\alpha^2 + c_0\alpha^3 \Rightarrow r = c_3 + c_2\alpha + c_1\alpha^2 + c_0\alpha^3$.

And we have set $P = (x - \alpha)T + r$ where $\alpha = \frac{b}{a}$. So we can put P the way below, too.

$$P = a \cdot \frac{1}{a}(x - \alpha)T + r = \frac{1}{a}(ax - a\alpha)T + r = (ax - a \cdot \frac{b}{a}) \cdot \frac{1}{a} \cdot T + r = (ax - b) \cdot \frac{1}{a} \cdot T + r.$$

And also, we have $T = b_0x^2 + b_1x + b_2$, too.

So we get $P = (ax - b) \cdot \frac{1}{a} \cdot T + r = (ax - b) \cdot \frac{1}{a} \cdot (b_0x^2 + b_1x + b_2) + r$. Besides, we have

$b_0 = c_0$, $b_1 = c_1 + c_0\alpha$, $b_2 = c_2 + c_1\alpha + c_0\alpha^2$, and $r = c_3 + c_2\alpha + c_1\alpha^2 + c_0\alpha^3$.

So doing substitutions for b_k where $k = 0$, 1, and 2. we get

$$P = (ax - b)\frac{1}{a}\{c_0x^2 + (c_1 + c_0\alpha)x + (c_2 + c_1\alpha + c_2\alpha^2)\} + (c_3 + c_2\alpha + c_1\alpha^2 + c_0\alpha^3).$$

Thus now, getting back to this: $P = (ax - b)Q + R$, we get

$$Q = \frac{1}{a}\{c_0x^2 + (c_1 + c_0\alpha)x + c_2 + c_1\alpha + c_0\alpha^2\}, \text{ and } R = c_3 + c_2\alpha + c_1\alpha^2 + c_0\alpha^3.$$

What if we want to divide P by just a simple polynomial as $x - 2$?

We can put P in terms of $(x - \alpha)$, and we can simply put it the way as follows.

$$P = (ax - b)\frac{1}{a}\{c_0x^2 + (c_1 + c_0\alpha)x + c_2 + c_1\alpha + c_0\alpha^2\} + c_3 + c_2\alpha + c_1\alpha^2 + c_0\alpha^3$$

$$= (x - \frac{b}{a})\{c_0x^2 + (c_1 + c_0\alpha)x + c_2 + c_1\alpha + c_0\alpha^2\} + c_3 + c_2\alpha + c_1\alpha^2 + c_0\alpha^3.$$

And we know $\alpha = \frac{b}{a}$.

So we get $P = (x - \alpha)\{c_0x^2 + (c_1 + c_0\alpha)x + c_2 + c_1\alpha + c_0\alpha^2\} + c_3 + c_2\alpha + c_1\alpha^2 + c_0\alpha^3$.

Now, if for instance, we set $\alpha = 2$, the quotient is $c_0x^2 + (c_1 + 2c_0)x + c_2 + 2c_1 + 4c_0$, and we can say that the remainder is $c_3 + 2c_2 + 4c_1 + 8c_0$.

So setting $P = (x - d)q + R$, and $P = c_0x^3 + c_1x^2 + c_2x + c_3$, we can see that

the quotient is $q = c_0x^2 + (c_1 + c_0d)x + c_2 + c_1d + c_0d^2$,

and the remainder is $R = c_3 + c_2d + c_1d^2 + c_0d^3$.

What if $P = c_0x^4 + c_1x^3 + c_2x^2 + c_3x + c_4$, though?

Then, we get $q = c_0x^3 + (c_1 + c_0d)x^2 + (c_2 + c_1d + c_0d^2)x + c_3 + c_2d + c_1d^2 + c_0d^3$, which is the quotient, and $R = c_4 + c_3d + c_2d^2 + c_1d^3 + c_0d^4$, which is the remainder.

So the division looks quite complicated, doesn't it?

We can do the division above using a method called a synthetic division, and using the synthetic division, we can divide $P = c_0x^3 + c_1x^2 + c_2x + c_3$ by $(x - d)$ the way as follows.

d	c_0	c_1	c_2	c_3
		c_0d	$c_1d + c_0d^2$	$c_2d + c_1d^2 + c_0d^3$
	c_0	$c_1 + c_0d$	$c_2 + c_1d + c_0d^2$	$c_3 + c_2d + c_1d^2 + c_0d^3$

What's going on?

To begin with, put below the horizontal line, the coefficient c_0 of x^3 in the polynomial P.

Next, multiplying c_0 by d, we get c_0d, which is put below the coefficient c_1 of x^2 in P.

Then, adding together c_1 and c_0d, we get $c_1 + c_0d$, which is put below the horizontal line.

Next, multiplying $(c_1 + c_0d)$ by d, we get $c_1d + c_0d^2$, which is put below the coefficient c_2 of x in P.

Then, adding together c_2 and $(c_1d + c_0d^2)$, we get $c_2 + c_1d + c_0d^2$, which is put below the horizontal line.

Next, multiplying $(c_2 + c_1d + c_0d^2)$ by d, we get $c_2d + c_1d^2 + c_0d^3$, which is put below the constant term c_3 in P.

Then, adding together c_3 and $(c_2d + c_1d^2 + c_0d^3)$, we get $c_3 + c_2d + c_1d^2 + c_0d^3$, which is put below the horizontal line, and is the remainder.

So setting $P = (x - d)q + R$, we can say that q is the quotient and that R is the remainder, and reflecting the facts stated in the steps above, we can say that

the first term c_0 below the horizontal line is the coefficient of x^2 in the quotient q,

the second term $(c_1 + c_0d)$ below the line is the coefficient of x in q, and

the third term $(c_2 + c_1d + c_0d^2)$ below the line is the constant term in q.

That is, $q = c_0x^2 + (c_1 + c_0d)x + c_2 + c_1d + c_0d^2$, and $R = c_3 + c_2d + c_1d^2 + c_0d^3$.

And of course, if the remainder R is 0, the binomial $(x - d)$ divides the polynomial P, and of course, so does the quotient q. It's simply because if $R = 0$, we get $P = (x - d)q$. So $(x - d)$ and q both are divisors of P. So what?

Using the fact, we can solve many equations as $5x^3 + 2x^2 - x - 6 = 0$.

We are going to cover the actual steps in the synthetic division, and take a specific example in the next section.

1.4. Polynomial Arithmetic 5

In this section, we are going to take actual steps in the synthetic division where we divide a polynomial $P = c_0x^3 + c_1x^2 + c_2x + c_3$ by a binomial $x - d$.

To begin with, dividing P by $x - d$, and assuming q is the quotient, and R is the remainder, we can set $P = (x - d)q + R$.

Then, we get $q = c_0x^2 + (c_1 + c_0d)x + c_2 + c_1d + c_0d^2$, and $R = c_3 + c_2d + c_1d^2 + c_0d^3$.

And of course, if the remainder R is 0, the binomial $(x - d)$ divides the polynomial P, and of course, so does the quotient q. It's simply because if $R = 0$, we get $P = (x - d)q$. So $(x - d)$ and q both are divisors of P.

And in this case, if $x = d$, we get $P = 0$, since $(x - d)q = (d - d)q = 0$.

So using the fact above, we can solve many equations as $x^3 - 2x^2 + 2x - 1 = 0$.
In fact, $x - 1$ divides $x^3 - 2x^2 + 2x - 1$, because $1^3 - 2 \cdot 1^2 + 2 - 1 = 1 - 2 + 2 - 1 = 0$.
And we can set $x^3 - 2x^2 + 2x - 1 = (x - 1)q = 0$, where q is a polynomial of degree 2.

And finding such a polynomial as the quotient q, we can use a method called a synthetic division, which is shown below.

d	c_0	c_1	c_2	c_3
		c_0d	$c_1d + c_0d^2$	$c_2d + c_1d^2 + c_0d^3$
	c_0	$c_1 + c_0d$	$c_2 + c_1d + c_0d^2$	$c_3 + c_2d + c_1d^2 + c_0d^3$

And doing the division the way above, we can take the steps the way as follows.

To begin with, put below the horizontal line, the coefficient c_0 of x^3 in the polynomial P.

d	c_0	c_1	c_2	c_3
	c_0			

Next, multiplying c_0 by d, we get c_0d, which is put below the coefficient c_1 of x^2 in P.

d	c_0	c_1	c_2	c_3
		c_0d		
	c_0			

Then, adding together c_1 and c_0d, we get $c_1 + c_0d$, which is put below the horizontal line.

d	c_0	c_1	c_2	c_3
		c_0d		
	c_0	$c_1 + c_0d$		

Next, multiplying $(c_1 + c_0d)$ by d, we get $c_1d + c_0d^2$, which is put below the coefficient c_2 of x in P.

d	c_0	c_1	c_2	c_3
		c_0d	$c_1d + c_0d^2$	
	c_0	$c_1 + c_0d$		

Then, adding together c_2 and $(c_1d + c_0d^2)$, we get $c_2 + c_1d + c_0d^2$, which is put below the horizontal line.

d	c_0	c_1	c_2	c_3
		c_0d	$c_1d + c_0d^2$	
	c_0	$c_1 + c_0d$	$c_2 + c_1d + c_0d^2$	

Next, multiplying ($c_2 + c_1d + c_0d^2$) by d, we get $c_2d + c_1d^2 + c_0d^3$, which is put below the constant term c_3 in P.

d	c_0	c_1	c_2	c_3
		c_0d	$c_1d + c_0d^2$	$c_2d + c_1d^2 + c_0d^3$
	c_0	$c_1 + c_0d$	$c_2 + c_1d + c_0d^2$	

Then, adding together c_3 and ($c_2d + c_1d^2 + c_0d^3$), we get $c_3 + c_2d + c_1d^2 + c_0d^3$, which is put below the horizontal line, and is the remainder.

d	c_0	c_1	c_2	c_3
		c_0d	$c_1d + c_0d^2$	$c_2d + c_1d^2 + c_0d^3$
	c_0	$c_1 + c_0d$	$c_2 + c_1d + c_0d^2$	$c_3 + c_2d + c_1d^2 + c_0d^3$

So the quotient q is as follows.

The first term c_0 below the horizontal line is the coefficient of x^2 in q.

The second term ($c_1 + c_0d$) below the line is the coefficient of x in q.

The third term ($c_2 + c_1d + c_0d^2$) below the line is the constant term in q.

That is, $q = c_0x^2 + (c_1 + c_0d)x + c_2 + c_1d + c_0d^2$, and $R = c_3 + c_2d + c_1d^2 + c_0d^3$.

So assuming $P = c_0x^3 + c_1x^2 + c_2x + c_3$, dividing P by $x - d$, and assuming q is the quotient, and R is the remainder, we get $P = (x - d)q + R$, where q and R are as above.

And putting together the entire steps in a sequence, we can put the sequence the way as follows.

d	c_0	c_1	c_2	c_3
	c_0			

d	c_0	c_1	c_2	c_3
		c_0d		
	c_0			

d	c_0	c_1	c_2	c_3
		c_0d		
	c_0	$c_1 + c_0d$		

d	c_0	c_1	c_2	c_3
		c_0d	$c_1d + c_0d^2$	
	c_0	$c_1 + c_0d$		

d	c_0	c_1	c_2	c_3
		c_0d	$c_1d + c_0d^2$	
	c_0	$c_1 + c_0d$	$c_2 + c_1d + c_0d^2$	

d	c_0	c_1	c_2	c_3
		c_0d	$c_1d + c_0d^2$	$c_2d + c_1d^2 + c_0d^3$
	c_0	$c_1 + c_0d$	$c_2 + c_1d + c_0d^2$	

d	c_0	c_1	c_2	c_3
		c_0d	$c_1d + c_0d^2$	$c_2d + c_1d^2 + c_0d^3$
	c_0	$c_1 + c_0d$	$c_2 + c_1d + c_0d^2$	$c_3 + c_2d + c_1d^2 + c_0d^3$

Let's now, for instance, divide $P = 2x^4 + 3x^3 + 5x^2 + 7x$ by $x - 2$.
Then, setting $P = (x - d)q + R$, we have $d = 2$, $c_0 = 2$, $c_1 = 3$, $c_2 = 5$, $c_3 = 7$, and $c_4 = 0$.
So doing the synthetic division, we get the steps as follows.

2	**2**	**3**	**5**	**7**	**0**
	2				

2	**2**	**3**	**5**	**7**	**0**
		4			
	2	**7**			

2	**2**	**3**	**5**	**7**	**0**
		4	**14**		
	2	**7**			

2	**2**	**3**	**5**	**7**	**0**
		4	**14**		
	2	**7**	**19**		

2	**2**	**3**	**5**	**7**	**0**
		4	**14**	**38**	
	2	**7**	**19**		

2	**2**	**3**	**5**	**7**	**0**
		4	**14**	**38**	
	2	**7**	**19**	**45**	

2	**2**	**3**	**5**	**7**	**0**
		4	**14**	**38**	**90**
	2	**7**	**19**	**45**	**90**

So the quotient is $2x^3 + 7x^2 + 19x + 45$, and the remainder is **90**.

That is, $2x^4 + 3x^3 + 5x^2 + 7x = (x - 2)(2x^3 + 7x^2 + 19x + 45) + 90$.

Let's next, for another instance, divide $P = 2x^4 + 3x^3 + 5x^2 + 7x$ by $x + 2$.
Then, we have $d = -2$, $c_0 = 2$, $c_1 = 3$, $c_2 = 5$, $c_3 = 7$, and $c_4 = 0$.
So doing the synthetic division, we get the steps as follows.

-2	2	3	5	7	0
	2				

-2	2	3	5	7	0
		-4			
	2	-1			

-2	2	3	5	7	0
		-4	2		
	2	-1			

-2	2	3	5	7	0
		-4	2		
	2	-1	7		

-2	2	3	5	7	0
		-4	2	-14	
	2	-1	7		

-2	2	3	5	7	0
		-4	2	-14	14
	2	-1	7	-7	14

So the quotient is $2x^3 - x^2 + 7x - 7$, and the remainder is 14.
That is, $2x^4 + 3x^3 + 5x^2 + 7x = (x + 2)(2x^3 - x^2 + 7x - 7) + 14$.

Suppose next, $\boldsymbol{A = 7x^8 - 4x^7 + 9x^5 + 12x^3 + 14}$, $\boldsymbol{B = 3x^3 + 2x + 1}$,
and we want to divide $\boldsymbol{A}$ by $\boldsymbol{B}$.

Then, of course, we can put that division this way: $\dfrac{\boldsymbol{A}}{\boldsymbol{B}} = \dfrac{7x^8 - 4x^7 + 9x^5 + 12x^3 + 14}{3x^3 + 2x + 1}$.
How do we get the quotient and the remainder, though?

Note:
The degree of a polynomial as $\boldsymbol{A}$ above is the largest exponent to which the variable $\boldsymbol{x}$ is raised. So for instance, $\boldsymbol{2x^8 - 3x^2 + 1}$ is a polynomial of degree 8.

Doing a division in numbers, we normally begin with eliminating the highest digit in the dividend, and move on to the next lower digit.

The same is true for a polynomial, too.
So we begin with the removal of the term of the highest degree in the polynomial, which we divide, of course, and then, we move on to the next lower.

Thus, among terms in the polynomial, we may want to keep degrees in order from the highest to the lowest so that we do not get confused. Putting 0s in front of the terms of missing degrees if any in the polynomial, we can avoid such a confusion.

Now, the term with the highest degree in the dividend $\boldsymbol{A}$ is $\boldsymbol{7x^8}$, which has to be thus, removed first. On the other hand, the term with the highest degree in the divisor $\boldsymbol{B}$ is $\boldsymbol{3x^3}$.

So in the beginning, we need to multiply the divisor by $\dfrac{7}{3}x^5$ to remove $\boldsymbol{7x^8}$.

That is, we do this operation first: $(3x^3 + 2x + 1)\dfrac{7}{3}x^5 = 7x^8 + \dfrac{14}{3}x^6 + \dfrac{7}{3}x^5$, which is thus, the product of the divisor and $\dfrac{7}{3}x^5$. So the first step is as follows.

$$
\begin{array}{r}
\dfrac{7}{3}x^5 \\
3x^3 + 2x + 1 \;\overline{)\; 7x^8 - 4x^7 + 0x^6 + 9x^5 + 0x^4 + 12x^3 + 0x^2 + 0x + 14} \\
\underline{\quad 7x^8 + \dfrac{14}{3}x^6 + \dfrac{7}{3}x^5 \quad}
\end{array}
$$

Removing the term $7x^8$ from the polynomial A, we subtract $7x^8 + \frac{14}{3}x^6 + \frac{7}{3}x^5$ from A.

That is, $A - (7x^8 + \frac{14}{3}x^6 + \frac{7}{3}x^5) = 7x^8 - 4x^7 + 9x^5 + 12x^3 + 14 - (7x^8 + \frac{14}{3}x^6 + \frac{7}{3}x^5)$

$= -4x^7 - \frac{14}{3}x^6 + (9 - \frac{7}{3})x^5 + 12x^3 + 14 = -4x^7 - \frac{14}{3}x^6 + \frac{20}{3}x^5 + 12x^3 + 14$. So we get

$$
\begin{array}{r}
\frac{7}{3}x^5 \\
3x^3 + 2x + 1 \;\overline{)\; 7x^8 - 4x^7 + 0x^6 + 9x^5 + 0x^4 + 12x^3 + 0x^2 + 0x + 14} \\
\underline{7x^8 + 0x^7 + \frac{14}{3}x^6 + \frac{7}{3}x^5} \\
-4x^7 - \frac{14}{3}x^6 + \frac{20}{3}x^5 + 12x^3 + 14
\end{array}
$$

In the next step, assuming $D = -4x^7 - \frac{14}{3}x^6 + \frac{20}{3}x^5 + 12x^3 + 14$, we want to divide D by the divisor $3x^3 + 2x + 1$.

Then, we want to remove $-4x^7$ from D.

In other words, we want to multiply the divisor by $-\frac{4}{3}x^4$ to produce $-4x^7$.

That is, we want to get $(3x^3 + 2x + 1)(-\frac{4}{3}x^4) = -4x^7 - \frac{8}{3}x^5 - \frac{4}{3}x^4$.

So removing the term $-4x^7$ from D, we subtract $(-4x^7 - \frac{8}{3}x^5 - \frac{4}{3}x^4)$ from D.

Then, we get

$D - (-4x^7 - \frac{8}{3}x^5 - \frac{4}{3}x^4) = -4x^7 - \frac{14}{3}x^6 + \frac{20}{3}x^5 + 12x^3 + 14 - (-4x^7 - \frac{8}{3}x^5 - \frac{4}{3}x^4)$

$= -\frac{14}{3}x^6 + \frac{20+8}{3}x^5 + \frac{4}{3}x^4 + 12x^3 + 14 = -\frac{14}{3}x^6 + \frac{28}{3}x^5 + \frac{4}{3}x^4 + 12x^3 + 14$.

So we get

$$
\begin{array}{r}
\frac{7}{3}x^5 - \frac{4}{3}x^4 \\
3x^3 + 2x + 1 \;\overline{)\; 7x^8 - 4x^7 + 0x^6 + 9x^5 + 0x^4 + 12x^3 + 0x^2 + 0x + 14} \\
\underline{7x^8 + 0x^7 + \frac{14}{3}x^6 + \frac{7}{3}x^5} \\
-4x^7 - \frac{14}{3}x^6 + \frac{20}{3}x^5 + 12x^3 + 14 \\
\underline{-4x^7 + 0x^6 - \frac{8}{3}x^5 - \frac{4}{3}x^4} \\
-\frac{14}{3}x^6 + \frac{28}{3}x^5 + \frac{4}{3}x^4 + 12x^3 + 14
\end{array}
$$

Then, we repeat the same process.

So assuming next, $E = -\frac{14}{3}x^6 + \frac{28}{3}x^5 + \frac{4}{3}x^4 + 12x^3 + 14$, we want to divide E by the divisor $3x^3 + 2x + 1$. Then, we want to remove $-\frac{14}{3}x^6$ from E. That is, we want to multiply the divisor by $-\frac{14}{9}x^3$ to produce $-\frac{14}{3}x^6$.

So we want to get this time $(3x^3 + 2x + 1)(-\frac{14}{9}x^3) = -\frac{14}{3}x^6 - \frac{24}{9}x^4 - \frac{14}{9}x^3$.

Removing the term $-\frac{14}{3}x^6$ from E, we subtract $-\frac{14}{3}x^6 - \frac{24}{9}x^4 - \frac{14}{9}x^3$ from E.

So we get $E - (-\frac{14}{3}x^6 - \frac{24}{9}x^4 - \frac{14}{9}x^3)$

$$= -\frac{14}{3}x^6 + \frac{28}{3}x^5 + \frac{4}{3}x^4 + 12x^3 + 14 - (-\frac{14}{3}x^6 - \frac{24}{9}x^4 - \frac{14}{9}x^3)$$

$$= \frac{28}{3}x^5 + (\frac{4}{3} + \frac{28}{9})x^4 + (12 + \frac{14}{9})x^3 + 14 = \frac{28}{3}x^5 + \frac{40}{9}x^4 + \frac{122}{9}x^3 + 14.$$

Then, we get

$$
\begin{array}{r|l}
 & \frac{7}{3}x^5 - \frac{4}{3}x^4 - \frac{14}{9}x^3 \\
\hline
3x^3 + 2x + 1 & 7x^8 - 4x^7 + 0x^6 + 9x^5 + 0x^4 + 12x^3 + 0x^2 + 0x + 14 \\
 & \underline{\;7x^8 + 0x^7 + \frac{14}{3}x^6 + \frac{7}{3}x^5\;} \\
 & \quad -4x^7 - \frac{14}{3}x^6 + \frac{20}{3}x^5 + 12x^3 + 14 \\
 & \quad \underline{-4x^7 + 0x^6 - \frac{8}{3}x^5 - \frac{4}{3}x^4} \\
 & \quad\quad -\frac{14}{3}x^6 + \frac{28}{3}x^5 + \frac{4}{3}x^4 + 12x^3 + 14 \\
 & \quad\quad \underline{-\frac{14}{3}x^6 + 0x^5 - \frac{24}{9}x^4 - \frac{14}{9}x^3} \\
 & \quad\quad\quad \frac{28}{3}x^5 + \frac{40}{9}x^4 + \frac{122}{9}x^3 + 14
\end{array}
$$

Now, we can continue with the rest of the divisions in the same manner until the degree of the remainder is less than the degree of the divisor. Then, we get

$$
\begin{array}{r|l}
 & \frac{7}{3}x^5-\frac{4}{3}x^4-\frac{14}{9}x^3+\frac{28}{9}x^2+\frac{40}{27}x+\frac{22}{9} \\
\hline
3x^3+2x+1 & 7x^8-4x^7+0x^6+9x^5+0x^4+12x^3+0x^2+0x+14 \\
 & 7x^8+0x^7+\frac{14}{3}x^6+\frac{7}{3}x^5 \\
\hline
 & -4x^7-\frac{14}{3}x^6+\frac{20}{3}x^5+12x^3+14 \\
 & -4x^7+0x^6-\frac{8}{3}x^5-\frac{4}{3}x^4 \\
\hline
 & -\frac{14}{3}x^6+\frac{28}{3}x^5+\frac{4}{3}x^4+12x^3+14 \\
 & -\frac{14}{3}x^6+0x^5-\frac{24}{9}x^4-\frac{14}{9}x^3 \\
\hline
 & \frac{28}{3}x^5+\frac{40}{9}x^4+\frac{122}{9}x^3+14 \\
 & \frac{28}{3}x^5+0x^4+\frac{56}{9}x^3+\frac{28}{9}x^2 \\
\hline
 & \frac{40}{9}x^4+\frac{22}{3}x^3-\frac{28}{9}x^2+14 \\
 & \frac{40}{9}x^4+0x^3+\frac{80}{27}x^2+\frac{40}{27}x \\
\hline
 & \frac{22}{3}x^3-\frac{164}{27}x^2-\frac{40}{27}x+14 \\
 & \frac{22}{3}x^3+0x^2+\frac{44}{9}x+\frac{22}{9} \\
\hline
 & -\frac{164}{27}x^2-\frac{172}{9}x+\frac{104}{9}
\end{array}
$$

Therefore, we can see that $A = BQ + R$ where

$Q = \frac{7}{3}x^5-\frac{4}{3}x^4-\frac{14}{9}x^3+\frac{28}{9}x^2+\frac{40}{27}x+\frac{22}{9}$, and $R = -\frac{164}{27}x^2-\frac{172}{9}x+\frac{104}{9}$.

That is,

$$7x^8 - 4x^7 + 9x^5 + 12x^3 + 14$$

$$= (3x^3 + 2x + 1)(\frac{7}{3}x^5 - \frac{4}{3}x^4 - \frac{14}{9}x^3 + \frac{28}{9}x^2 + \frac{40}{27}x + \frac{22}{9}) - \frac{164}{27}x^2 - \frac{172}{9}x + \frac{104}{9}.$$

Let's try another simple example.

Dividing $A = 2x^5 - 3x^4 + 16x^3 - 19x^2 + 40$ by $B = 2x^3 - 3x^2 + 5$, we get

$$\begin{array}{r} x^2 + 8 \\ 2x^3 - 3x^2 + 5 \;\overline{)\; 2x^5 - 3x^4 + 16x^3 - 19x^2 + 40} \\ \underline{2x^5 - 3x^4 + 0x^3 + 5x^2} \\ 16x^3 - 24x^2 + 40 \\ \underline{16x^3 - 24x^2 + 40} \\ 0 \end{array}$$

So we can see that $\frac{A}{B} = \frac{2x^5 - 3x^4 + 16x^3 - 19x^2 + 40}{2x^3 - 3x^2 + 5} = x^2 + 8.$

In this case, the remainder is 0, so B divides A, and also, can be a divisor of A. And of course, the quotient ($x^2 + 8$) can be a divisor, too.

Thus, if we have $P = DQ + R$ where $R \neq 0$, D cannot be a divisor, and neither can Q.

If however, we have $P = DQ$, D and Q both can be divisors.

We can use a different word for a divisor, and it is called a factor. So factors are divisors, which can divide a dividend with 0 remainder. What divisor then is a factor?

It is the core of this book, and is covered in the section, "**What is a factorization?**"

By the way, we may want to check out a calculation tool, which is quite often used in fractional arithmetic. And the tool is as follows.

$\frac{C}{AB} = \frac{C}{B-A}\left(\frac{1}{A} - \frac{1}{B}\right)$, where $A \neq B$, and A and B both $\neq 0$, of course. How?

To begin with, we can have $\frac{1}{A} - \frac{1}{B} = \frac{B}{AB} - \frac{A}{AB} = \frac{B-A}{AB}$.

So we get $\frac{C}{B-A}\left(\frac{1}{A} - \frac{1}{B}\right) = \frac{C}{B-A} \cdot \frac{B-A}{AB} = \frac{C}{AB}$, and thus, $\frac{C}{AB} = \frac{C}{B-A}\left(\frac{1}{A} - \frac{1}{B}\right)$.

For instance, $\frac{2}{21} = \frac{2}{3 \cdot 7} = \frac{2}{7-3}\left(\frac{1}{3} - \frac{1}{7}\right) = \frac{2}{4}\left(\frac{1}{3} - \frac{1}{7}\right) = \frac{1}{2}\left(\frac{1}{3} - \frac{1}{7}\right)$.

$$\frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \ \ldots \ + \frac{1}{99 \cdot 100}$$

$$= \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \ \ldots \ - \frac{1}{98} + \frac{1}{98} - \frac{1}{99} + \frac{1}{99} - \frac{1}{100} = \frac{1}{2} - \frac{1}{100}.$$

2. What is a factorization?

It is often called factoring, too.
Normally, doing a factorization, we factorize (or factor) integers or polynomials.
Factorizing (factoring) an integer or a polynomial, we put it in terms of its *factors*, that is, we express it using its factors.
And putting it in terms of its factors, we put them in *a form of a product*.

So for instance, if $\boldsymbol{(x-a)}$ and $\boldsymbol{(x-b)}$ are factors of $\boldsymbol{P = x^2 - (a+b)x + ab}$, and if factorizing $\boldsymbol{P}$, we put $\boldsymbol{P}$ this way: $\boldsymbol{(x-a)(x-b)}$, which is in a form of a product, and we can set $\boldsymbol{x^2 - (a+b)x + ab = (x-a)(x-b)}$. What then is a factor?

Factors are divisors. They are not just divisors though. What divisors then are factors?

For instance, if we make an integer taking a product of integers, the integers we take the product of are divisors of the integer we make, and are called the factors of the integer.

So for instance, factorizing 12, we can put it this way: $12 = 3 \cdot 4$, or this way: $12 = 2 \cdot 2 \cdot 3$.
And in the case where $12 = 3 \cdot 4$, we say that 3 and 4 are factors of 12, and in the case of $12 = 2 \cdot 2 \cdot 3$, we say that the factors of 12 are two of 2s and a 3.

What then about the cases as follows? $12 = 1 \cdot 12$, $12 = 1 \cdot 3 \cdot 4$, and $12 = 1 \cdot 2 \cdot 2 \cdot 3$

Technically, 1 can be a factor, too, and in fact, can be a factor of every integer.
And technically also, an integer itself can be a factor of itself, too, simply because every integer can be the product of 1 and itself.

So anyway, factorizing, we find factors, and put them in a *form of a product*.

Usually though, they are not just factors. Normally, if we are asked to do a factorization, we are to do a full factorization unless told otherwise. What then is a full factorization?

Doing a full factorization, we find all the *prime* factors applicable, and put in a form of a product all those prime factors. So such a factorization is called a *prime factorization* or a *prime decomposition*, too. What do we mean by though, factors and prime factors?

Factors are divisors. Not all divisors are factors though.

A divisor can be negative as well as positive, but a factor is positive, and is a divisor. So saying a factor, we mean a positive divisor.

For instance, *all the divisors* of 12 are ±1, ±2, ±3, ±4, ±6, and ±12. So 12 has 12 divisors.

However, all the integers that can be factors of 12 are 1, 2, 3, 4, 6, and 12, each of which is a divisor of 12. It is *not* the case though, 12 has 6 factors. The biggest number of factors 12 can have is 4, since we have $12 = 1 \cdot 2 \cdot 2 \cdot 3$. So 12 can have at most 4 factors.

So factors are divisors, but not all divisors are factors.

Talking about factorizations however, saying just a divisor, we mean a positive divisor, and for simplicity, we do not take 1 as a divisor unless told otherwise. And the same is true for a factor, too.

So we don't normally take 1 as a factor unless told otherwise or unless 1 is absolutely necessary. Thus, if an integer is said to have no factor, it can be divided by 1 and itself only. So finding such a factor cannot be an issue, and is not worth a talk.

What's important is to find all the integers that can be factors or prime factors.
What then, is a prime factor?

A prime factor is a factor that can be divided by 1 and itself only. So for instance, if 2 is a factor, the factor is a prime factor, because it can be divided by 1 and itself only.

Usually, factorizing an integer, we find all its prime factors, and put them in a form of a product. So just saying factorizations, we mean full factorizations (prime factorizations or prime decompositions).

Normally though, we don't factorize every integer. That is, we factorize some integers.
What integer then do we factorize?

Many integers are products of integers.

So an integer can be a product of other integers.

For instance, 8 is 2·4 or 2·2·2.

What then do we call such an integer as 8?

It is called a *composite integer*, often called a composite number, too.

So a composite integer is a product of other integers, and can be, for instance, 9 or 12.
And most integers are composite, and thus, are composite integers.
What then about integers not composite?

Those integers are said to be prime, and thus, are called prime integers, often called prime numbers, too. And briefly, we just call them primes. What then is a prime?

A prime is an integer that can be divided by 1 and itself only.

So 2 is the only prime even. And all the other primes are odd.
Among those odd primes, we have, for instance, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, etc.

So some integers called primes get combined via multiplications, and make a composite integer.

Factorizing thus, we do a full factorization,
and factorizing an integer, we factorize a composite integer,
find all its prime factors,
and put them in a form of a product.

So for instance, doing the full factorization of 12, we get $12 = 2\cdot2\cdot3 = 2^2 3$, which is in a form of a product. How then can we find those prime factors?

If putting in a form of a product, all the prime factors of an integer, what do we get?
That is, taking the product of all the prime factors of an integer, what do we get?

We get a composite integer, which is the very integer factorized.
So taking the product of all the prime factors of an integer ***A***, for instance, we get the integer ***A***.

Thus, we can find the prime factors doing divisions. So via divisions, we can find all those prime factors applicable, the product of which is the integer to be factorized.

And a prime factor is a prime integer, just called a prime, too.

So finding prime factors, we begin with division by the smallest prime 2,
if 2 divides, take 2 as a factor,
and then, move on to the division of the quotient by the next larger prime, which is 3.
If however, 2 doesn't divide, then do not take 2 as a factor,
and then, move on to the division by the next larger prime, which is 3.
Then, the same process goes on until the quotient is a prime.

Thus, putting threads together, if factorizing an integer,
we factorize a composite integer ***B***, for instance.
And factorizing ***B***, we find all the prime factors applicable,
that is, all the primes that can divide ***B***,
and put them in a form of a product,
since the product of all those primes is the composite integer ***B***.

So factorizing an integer,
we find all the primes that divide the integer using divisions,
take those divisors as factors, and paste them together using multiplication signs as · or x.

So for instance, factorizing 12, we can put it this way: $12 = 2 \cdot 2 \cdot 3$ or $12 = 2 \times 2 \times 3$.
And those factors are prime factors, of course.

Next, as in the case of the example above, some prime factors can be the same.
It can even be the case in fact, all prime factors are the same, too.

So for instance, we can have $36 = 2 \cdot 2 \cdot 3 \cdot 3$, $2 \cdot 3 \cdot 2 \cdot 3$, or $3 \cdot 3 \cdot 2 \cdot 2$, and $8 = 2 \cdot 2 \cdot 2$.

In the case of 36, the prime factor 2 repeats, and so does the prime factor 3.
And in the case of 8, 2 is the only prime factor, and repeats itself.

And also, if for instance, the integer to be factorized is very big as 120932352, the factorization, that is, the product form will be too lengthy, and take up space too much.

In fact, $120932352 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$.
Usually therefore, we show a factorization using power notation.
So we normally put the factorizations above the way as follows.

$$36 = 2^2 3^2,\ 8 = 2^3, \text{ and } 120932352 = 2^{11} 3^{10}.$$

So if an integer is fully factorized, it is expressed in terms of primes only, some of or all of which can be the same.

Thus, doing a full factorization, we find all the prime factors applicable, and put them in a form of a product. And it is often the case where it is a product of powers.
For instance, 1, 2, 3, 4, 6, 9, 12, 18, and 36 are all the integers that can be factors of 36.

And we have $4 = 2 \cdot 2$, $6 = 2 \cdot 3$, $9 = 3 \cdot 3$, $12 = 2 \cdot 2 \cdot 3$, and $18 = 2 \cdot 3 \cdot 3$.
So it has two kinds in prime factors, and the two are 2 and 3.
And we usually put it this way: $36 = 2^2 3^2$, and call it the (full) factorization of 36.

So in fact, we don't have to find all the divisors of the integer to be factorized.
We have only to find all its prime factors.

And note that all the prime factors of 36 are 2 of 2s and 2 of 3s, but the kinds in prime factors are 2 and 3.

Therefore, factorizing 36, we put all its prime factors in a product form, so we put together 2s and 3s the way as follows: $2 \cdot 2 \cdot 3 \cdot 3$, which is usually put this way: $2^2 3^2$.

Thus in short, factorizing 36, we get $36 = 2^2 3^2$.
And basically, the same is true, also, for polynomial factorizations, which will begin to get covered in the next section.

3.0. Factorizing Polynomials 1

Factorizing a polynomial, too, we find its factors, and put them in a form of a product. What then is a factor of a polynomial?

A factor is a divisor. So factorizing a polynomial, too, we find its divisors, and put them in a form of a product. What divisor though?

Such a divisor is not just an integer but a math expression, too, which can be a monomial or even a polynomial, too.
So a factor of a polynomial can be an integer, a monomial, or a polynomial.

And normally, factorizing, we do a full factorization unless told otherwise.

So factorizing a polynomial, too, we want to find all its prime factors, and put them in a form of a product. What then is a prime factor of a polynomial?

If an integer or an expression is prime, it has as divisors 1 and itself only.

So a prime factor of a polynomial, too, is a factor that can be divided by 1 and itself only.
Anyway, either prime or not, factors are divisors. So factorizing a polynomial, we want to find first, its divisors. What then can be such a divisor?

Unlike an integer factorization, such a divisor can be an integer or a math expression, which is a monomial or a polynomial. How then can we find such a divisor?

Factorizing a polynomial, we don’t just look for a divisor that can divide the whole polynomial at once.

We look for a divisor that can divide *some of* or *all of* the terms in the polynomial.
What kind of divisor is that?

We call it a *common* divisor.
If an integer or an expression divides every term in a polynomial, it divides the polynomial, and thus, is a divisor of the polynomial.

Then, such a divisor is said to be common to every term in the polynomial, and thus, is called a *common divisor*. And a common divisor can be an integer or an expression, which is a monomial or a polynomial.

For instance, 4 is a divisor common to every term in a polynomial $\mathbf{4x + 8}$.

That's because 4 divides $\mathbf{4x}$, which is one term, and 4 divides 8, too, which is the other term. That is to say that 4 divides every term in the polynomial $\mathbf{4x + 8}$.

So we can say that 4 can divide the polynomial $\mathbf{4x + 8}$. And in fact, dividing $\mathbf{4x + 8}$ by 4, we get $\boldsymbol{x + 2}$ as the quotient, and get no remainder, that is, the remainder is 0.

Thus, we can set $\mathbf{4x + 8 = 4(x + 2) = 2^2(x + 2)}$, which is in a form of a product.

Normally though, we just put it this way: $\mathbf{4x + 8 = 4(x + 2)}$.
What then about a polynomial $\boldsymbol{x^2 + 2x}$?

We know $\boldsymbol{x}$ divides $\boldsymbol{x^2}$, which is a term in $\boldsymbol{x^2 + 2x}$, and $\boldsymbol{x}$ divides the other term, too, which is $\mathbf{2x}$. So $\boldsymbol{x}$ is a divisor common to every term in the polynomial $\boldsymbol{x^2 + 2x}$.

That is to say that $\boldsymbol{x}$ divides every term in the polynomial $\boldsymbol{x^2 + 2x}$.
So we can say that $\boldsymbol{x}$ can divide the polynomial $\boldsymbol{x^2 + 2x}$.

And in fact, dividing $\boldsymbol{x^2 + 2x}$ by $\boldsymbol{x}$, we get $\boldsymbol{x + 2}$ as the quotient, and the remainder is 0.
So we can set $\boldsymbol{x^2 + 2x = x(x + 2)}$, which is in a form of a product.

Thus, a divisor that divides every term in a polynomial is common to all the terms in the polynomial, and is a *common divisor* (of all the terms in the polynomial). And such a divisor divides the polynomial.

So a divisor common to every term in a polynomial is a divisor of the polynomial.
Normally thus, we begin with a divisor common to all the terms in the polynomial.
What then about a factor of a polynomial?

We know a factor is a divisor.
So finding such a common divisor as stated above, we get a factor of a polynomial.
Normally though, since a factor is a divisor, we use the two terminologies 'factor' and 'divisor' interchangeably.

So using the terminology 'factor', too, we can reiterate the same story about a divisor and a common divisor described above. So let's see now, how it can get reiterated.

If an integer or an expression divides every term in a polynomial, it divides the polynomial, and thus, is a factor of the polynomial.

Then, such a *factor* is *common to every term* in the polynomial, and thus, is a *common factor* of all the terms in the polynomial. And such a common factor can be an integer or an expression, which is a monomial or a polynomial.

For instance, x is a factor common to all the terms in a polynomial $xy + xz$.
So x divides every term in the polynomial $xy + xz$.
That is, x divides xy, and also, divides xz.

So x is a factor of xy, and is a factor of xz, too, and thus, is a factor common to both terms xy and xz. So x is a common factor of all the terms in $xy + xz$, and divides $xy + xz$.

Thus, a common factor is a common divisor, and divides the polynomial.
Then, the quotient is $y + z$, and the remainder is 0.

And in fact, we can get $xy + xz = x(y + z)$, which is in a form of a product.
So x is a factor of the polynomial $xy + xz$.

And the same is true for the quotient $y + z$, too, because it can divide the polynomial above. So the quotient is a factor, too, which is a polynomial, too, in this case.

And thus, a factor of a polynomial can be not only an integer or a monomial but a polynomial, too. So factorizing a polynomial, what should we begin with?

It is a divisor or a factor common to all the terms in the polynomial.
What if we cannot find it, though? Is the polynomial then, not factorable?

It is *not always* the case. What then can we do?

Suppose for instance, we want to factorize a polynomial $xy + 2y + 3x + 6$, and call it B.
Then first, we can put it this way: $B = xy + 2y + 3x + 6$.

We cannot see though, any common divisor, that is, we can see no common factor.
We can take however, the polynomial B as a sum of two small polynomials.

One is $xy + 3x$, and the other is $2y + 6$. That is, we can put B the way as follows.

$$B = (xy + 3x) + (2y + 6).$$

Then, we can see that each small polynomial can have a divisor, which is common to all the terms in each small polynomial, of course. That is to say that a part of a polynomial can be another polynomial that can have a divisor.

For instance, $(xy + 3x)$ is a part of B, and x is (a divisor) common to all terms in $xy + 3x$, and thus, is a divisor of $xy + 3x$. (Note that when x is a divisor (or a factor) common to terms, we often just say that x is common to the terms.)

So we can put $(xy + 3x)$ this way $x(y + 3)$. What then can we do?

We can begin with examining divisors (or factors) of each term.
And then, we can break the polynomial into parts so that all the parts can have their own divisors, and also, can share the same divisor, too.

That is, we start with examining each term, and then, partition the polynomial so that all the parts can share the same divisor.

For instance, examining each term in $B = (xy + 3x) + (2y + 6)$, we can notice that $y + 3$ is a divisor of $(xy + 3x)$, and is a divisor of $(2y + 6)$, too. How?

First, we can get $xy + 3x = x(y + 3)$, since x is common to xy and $3x$.
Next, we can get $2y + 6 = 2(y + 3)$, since 2 is common to $2y$ and 6.

And since we have $xy + 3x = x(y + 3)$, we can say that not only x but $(y + 3)$, too, is a divisor of the polynomial $xy + 3x$, because $(y + 3)$ divides $xy + 3x$, and so does x.

Also, since we have $2y + 6 = 2(y + 3)$, we can say that not only 2 but $(y + 3)$, too, is a divisor of the polynomial $2y + 6$, because $(y + 3)$ divides $2y + 6$, and so does 2.

So the two parts $(xy + 3x)$ and $(2y + 6)$ share the same divisor, which is $y + 3$.
What divisor then is the same divisor?

It is a divisor common to all the two parts $(xy + 3x)$ and $(2y + 6)$, the sum of which is B.
So what?

Taking as a term each of the two parts, we can say that the polynomial B has two terms, one is $(xy + 3x)$, and the other is $(2y + 6)$.

And we know $(y + 3)$ is common to both terms $(xy + 3x)$ and $(2y + 6)$.
And we have $B = (xy + 3x) + (2y + 6)$.

So we can say that the polynomial B has a divisor common to all the two terms, and the common divisor is $(y + 3)$, which is thus, a factor of the polynomial B.

Thus, putting threads together, we can put the ideas above the way as follows.

Setting $B = (xy + 3x) + (2y + 6)$, and noticing x is common to every term in $(xy + 3x)$, and 2 is common to every term in $(2y + 6)$, we can see that $(y + 3)$ is a factor of B, since it is a divisor common to all the two parts $(xy + 3x)$ and $(2y + 6)$.

In other words, we can get $B = x(y + 3) + 2(y + 3)$, where $(y + 3)$ is a common divisor.

Therefore, $(y + 3)$ divides B, and thus, is a factor of B.
So putting threads together, we get

$$B = xy + 3x + 2y + 6 = (xy + 3x) + (2y + 6) = x(y + 3) + 2(y + 3) = (y + 3)(x + 2).$$

Thus, we get $xy + 3x + 2y + 6 = (y + 3)(x + 2)$, which is in a form of a product.
And we know $y + 3$ has as divisors 1 and itself only, and thus, is prime. And the same is true for $x + 2$, too. So we can now say that B is (fully) factorized to $(y + 3)(x + 2)$.

Suppose for another instance, $\boldsymbol{a}$ is a divisor common to all the terms in a polynomial $\boldsymbol{P}$.

Then, $\boldsymbol{a}$ divides $\boldsymbol{P}$, and thus, is a factor of $\boldsymbol{P}$, so $\boldsymbol{P}$ can be set equal to a product of the factor $\boldsymbol{a}$ and another polynomial called $\boldsymbol{Q}$, for instance.

That is to say that $\boldsymbol{P}$ is now set equal to $\boldsymbol{aQ}$. In other words, we get $\boldsymbol{P = aQ}$.

Suppose next, $\boldsymbol{b}$ is a divisor common to all the terms in the polynomial $\boldsymbol{Q}$ above.

Then, $\boldsymbol{b}$ divides $\boldsymbol{Q}$, and thus, is a factor of $\boldsymbol{Q}$, so $\boldsymbol{Q}$ can be set equal to a product of $\boldsymbol{b}$ and another polynomial called $\boldsymbol{R}$, for instance. So $\boldsymbol{Q}$ is now set equal to $\boldsymbol{bR}$.

That is, we get $\boldsymbol{Q = bR}$.

Then, since $\boldsymbol{P = aQ}$, and $\boldsymbol{Q = bR}$, we get $\boldsymbol{P = abR}$. So $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{R}$ are factors of $\boldsymbol{P}$.

Suppose now, $\boldsymbol{R}$ is $\boldsymbol{2x^2 + 1}$.
Then, no divisor is common to all the terms in $\boldsymbol{R}$, so $\boldsymbol{R}$ has as divisors 1 and itself only.

So $\boldsymbol{R}$ is a prime factor. And in fact, $\boldsymbol{a}$ and $\boldsymbol{b}$ are prime factors, too, since each is a factor, and has as divisors 1 and itself only.

Thus, all the prime factors of $\boldsymbol{P}$ are $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{R}$, so $\boldsymbol{P}$ is now fully factorized to $\boldsymbol{abR}$.

In sum, we get $\boldsymbol{P = aQ = abR = ab(2x^2 + 1)}$. So before factorization, $\boldsymbol{P = 2abx^2 + ab}$.

Suppose this time, $\boldsymbol{R}$ is $\boldsymbol{2x + 6}$.
Then, $\boldsymbol{R}$ can be put in $\boldsymbol{2(x + 3)}$ since 2 is a divisor (or factor) common to $\boldsymbol{2x}$ and $\boldsymbol{6}$.

Suppose also, $\boldsymbol{c}$ is $\boldsymbol{x + 3}$. Then, we get $\boldsymbol{R = 2c}$.

And we know $\boldsymbol{c = x + 3}$ is prime, since no divisor is common to all the terms in $\boldsymbol{c}$.

So all the prime factors of $\boldsymbol{P}$ are $\boldsymbol{2}$, $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$, and thus, we can say that $\boldsymbol{P}$ gets (fully) factorized to $\boldsymbol{2abc}$.

In sum, we get $\boldsymbol{P = aQ = abR = 2ab(x + 3)}$. So before factorization, $\boldsymbol{P = 2abx + 6ab}$.

3.1. Factorizing Polynomials 2

Factorizing a polynomial, we find its factors, and put them in a form of a product.
And a factor is a divisor. So factorizing a polynomial, we find its divisors, and put them in a form of a product.

Such a divisor can be not just an integer but a math expression, too, as $\boldsymbol{a}$, $\boldsymbol{x}$, $\boldsymbol{ax}$, $\boldsymbol{ax + b}$, $\boldsymbol{2x + y}$, and therefore, can be a monomial or even a polynomial, too.
So a factor of a polynomial can be an integer, a monomial, or a polynomial.

And normally, factorizing, we do a full factorization unless told otherwise.
So factorizing a polynomial, too, we want to find all its prime factors, and put them in a form of a product. What then is a prime factor of a polynomial?

If an integer or an expression is prime, it has as divisors 1 and itself only.
So a prime factor of a polynomial, too, is a factor that can be divided by 1 and itself only.
Anyway, either prime or not, factors are divisors. So factorizing a polynomial, we want to find first, its divisors.

Unlike an integer factorization, such a divisor can be an integer or a math expression, which is a monomial or a polynomial.
And factorizing a polynomial, we don't just look for a divisor that can divide the whole polynomial at once. What else then can we try finding?

We can try looking for a divisor that can divide *some of* or *all of* the terms in the polynomial. And we call it a *common* divisor.
If an integer or an expression can divide every term in a polynomial, it can divide the polynomial, and thus, is a divisor of the polynomial.

Then, such a divisor is said to be common to every term in the polynomial, and thus, is called a *common divisor*. And a common divisor can be an integer or an expression, which is a monomial or a polynomial.

Suppose for instance, $\boldsymbol{m}$ is a divisor common to some terms in a polynomial $\boldsymbol{S}$, and $\boldsymbol{n}$ is a divisor common to all the other terms in the polynomial $\boldsymbol{S}$.

Then, $\boldsymbol{S}$ is a sum of two products, one is a product of the divisor $\boldsymbol{m}$ and a polynomial, and the other is a product of the other divisor $\boldsymbol{n}$ and another polynomial.

Suppose next, the two polynomials in the two products above are the same, and each is called $\boldsymbol{T}$. What then is the polynomial $\boldsymbol{T}$?

The polynomial $\boldsymbol{T}$ is a divisor of $\boldsymbol{S}$, and thus, is a factor of $\boldsymbol{S}$, so $\boldsymbol{S}$ is the product of $\boldsymbol{T}$ and the sum of the two divisors $\boldsymbol{m}$ and $\boldsymbol{n}$. In other words, we get $\boldsymbol{S = mT + nT = T(m + n)}$.

Suppose next, $\boldsymbol{T}$ is a prime polynomial as $\boldsymbol{x^2 + y + 1}$.

A prime polynomial is a polynomial that has as divisors 1 and itself only.

Then, $\boldsymbol{T}$ and $(\boldsymbol{m + n})$ are prime factors of $\boldsymbol{S}$, which is thus, fully factorized to $\boldsymbol{T(m + n)}$.

So we can set $\boldsymbol{S = (m + n)(x^2 + y + 1)}$, which is called the factorization of $\boldsymbol{S}$.

And expanding (simplifying) the right hand side, we get $\boldsymbol{mx^2 + my + m + nx^2 + ny + n}$.

So we can say that before factorization, $\boldsymbol{S = mx^2 + my + m + nx^2 + ny + n}$.

Thus, we can get $\boldsymbol{mx^2 + my + m + nx^2 + ny + n = (m + n)(x^2 + y + 1)}$.

And we can say that the polynomial $\boldsymbol{S}$ is (fully) factorized to $\boldsymbol{(m + n)(x^2 + y + 1)}$.

So in a polynomial factorization, factors can be integers, monomials, and polynomials that can divide the polynomial.

Let’s now, for another instance, factorize a polynomial $\boldsymbol{U = 2v^2 + v}$.

First, v is a divisor common to all the terms in the polynomial U, so U can be divided by v, and therefore, we get $U = 2v \cdot v + 1 \cdot v = v(2v + 1) \Rightarrow U = v(2v + 1)$.

Next, v is a factor, and has no divisor other than 1 and itself, so v is a prime factor.

And next, the polynomial $(2v + 1)$ does not have any divisor common to all the terms in itself, but can divide U, so $(2v + 1)$ itself is a factor, and is prime, too. Thus, $(2v + 1)$ is a prime factor. Therefore, the (full) factorization of U is $v(2v + 1)$.

Suppose next, we want to factorize a polynomial $V = x^2 + xy + x + y$.
Then first, we try finding a divisor common to all the terms in V.
This time though, we cannot see such a divisor. What then should we do?

Factorizing polynomials, we basically use the ***three basic laws*** in arithmetic operations, which are as follows.

Distributive Law: $A(B + C) = AB + AC$, which is in fact, a factorization.

Associative Law: $A + B + C = (A + B) + C = A + (B + C)$, and $ABC = (AB)C = A(BC)$.

Commutative Law: $A + B = B + A$, and $AB = BA$.

So let's now, get back to the polynomial $V = x^2 + xy + x + y$, and factorize it.
Then first, we can notice that x can divide each of the two terms in $x^2 + xy$, so using the distributive law, we can take out x from $x^2 + xy$, and put the rest in a pair of parentheses.

That is, we get $V = x^2 + xy + x + y = x(x + y) + x + y \Rightarrow V = x(x + y) + x + y$.

Next, we can notice that $V = x(x + y) + (x + y) \cdot 1$.

Then again, since V is made of two terms, one is $x(x + y)$, and the other is $(x + y) \cdot 1$, we can see that $x + y$ can divide each of those two terms in V. So using the distributive law, we can take out $(x + y)$, too, and put the rest in a pair of parentheses.

That is, we get $V = x(x + y) + (x + y) \cdot 1 \Rightarrow V = (x + y)(x + 1)$.

Now, $(x + y)$ and $(x + 1)$ are factors, and are prime, too, so they are prime factors.

So the polynomial $V = x^2 + xy + x + y$ is factorized to a product of two polynomials, which are $x + y$ and $x + 1$. Therefore, the factorization of V is $(x + y)(x + 1)$.

So we get $V = x^2 + xy + x + y = x(x + y) + x + y = (x + y)(x + 1)$.

Let's now try another polynomial $W = yz + xy + y^2 + xz$.

We can notice first, that y is common to xy and y^2, and next, can notice that z is common to yz and xz. So?

So we can get $xy + y^2 = y(x + y)$, and $yz + xz = z(y + x)$.

Then, we can see that $(x + y)$ is common to both cases. What then?

We can factorize the polynomial the way as follows.

$W = yz + xy + y^2 + xz = (xy + y^2) + (yz + xz) = y(x + y) + z(y + x) = (x + y)(y + z)$.

So we get $yz + xy + y^2 + xz = (x + y)(y + z)$. Thus, W gets factorized to $(x + y)(y + z)$.

Let's next, try another polynomial $x^2 + 2x + 1$.

First, in $x^2 + 2x + 1$, we can notice that x can be a divisor common to two terms x^2 and $2x$. So taking x out of both of the terms, that is, dividing each of the two terms by x, we get $x + 2$ as the sum of the quotients.

Thus, we get $x(x + 2)$, together with 1, that is, we get $x^2 + 2x + 1 = x(x + 2) + 1$.

However, $x + 2$ has not much to do with 1 in $x(x + 2) + 1$.
What should we do then?

An expression can be put a lot of ways. The same is true for a number, too. For instance,

$$10 = 5 + 5 = 3 + 7 = 12 - 2 = \frac{20}{2} = \frac{1.2}{0.12} = \dots$$

$$5x = 2x + 3x = 7x - 2x = 5x + x - x = 5x + 2x - 2x = \frac{10x}{2} = \frac{5x(y+1)}{y+1} = \dots$$

And doing a problem in math, we break it into parts, and then, put them together.

We don't just break it though. We want to break it into parts so that the parts can work. That is, we break it into working parts. In this case then, such working parts should be able to give us a factor. How then should we break the polynomial $x^2 + 2x + 1$?

We can break it into $(x^2 + x)$ and $(x + 1)$.

Then, we get $x^2 + 2x + 1 = (x^2 + x) + (x + 1)$.

Thus, we can get $x(x + 1)$, together with $(x + 1)$.

That is, we get $x^2 + 2x + 1 = (x^2 + x) + (x + 1) = x(x + 1) + x + 1$,

which is $x(x + 1) + (x + 1)\cdot 1$.

So $x + 1$ is a divisor common to both of the terms, $x(x + 1)$ and $(x + 1)\cdot 1$.

Thus, we get $x^2 + 2x + 1 = (x + 1)(x + 1) = (x + 1)^2$.

Now, what's the opposite of a factorization?

It's an expansion. Many people call it simplification, too, though.

Usually, a polynomial is given in either of two forms: *product* or *expansion*.

That is to say that *fully* or *partially*, it's been either factorized or expanded (simplified).

Suppose that a polynomial is given in a product form.

Then, it is in a product form, where expressions look pasted together by multiplications as in $\frac{2}{3}kx^2(xy + y)$ or $2x(x + 1)$, and those expressions can be polynomials, monomials, or numbers.

Then, expanding or simplifying the polynomial, we remove the brackets (parentheses). We don't just erase the brackets, of course.

For instance, given a polynomial $2(x + 3y)(x^2 + y + 1)$, which is in a product form, and is in fact, a polynomial fully factoriazed, how then do we get the expansion?

What are we looking for when factorizing a polynomial?

It is a set of factors, which are divisors, so we mainly do divisions doing factorizations. And expansions (simplifications) are reverse operations of factorizations.

So expanding or simplifying a polynomial factorized fully or not, we do multiplications, and thus, multiply out the factors. By the way, prime factors are often just called factors, too, for simplicity, if no confusion is involved, of course.

How then do we do the multiplications, doing the expansion or the simplification?

We multiply out the factors.
And multiplying out factors, we multiply all the terms in all the factors in a *league*.
So for instance, expanding (simplifying) a polynomial $(x + y)(u + v + 1)$, we get

$$(x + y)(u + v + 1) = (x + y)u + (x + y)v + (x + y) = xu + yu + xv + yv + x + y.$$

Let's for more examples, expand polynomials as follows.

$z(xy + y)(x + z)$, $(x + y)(y + z)(z - x)$, and $(x + y + z)(x - y)$.

Then, we get

$$z(xy + y)(x + z) = (xyz + yz)(x + z) = (xyz + yz)x + (xyz + yz)z = x^2yz + xyz + xyz^2 + yz^2.$$

$$(x + y)(y + z)(z - x) = \{x(y + z) + y(y + z)\}(z - x) = (xy + xz + y^2 + yz)(z - x)$$

$$= (xy + xz + y^2 + yz)\{z + (-x)\} = (xy + xz + y^2 + yz)z + (xy + xz + y^2 + yz)(-x)$$

$$= (xy + xz + y^2 + yz)z - (xy + xz + y^2 + yz)x = xyz + xz^2 + y^2z + yz^2 - x^2y - x^2z - y^2x - xyz$$

$$= xz^2 + y^2z + yz^2 - x^2y - x^2z - y^2x.$$

$$(x + y + z)(x - y) = (x + y + z)x - (x + y + z)y = x^2 + xy + xz - (xy + y^2 + yz)$$

$$= x^2 + xy + xz - xy - y^2 - yz = x^2 + xz - y^2 - yz.$$

And doing the operations above backward, we can see better how factorizations can go. Now, factorizations of some polynomials are frequently used when we do algebra. They are called *factorization identities*, which are often called factorization formulas, too. And the factorization identities often used are as follows.

$$(x + y)^2 = x^2 + 2xy + y^2 \qquad (x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x + y)(x - y) = x^2 - y^2 \qquad (x + y)(x^2 - xy + y^2) = x^3 + y^3$$

$$(x^2 + xy + y^2)(x^2 - xy + y^2) = x^4 + x^2y^2 + y^4$$

$$(x + a)(x + b) = x^2 + (a + b)x + ab. \qquad (ax + b)(cx + d) = acx^2 + (ad + bc)x + bd.$$

$$(x + a)(x + b)(x + c) = x^3 + (a + b + c)x^2 + (ac + bc + ca)x + abc.$$

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca).$$

$$(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) = a^3 + b^3 + c^3 - 3abc.$$

And of course, we are going to see how we can get the left hand side from the right hand side of each of all the identities above when doing the sets of **Examples**.

And doing some algebra to some of those above, we can get some more identities.

Changing y with $-y$, we can get

$(x + y)^2 = x^2 + 2xy + y^2 \Rightarrow (x - y)^2 = x^2 - 2xy + y^2$.

$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 \Rightarrow (x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$.

$(x + y)(x^2 - xy + y^2) = x^3 + y^3 \Rightarrow (x - y)(x^2 + xy + y^2) = x^3 - y^3$.

And by the same token, we can get

$(x + a)(x + b) = x^2 + (a + b)x + ab \Rightarrow (x - a)(x - b) = x^2 - (a + b)x + ab$.

$(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd \Rightarrow (ax - b)(cx - d) = acx^2 - (ad + bc)x + bd$.

$(x + a)(x + b)(x + c) = x^3 + (a + b + c)x^2 + (ac + bc + ca)x + abc \Rightarrow$

$(x - a)(x - b)(x - c) = x^3 - (a + b + c)x^2 + (ac + bc + ca)x - abc$.

And we can put together some of those above the way as follows.

To begin with, we have $(x + y)^2 = x^2 + 2xy + y^2$, and $(x - y)^2 = x^2 - 2xy + y^2$.

So putting both together, we can have $(x \pm y)^2 = x^2 \pm 2xy + y^2$.

Next, we have $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$, and $(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$.

So putting both together, we can have $(x \pm y)^3 = x^3 \pm 3x^2y + 3xy^2 \pm y^3$.

And we can have

$x^3 + 3x^2y + 3xy^2 + y^3 = x^3 + 3xy(x + y) + y^3$

$x^3 - 3x^2y + 3xy^2 - y^3 = x^3 - 3xy(x - y) - y^3$

So we can put it this way, too: $x^3 \pm 3x^2y + 3xy^2 \pm y^3 = x^3 \pm 3xy(x \pm y) \pm y^3$.

Next, we have $(x + y)(x^2 - xy + y^2) = x^3 + y^3$, and $(x - y)(x^2 + xy + y^2) = x^3 - y^3$.

So putting both together, we can have $x^3 \pm y^3 = (x \pm y)(x^2 \mp xy + y^2)$.

Next, we have

$(x + a)(x + b) = x^2 + (a + b)x + ab$,

$(x - a)(x - b) = x^2 - (a + b)x + ab$

So putting both together, we can have $(x \pm a)(x \pm b) = x^2 \pm (a + b)x + ab$.

And next, we have

$(x + a)(x + b)(x + c) = x^3 + (a + b + c)x^2 + (ac + bc + ca)x + abc$.

$(x - a)(x - b)(x - c) = x^3 - (a + b + c)x^2 + (ac + bc + ca)x - abc$.

So in sum, we have $(x \pm a)(x \pm b)(x \pm c) = x^3 \pm (a + b + c)x^2 + (ac + bc + ca)x \pm abc$.

Also, doing some algebra, we can get some useful expressions as follows.

$(x + y)^2 = x^2 + 2xy + y^2 \Rightarrow x^2 + y^2 = (x + y)^2 - 2xy$.

$(x - y)^2 = x^2 - 2xy + y^2 \Rightarrow x^2 + y^2 = (x - y)^2 + 2xy$.

$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 = x^3 + 3xy(x + y) + y^3 \Rightarrow x^3 + y^3 = (x + y)^3 - 3xy(x + y)$.

$(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3 = x^3 - 3xy(x-y) - y^3 \Rightarrow x^3 - y^3 = (x-y)^3 + 3xy(x-y)$.

Besides, we can put $a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$ this way: $a^3 + b^3 + c^3 - 3abc = \frac{1}{2}(a+b+c)\{(a-b)^2 + (b-c)^2 + (c-a)^2\}$.

GCD and LCM 1

Like numbers and formulas, LCM and GCD are tools we can use solving problems, and are quite handy in many cases.
Doing algebra, we often use GCD and LCM of polynomials, as well as integers.
So to begin with, what do we mean by GCD?

It is the acronym of Greatest Common Divisor. So it is a divisor common and the largest. Since a factor is a divisor, GCD can be called GCF, Greatest Common Factor, too. And finding GCD, we usually work with integers, monomials, or polynomials.

If GCD is an integer, it is the largest divisor that can divide every integer in a group. And the same is true for a set of monomials or a set of polynomials, too. So if GCD is a monomial or polynomial, it is the largest divisor that can divide every monomial or polynomial in a group. Why GCD though? What's the use?

Suppose for instance, we want to distribute to students 576 apples, 360 pears, 180 bananas, and 252 oranges. Suppose also, we want to distribute all the fruit evenly in each kind. Thus, each student gets the same amount of each fruit. So amounts in kinds are different, but the total amount each student gets is the same. How then can we do so?

We might consider cases the way as follows.
The first case is that one student gets them all.
The second is that two students are recipients, and each student gets 288 apples, 180 pears, 90 bananas, and 126 oranges.
The third is that three students receive, and each student gets 192 apples, 120 pears, 60 bananas, and 84 oranges.
The fourth is that four students receive, and each student gets 144 apples, 90 pears, 45 bananas, and 63 oranges.
The sixth is that six students receive, and each student gets 96 apples, 60 pears, 30

bananas, and 42 oranges. And so forth. Why not five students, though?
It's because 5 doesn't divide 576 and 252, so 5 does not divide every number of fruit.
What then does the number of students mean in each case stated above?

It is a divisor common to all the integers, each of which is the number of each fruit. So the number of students has to be a divisor common to all these: 576, 360, 180, and 252.
What then is the total number of the cases we need to consider?

Many more that 30 cases. Way too many. So we may want to find a different way.
A tool called a common divisor can help.

A common divisor number of students can get the same amount of each fruit.
So if a common divisor is 3, each of 3 students gets a third of each fruit.
And it is one of options we can take. What then is the total number of options?

It is the number of the common divisors. In fact, we get 9 options, since there are 9 common divisors. What if now, we want to maximize the number of recipients?

Then, we need to use another tool called GCD.
So we want to find the greatest of all the divisors common to all the numbers of fruit.
How can we find it, though?

GCD is a common divisor, so first, we want to find divisors common to all the numbers.
And then, find the greatest of those common. What then should we do?

We can find such divisors efficiently using factors of each number, since factors are divisors. So prior to common divisors, we want to find factors of all the numbers. Thus, we want to begin with factorizing all the numbers. Then, we get

$$576 = 2^6 3^2, \quad 360 = 2^3 3^2 5, \quad 180 = 2^2 3^2 5, \text{ and } 252 = 2^2 3^2 7.$$

They are prime factorizations. A prime factorization of an integer puts the integer in a power of a prime such as 3^2, or in a product of such powers such as $2^3 3^2$.

Now that all the integers have been fully factorized, we can get the GCD. How?

First, find all the prime factors common to all the integers. Then, we get 2 and 3.
Next, find the smallest exponent to apply to each of the prime factors common. Then, 2 is the smallest for the prime factor 2, and also, 2 is the smallest for the prime factor 3.

Then, making the power of each common prime factor applying its smallest exponent, and taking the product of all the powers made, we get the GCD. Then, the GCD is $2^2 3^2$, which is 36, which is therefore, is the maximum number of the recipients.
How much fruit then each of those 36 students can get?

Setting GCD = G, we can put the numbers the way as follows.

Apples:	$576 = 2^6 3^2 = 2^2 3^2 2^4 = 16G.$
Pears:	$360 = 2^3 3^2 5 = 2^2 3^2 2 \cdot 5 = 10G.$
Bananas:	$180 = 2^2 3^2 5 = 5G.$
Oranges:	$252 = 2^2 3^2 7 = 7G.$

Thus, each student gets 16 apples, 10 pears, 5 bananas, and 7 oranges.

So distributing items evenly in each kind,
and maximizing the recipients,
we want to find the GCD of the numbers of items in all kinds.

In other words, distributing items equally in each kind,
and minimizing each number of items in each kind each recipient gets,
we want to find the GCD of the numbers of items in all kinds.

How then actually can we get the GCD?

First, we get all the integers prime factorized. How?

Keep dividing each integer by a prime until the quotient is a prime. How?

Start divisions by the smallest prime 2, and if it doesn't divide, move on to 3, but if it doesn't either, move on to the next smallest until the prime that can divide gets found.

And then, keep doing divisions until the quotient is a prime or the divisor doesn't divide.

If the quotient is a prime, the factorization is done, but if divisor doesn't divide, find the

next smallest prime, and then keep doing divisions until the quotient is a prime.
So for instance, factorizing 21780, we can do divisions the way as follows.

In the table below, 19890 is the quotient after the division of 21780 by the first 2, and 5445 is the quotient after the division of 19890 by the second 2 below the first 2.
So the numbers in the first column are primes that are divisors.

2	21780
2	19890
3	5445
3	1815
5	605
11	121
	11

Second, we put the integer in a power, or put a product of powers each of which is a power of a prime. Then, using the example above, we get $21780 = 2^2 3^2 5 \cdot 11$, where technically, 11 can be a power, since $11 = 11^1$. And for more instances, we can have $14641 = 11^4$, and $250 = 2 \cdot 5^3$. So after factoring all the 4 numbers of fruit, we get

$$576 = 2^6 3^2, \quad 360 = 2^3 3^2 5, \quad 180 = 2^2 3^2 5, \text{ and } 252 = 2^2 3^2 7.$$

And we can do such a factorization fast to every integer in a group the way as follows.

2	576	360	180	252
2	288	180	90	126
2	144	90	45	63
2	72	45		
2	36			
2	18			
3	9			
	3	↓	↓	↓
3		45	45	63
3		15	15	21
		5	5	7

In the table above, the first column is made of prime factors, which are divisors, so that all the integers in each row get divided by the prime factor in the row.
At the top of each subsequent column is each of all the integers in the group.

And at the bottom of each column is a prime.
So we can express each integer the way as follows.

576 = 2 x 2 x 2 x 2 x 2 x 2 x 3 x 3 = $2^6 3^2$, so 576 has 6 of 2s and 2 of 3s.

360 = 2 x 2 x 2 x 3 x 3 x 5 = $2^3 3^2 5$, so 360 has 3 of 2s, 2 of 3s, and a 5.

180 = 2 x 2 x 3 x 3 x 5 = $2^2 3^2 5$, so 180 has 2 of 2s, 2 of 3s, and a 5.

252 = 2 x 2 x 3 x 3 x 7 = $2^2 3^2 7$, so 152 has 2 of 2s, 2 of 3s, and a 7.

Making the product of all the common factors, we get the GCD. How?

Examining all the factorizations, we can see that every integer in the group commonly has 2 of 2s and 2 of 3s.
So all the common factors are 2 of 2s and 2 of 3s.
Therefore, since some factors repeat themselves in the product, we make powers.

Now, the GCD needs to have ***all*** the prime factors ***common*** to ***every*** integer, since it's the greatest and common.

So making a power, we use as the exponent the number of the same prime factors common to all the integers in the group, because the GCD divides all those integers.

Examining thus, all the factorizations, we can see that 2 of 2s and 2 of 3s are common, the exponent for the factor 2 is 2, and also, the exponent for the factor 3 is 2.

So GCD = $2^2 3^2$.
And examining the factorizations 576 = $2^6 3^2$, 360 = $2^3 3^2 5$, 180 = $2^2 3^2 5$, and 252 = $2^2 3^2 7$, we can get the GCD the way as follows, too.

First, find the prime common to all the integers,
Second, find the smallest exponent applied to the prime,
Third, make a power using as the base the common prime found, and using as the exponent the smallest exponent found.
Next, if any, with another prime common to all the integers, do the same processes above.

And if no more prime common to all the integers, take the product of all the powers.
So GCD is the greatest common divisor, the largest divisor common to all in a group.
What then about the smallest?

We can call it LCD. That is, the least common divisor
Technically, 1 is the LCD, because 1 is the smallest (positive) of all divisors common to all integers. Excluding 1 though, what can be the LCD?

The least means the smallest, and it divides all the integers in the group. So LCD is the smallest divisor common to all the integers in the group. How then to get it?

We know a factor is a divisor, and getting back to the table above, we have
$576 = 2^6 3^2$ apples, $360 = 2^3 3^2 5$ pears, $180 = 2^2 3^2 5$ bananas, and $252 = 2^2 3^2 7$ oranges.
Then, 2 and 3 are all prime factors common to all the integers, and 2 is the smaller.
Thus, the smallest prime factor common to all is 2, which is therefore, the LCD.
So what is the LCD about?

Distributing items equally in each kind,
and minimizing the number of recipients,
we want to find the LCD of the numbers of items in all kinds.

That is, distributing items equally in each kind,
and maximizing each number of items in each kind each recipient gets,
we want to find the LCD of the numbers of items in all kinds.
What then is LCM?

It is the acronym of Lest Common Multiple. So it is a multiple common and the smallest.
As in the case of GCD, when working with a set of objects, we often consider LCM, and the objects are usually integers, monomials, or polynomials.

If LCM is an integer, it is common to all the integers in a group. Since it is a multiple, every integer in the group can divide it, and is the smallest that can be divided by every integer in the group.

And the same is true for a set of monomials or a set of polynomials, too. Thus, if LCM is a monomial or polynomial, it is the smallest that can be divided by every monomial or

polynomial in the group. Why do we need LCM, though?
Suppose we want to distribute items evenly to a number of groups, in each of which though, the number of members can be different, but every member has to get the same amount. Suppose also, we want to minimize the amount each group gets.

For instance, one of three groups has 2 members, another group has 3, and the other has 5. In each group however, the amount every member gets is the same, and the total amount given to each group is the same, too. And also, we want to minimize the total amount each group gets.

Then, we want to find the least number that the number of members in every group can divide, and the least is LCM.

Suppose for instance, we want to distribute pencils to four schools, one of which has 576 students, another has 360, another has 180, and the other has 252, but want to allocate pencils evenly to all the schools, in each of which every student gets the same amount.

So since each school has a different number of students, the amount a student gets in one school is different from the amount a student gets in any of the other schools.
Then, how many pencils can each school get?

Every school gets the same amount, of course, which however, we want to find.

So to begin with, suppose $\boldsymbol{X}$ is the number of pencils each school gets.

Then, since in each school, every student gets the same amount, the number of students in each school divides $\boldsymbol{X}$. So $\boldsymbol{X}$ is a multiple of the number of students in each school.
That is, $\boldsymbol{X}$ is a multiple common to the numbers of students in all the schools.
What particular number then can $\boldsymbol{X}$ be?

Since each number of students is an integer, $\boldsymbol{X}$ is a multiple of an integer, and thus, can be infinitely large, so there is no maximum for it. There does exist the minimum, though.
What then is the minimum?

It is the minimum of $\boldsymbol{X}$, of course. So to begin with, $\boldsymbol{X}$ is a multiple common to the numbers of students in all the schools, and is the smallest.

In other words, the number of students in every school divides $\boldsymbol{X}$, and $\boldsymbol{X}$ is the smallest.

Thus, ***X*** is the smallest of all multiples common to the numbers of students in all the schools, and is the LCM. How can we find it, though?

Suppose ***A*** is a multiple of ***B***, and is a multiple of ***C***, too.
Then, ***A*** is a multiple common to ***B*** and ***C***. Thus, ***B*** can divide ***A***, and so can ***C***.
So do we have to have $A = BC$?

Not necessarily.
Suppose now, $B = uv$, and $C = vw$.
Then, ***u*** and ***v*** are prime factors of ***B***, and ***v*** and ***w*** are prime factors of ***C***.
Both of ***B*** and ***C*** can divide not only $uvvw = uv^2w$, but uvw, too, which is not BC.

Suppose next, $B = uv^2$, and $C = vw$.
Then again, ***u*** and ***v*** are prime factors of ***B***, and ***v*** and ***w*** are prime factors of ***C***.
Both of ***B*** and ***C*** can divide $u^2v^2w^2$, u^2v^2w, and uv^2w, but not uvw.
That's because $B = uv^2$ cannot divide uvw..

Suppose next, $B = uv^2$, and $C = vw^2$.
Then again, ***u*** and ***v*** are prime factors of ***B***, and ***v*** and ***w*** are prime factors of ***C***.
Both of ***B*** and ***C*** can divide $u^2v^3w^3$, $u^2v^2w^2$, and uv^2w^2, but not uv^2w.
That's because $C = vw^2$ cannot divide uv^2w.

Suppose next, $B = uv^2w$, and $C = vw^2$.
Then, ***u***, ***v***, and ***w*** are prime factors of ***B***, and ***v*** and ***w*** are prime factors of ***C***.
Both of ***B*** and ***C*** can divide $u^2v^3w^3$, $u^2v^2w^2$, and uv^2w^2, but not uv^2w.
That's because $C = vw^2$ cannot divide uv^2w.

Suppose next, $B = uv^2ws$, and $C = vw^2t$.
Then, ***u***, ***v***, ***w*** and ***s*** are prime factors of ***B***, and ***v***, ***w***, and ***t*** are prime factors of ***C***.
Both of ***B*** and ***C*** can divide $u^2v^3w^3s^2t^2$, $u^2v^2w^2s^2t$, and uv^2w^2st, but not uv^2ws.

That's because $\boldsymbol{C = vw^2t}$ cannot divide $\boldsymbol{uv^2ws}$.

Therefore, a multiple common to a set of monomials has ***at least*** all prime factors all the monomials have, and the exponent to be used for each prime factor is ***at least*** the largest of all used for the prime factor.

Now, in the case of $\boldsymbol{uv^2ws}$ and $\boldsymbol{vw^2t}$,
all prime factors the two monomials $\boldsymbol{uv^2ws}$ and $\boldsymbol{vw^2t}$ have are ***u***, ***v***, ***w***, ***s***, and ***t***,
1 is the largest exponent used for ***u***,
2 is the one for each of ***v*** and ***w***,
and 1 is the one for each of ***s*** and ***t***.

So a multiple common to $\boldsymbol{uv^2ws}$ and $\boldsymbol{vw^2t}$ has as prime factors ***at least u***, ***v***, ***w***, ***s***, and ***t***, and the exponents to be used for the prime factors are ***at least*** 1, 2, 2, 1, and 1 respectively in the order of ***u***, ***v***, ***w***, ***s***, and ***t***.

Therefore, the multiple the ***least*** and common to a set of monomials has *all prime factors* all the monomials have,

and the *exponent* to be used for each prime factor is *the largest* of all used for the prime factor.

Thus, the LCM of $\boldsymbol{uv^2ws}$ and $\boldsymbol{vw^2t}$ is $\boldsymbol{uv^2w^2st}$.

The same is true for integers and polynomials, too. So solving the problem with the pencil distribution, we want to begin with factorizing all the numbers of students.

Factorizing them all, we can see one of the four schools has $576 = 2^6 3^2$ students, another has $360 = 2^3 3^2 5$, another has $180 = 2^2 3^2 5$, and the other has $252 = 2^2 3^2 7$.

Thus, all prime factors the product of all the numbers of students has are 2, 3, 5, and 7, and the largest exponents used for the prime factors are 6, 2, 1, and 1.

So the least common multiple of all the numbers of the students is $2^6 3^2 5 \cdot 7$, which is ***X***, which is the minimum number of pencils each school gets.

Thus, since every school gets $2^6 3^2 5 \cdot 7 = 20160$ pencils, we can see that

Each student attending the school of $576 = 2^6 3^2$ students gets $5 \cdot 7 = 35$ pencils.
Each student attending the school of $360 = 2^3 3^2 5$ students gets $2^3 \cdot 7 = 56$ pencils.
Each student attending the school of $180 = 2^2 3^2 5$ students gets $2^4 \cdot 7 = 112$ pencils.
Each student attending the school of $252 = 2^2 3^2 7$ students gets $2^4 \cdot 5 = 80$ pencils.

What if there is another school with 180 students?

Then, we just add another 20160 pencils.

That's not the only case where we can use LCM, of course.

We can use LCM when adding together fractions where denominators are different. Of course, the same is true for subtractions with such fractions, too, since subtractions are additions of the negatives. In such cases though, we often use just a common multiple.

Taking the LCM of all the denominators different, and using the LCM as the common denominator, we can add the fractions.

So for instance, adding together $\frac{1}{2}$, $\frac{5}{3}$, and $\frac{7}{4}$, we can take the sum the way as follows.

Taking the LCM of 2, 3, and 4, which is 12, and using it as the common denominator of the fractions, we get $\frac{1}{2} = \frac{6}{12}$, $\frac{5}{3} = \frac{20}{12}$, and $\frac{7}{4} = \frac{21}{12}$, so adding them up, we get $\frac{6+20+21}{12} = \frac{47}{12}$, which is the sum of $\frac{1}{2}$, $\frac{5}{3}$, and $\frac{7}{4}$.

Just taking the product of 2, 3, and 4 though, and using it as the common denominator, we can get the same result, too, of course, but calculation gets longer. And the same is true, also, for fractions made of monomials and polynomials. So for instance, adding

together $\frac{1}{2}$, $\frac{5a}{4c^2}$, and $\frac{2b(a+1)}{3c(a+b)}$, we can take the sum the way as follows.

Taking the LCM of **2**, $4c^2$, and $3c(a+b)$, and using it as the common denominator of the fractions, we get $\frac{1}{2}=\frac{6c^2(a+b)}{12c^2(a+b)}$, $\frac{5a}{4c^2}=\frac{15a(a+b)}{12c^2(a+b)}$, and $\frac{2b(a+1)}{3(a+b)}=\frac{8bc^2(a+1)}{12c^2(a+b)}$, so adding them up, we get $\frac{6c^2(a+b)+15a(a+b)+8bc^2(a+1)}{12c^2(a+b)}$, which is the sum of $\frac{1}{2}$, $\frac{5a}{4c^2}$, and $\frac{2b(a+1)}{3c(a+b)}$.

How is the LCM $12c^2(a+b)$, though?

Doing algebra, we often use GCD and LCM of monomials or polynomials, as well as integers. In the section, **GCD and LCM 2**, we are going to look at how we can get the LCM above and how it works.

GCD and LCM 2

Let's now have a look at how $12c^2(a + b)$ can be the LCM of 2, $4c^2$, and $3c(a + b)$.
We are going to begin with GCD of polynomials.

Suppose that $k = 0, 1, 2, \ldots n$, where n is a nonnegative integer, and that for all k, all Q_k are polynomials, and are different from each other.

Suppose also that D is a polynomial, $M_0 = DQ_0$, and $M_1 = DQ_1$.

Then, M_0 is the product of D and Q_0, which are polynomials.
So M_0 is a polynomial, too, and is a multiple of D. Thus, D is a divisor of M_0.

And the same is true for M_1, too, so M_1 is a multiple of D, too.
So D is a divisor of M_1, as well as M_0. Thus, D is a divisor common to M_0 and M_1.

Now, suppose further, that there is no factor common to Q_0 and Q_1.

That is, other than 1, there is no divisor common to Q_0 and Q_1.
In other words, Q_0 has no factor that can be a factor of Q_1, and vice versa.

For instance, if $Q_0 = (x + 1)(x + 2)$, and $Q_1 = (x + 3)(x + 4)$, all factors Q_0 has are $x + 1$ and $x + 2$, so neither of those two factors can divide (that is, can be a factor of) Q_1.

Thus, other than 1 and D, there is no divisor common to M_0 and M_1.

So the only divisors common to M_0 and M_1 are 1 and D.
And D is the bigger divisor than 1. Why the bigger?

A divisor that has more divisors is the bigger divisor. And divisors of $\boldsymbol{D}$ can be 1 and itself, but 1 is the only divisor that can divide 1. So $\boldsymbol{D}$ is the bigger. And as stated above, other than 1 and $\boldsymbol{D}$, there is no divisor common to $\boldsymbol{M_0}$ and $\boldsymbol{M_1}$.

So $\boldsymbol{D}$ is a divisor common to $\boldsymbol{M_0}$ and $\boldsymbol{M_1}$, and the greatest.

Thus, $\boldsymbol{D}$ is the greatest divisor common to $\boldsymbol{M_0}$ and $\boldsymbol{M_1}$. So $\boldsymbol{D}$ is the GCD of $\boldsymbol{M_0}$ and $\boldsymbol{M_1}$.

Suppose this time, $\boldsymbol{M_k = DQ_k}$, where $\boldsymbol{k = 0, 1, 2, \ldots n}$, where $\boldsymbol{n}$ is an integer $\boldsymbol{\geq 0}$.

That is, $\boldsymbol{M_0 = DQ_0}$, $\boldsymbol{M_1 = DQ_1}$, $\boldsymbol{M_2 = DQ_2}$, …, $\boldsymbol{M_n = DQ_n}$.

Then, for each $\boldsymbol{k}$, $\boldsymbol{M_k}$ is a multiple of $\boldsymbol{D}$, so $\boldsymbol{D}$ is a divisor of $\boldsymbol{M_k}$.

Thus, $\boldsymbol{D}$ is a divisor of $\boldsymbol{M_k}$ for every $\boldsymbol{k}$. So $\boldsymbol{D}$ is a common divisor of $\boldsymbol{M_k}$ for all $\boldsymbol{k}$.

Now, suppose further, for a value of $\boldsymbol{k}$, $\boldsymbol{Q_k}$ has no factor common to all the other $\boldsymbol{Q_k}$.

For instance, $\boldsymbol{Q_2}$ has no factor common to all the others, which are $\boldsymbol{Q_0}$, $\boldsymbol{Q_1}$, $\boldsymbol{Q_3}$, $\boldsymbol{Q_4}$,…, and $\boldsymbol{Q_n}$. (That is, no factor of $\boldsymbol{Q_2}$ can be a factor common to $\boldsymbol{Q_0}$, $\boldsymbol{Q_1}$, $\boldsymbol{Q_3}$, $\boldsymbol{Q_4}$, …, and $\boldsymbol{Q_n}$.)

Thus, there is no factor common to $\boldsymbol{Q_k}$ for all $\boldsymbol{k}$.

So other than 1 and $\boldsymbol{D}$, there is no divisor common to $\boldsymbol{M_k}$ for all $\boldsymbol{k}$.

Then, $\boldsymbol{D}$ is not just a divisor common to $\boldsymbol{M_k}$ for all $\boldsymbol{k}$. What else, too, is it, then?

$\boldsymbol{D}$ is the greatest divisor, too. So $\boldsymbol{D}$ is the greatest divisor common to all $\boldsymbol{M_k}$.

That is, $\boldsymbol{D}$ is the GCD of $\boldsymbol{M_k}$ for all $\boldsymbol{k}$. How?

The only divisors common to all M_k are 1 and D, and D has more divisors than 1 has. A divisor that has more divisors is the bigger divisor. So D is the GCD.

Let's now, take more concrete examples. Suppose that $D = x + 1$, and that

$$Q_0 = (x + y)(y + 1)$$
$$Q_1 = (x + y)^2(y + 1)$$
$$Q_2 = (x + 2y)(y^2 + y + 4)$$
$$Q_3 = (x + 1)(x + y)(x + 3y).$$

Suppose also, that $M_k = DQ_k$, where $k = 0, 1, 2$, and 3, that is,

$$M_0 = DQ_0 = (x + 1)(x + y)(y + 1)$$
$$M_1 = DQ_1 = (x + 1)(x + y)^2(y + 1)$$
$$M_2 = DQ_2 = (x + 1)(x + 2y)(y^2 + y + 4)$$
$$M_3 = DQ_3 = (x + 1)(x + 1)(x + y)(x + 3y) = (x + 1)^2(x + y)(x + 3y).$$

Then, D is the greatest common divisor, that is, the GCD of M_k for all k. How?

To begin with, D is a divisor of every M_k for every k.

So D is a divisor common to all M_k for $k = 0, 1, 2$, and 3.

Next, there is no factor common to Q_0, Q_1, Q_2, and Q_3.

That is, for instance, all factors of Q_0 are $(x + y)$ and $(y + 1)$, and neither of all the factors can divide at the same time all of Q_1, Q_2, and Q_3.

And the same is true for each of Q_1, Q_2, and Q_3, too.

So other than 1 and D, there is no divisor common to all the M_k.

Next, D has more divisors than 1 has, and therefore, is the GCD.

Suppose this time, that $D = (x + 1)(x + 2)$, and that

$Q_0 = (x + y + z)(y + 1)$
$Q_1 = (x + z)(y + 1)^2$
$Q_2 = (x + y)(y^2 + x)$
$Q_3 = (x + 1)(x + y)$
$Q_4 = (x + 1)(x + 2)(x + y)(y + 1)$.

Suppose also, $M_k = DQ_k$, where $k = 0, 1, 2, 3$, and 4, that is,

$M_0 = DQ_0 = (x + 1)(x + 2)(x + y + z)(y + 1)$
$M_1 = DQ_1 = (x + 1)(x + 2)(x + z)(y + 1)^2$
$M_2 = DQ_2 = (x + 1)(x + 2)(x + y)(y^2 + x)$
$M_3 = DQ_3 = (x + 1)(x + 2)(x + 1)(x + y) = (x + 1)^2(x + 2)(x + y)$
$M_4 = DQ_4 = (x + 1)(x + 2)(x + 1)(x + 2)(x + y)(y + 1) = (x + 1)^2(x + 2)^2(x + y)(y + 1)$.

Then, D is the greatest common divisor, that is, the GCD of M_k for all k. How?

To begin with, D is a divisor of every M_k for every k.
So D is a divisor common to M_k for all k.

Next, $(x + 1)$ is a factor of D, and can divide M_k for all k, too.

The same is true for $(x + 2)$, too.

So $(x + 1)$ and $(x + 2)$ are divisors common to M_k for all k.

Next, there is no factor common to all Q_k, which are Q_0, Q_1, Q_2, Q_3, and Q_4.

That is, for instance, all factors of Q_3 are $(x + 1)$ and $(x + y)$, and neither of all the factors can be common to all of Q_0, Q_1, Q_2, and Q_4.

And the same is true for each of Q_0, Q_1, Q_2, and Q_4, too.

So other than 1, $(x + 1)$, $(x + 2)$, and $(x + 1)(x + 2) = D$, there is no divisor common to M_k for all k.

Now, the divisors of $x + 2$ is 1 and itself, and the same is true for $x + 1$, too, but those of D is $x + 1$ and $x + 2$, together with 1 and itself. The more divisors, the bigger divisor.

Among all the common divisors, D has the most number of divisors, so D is the GCD.

Suppose for another example, we have

$$M_0 = 144(x + 1)^2(x + y)^2(y + 1) = 2^4 3^2(x + 1)^2(x + y)^2(y + 1)$$
$$M_1 = -216(x + 1)(x + y)^3(y + 2) = -2^3 3^3(x + 1)(x + y)^3(y + 2)$$
$$M_2 = 360(x + 1)(x + y)^2(y + 3)^2 = 2^3 3^2 5(x + 1)(x + y)^2(y + 3)^2$$
$$M_3 = 504(x + 1)(x + y)^2(y^2 + y + 4) = 2^3 3^2 7(x + 1)(x + y)^2(y^2 + y + 4)$$
$$M_4 = -792(x + 1)^2(x + y)^3(2y + 5x) = -2^3 3^2 11(x + 1)^2(x + y)^3(2y + 5x)$$

Then, we can begin with finding divisors common to all M_k for $k = 0, 1, 2, 3$, and 4. For instance $x + 1$, $2(x + 1)$, $3(x + 1)$, etc. are common divisors. Looking for GCD though, we may not want to begin with just finding all divisors common. Why not?

There are quite a few, or rather, excessively many divisors common to all M_k.
What then do we need to begin with?

Let's talk about the case with integers, first. The GCD of a set of integers is basically a product of all the factors common to all the integers. So we want to begin with factorizing all the integers. What are all the common factors, though, stated above?

Suppose all the integers are now fully factorized, so for instance, we have $360 = 2^3 3^2 5$, $432 = 2^4 3^3$, and $504 = 2^3 3^2 7$. Then, two prime factors are common to all the integers.

And the two prime factors are 2 and 3. Then, the GCD is the product of two factors common to all the integers, and the two factors are two powers.

One of the two is a power where the base is one of the two prime factors, so is 2 or 3. And the exponent is the smallest of all used for the prime factor, so if the prime factor is 2, the exponent is 3, and if the prime factor is 3, the exponent is 2.

Thus, the GCD of 360, 432, and 504 is $2^3 3^2$, which is 72.

And let's see now if the two powers 2^3 and 3^2 are divisors common to 360, 432, and 504.

We have $360 = 2^3 3^2 5$, $432 = 2^4 3^3$, and $504 = 2^3 3^2 7$.

And each of 2^3 and 3^2 can divide $2^3 3^2 5$, can divide $2^4 3^3$, and also, can divide $2^3 3^2 7$.

So each of 2^3 and 3^2 can divide all the three integers 360, 432, and 504.

And if an integer is greater than $2^3 3^2$, the integer doesn't divide all of the three integers.

We know the two powers are two factors common to all the three integers.
So the product of all the factors common to all the integers is the greatest of all divisors common to all the integers, and thus, is the GCD. Why is such a product the greatest?

Let's take a look at the GCD of $360 = 2^3 3^2 5$, $432 = 2^4 3^3$, and $504 = 2^3 3^2 7$.

The GCD is $2^3 3^2$, which is the product of two powers 2^3 and 3^2.

In the power 2^3, the base is 2, which is a prime factor common to all the three integers, and the exponent is 3, which is the smallest of all used for the prime factor 2.

So in all the integers $360 = 2^3 3^2 5$, $432 = 2^4 3^3$, and $504 = 2^3 3^2 7$, among all the powers where the base is 2, 2^3 is the divisor the greatest and common to all the three integers.

In the power 3^2, the base is 3, which is a prime factor common to all the three integers, and the exponent is 2, which is the smallest of all used for the prime factor 3.

So in all the integers $360 = 2^3 3^2 5$, $432 = 2^4 3^3$, and $504 = 2^3 3^2 7$, among all the powers where the base is 3, 3^2 is the divisor the greatest and common to all the three integers.

Thus, among all integers, the integer $2^3 3^2$, which is 72 is the greatest of all divisors common to 360, 432, and 504, and therefore, is the GCD.
What then about the GCD of $360 = 2^3 3^2 5$, $720 = 2^4 3^2 5$, and $600 = 2^3 3 \cdot 5^2$?

Then, three prime factors are common to all the integers.
And the three are 2, 3 and 5. Then, the GCD is the product of three factors common to all the integers, and the three factors are three powers.

Each of the three common factors is a power where the base is one of the three prime factors, so is 2, 3, or 5.

And the exponent is the smallest of all used for the prime factor,
so if the prime factor is 2, the exponent is 3, if the prime factor is 3, the exponent is 1, and if the prime factor is 5, the exponent is 1. Why is 5 a power, though?

It's because technically, 5 is a power, too, since $5 = 5^1$, which is the first power of 5. For instance, 5^3 can be called the third power of 5, 5 to the third power, 5 to the third for short, and is usually just called 5 cubed.

Thus, now, taking the product of the thee powers, we get $2^3 3^1 5^1$, which is $2^3 3 \cdot 5$, which is 120, which is the GCD of 360, 720, and 600. And if an integer is greater than $2^3 3 \cdot 5$, the integer doesn't divide all of the three integers. Also, we know those powers are all the factors common to all the integers.

So the product of all the factors common to all the integers is the greatest of all divisors common to all the integers, and thus, is the GCD.
Why is such a product the greatest, though?

Let's take a look at the GCD of $360 = 2^3 3^2 5$, $720 = 2^4 3^2 5$, and $600 = 2^3 3 \cdot 5^2$.

The GCD is $2^3 3 \cdot 5$, which is the product of three powers 2^3, 3^1, and 5^1.

In the power 2^3, the base is 2, which is a prime factor common to all the integers, and the exponent is 3, which is the smallest of all used for the prime factor 2.

So in all the integers $360 = 2^3 3^2 5$, $720 = 2^4 3^2 5$, and $600 = 2^3 3 \cdot 5^2$, among all the powers where the base is 2, 2^3 is the divisor the greatest and common to all the three integers.

In the power 3^1, the base is 3, which is a prime factor common to all the integers, and the exponent is 1, which is the smallest of all used for the prime factor 3.

So in all the integers $360 = 2^3 3^2 5$, $720 = 2^4 3^2 5^1$, and $600 = 2^3 3^1 5^2$, among all the powers where the base is 3, 3^1 is the divisor the greatest and common to all the three integers.

And in the power 5^1, the base is 5, which is a prime factor common to all the integers, and the exponent is 1, which is the smallest of all used for the prime factor 5.

So in all the integers $360 = 2^3 3^2 5$, $720 = 2^4 3^2 5^1$, and $600 = 2^3 3^1 5^2$, among all the powers where the base is 5, 5^1 is the divisor the greatest and common to all the three integers.

Thus, among all integers, the integer $2^3 3^1 5^1$, which is 120 is the greatest of all the divisors common to 360, 720, and 600, and therefore, is the GCD.

So in short, finding the GCD of a set of integers,
we get all the prime factors common to all the integers,
and get the exponents the smallest of all used for the prime factors,
and then, take the product of all the powers. Then, the product is the GCD.

And the same is true for GCD of monomials and polynomials, too.

So finding the GCD of a set of polynomials, too,
we want to do it *prime factor by prime factor*,
and thus, we want to factorize all the polynomials first.

Now, getting back to the example, we have

$$\boldsymbol{M_0 = 2^4 3^2 (x+1)^2 (x+y)^2 (y+1)}, \quad \boldsymbol{M_1 = -2^3 3^3 (x+1)(x+y)^3 (y+2)},$$
$$\boldsymbol{M_2 = 2^3 3^2 5 (x+1)(x+y)^2 (y+3)^2}, \quad \boldsymbol{M_3 = 2^3 3^2 7 (x+1)(x+y)^2 (y^2+y+4)}, \text{ and}$$
$$\boldsymbol{M_4 = -2^3 3^2 11 (x+1)^2 (x+y)^3 (2y+5x)}.$$

To begin with, we factorize all the polynomials we want to find GCD of.
All the polynomials $\boldsymbol{M_k}$ are all fully factorized already, so we are ready to get the GCD.

Next, finding all the prime factors common to all $\boldsymbol{M_k}$, we get **2**, **3**, $\boldsymbol{x + 1}$, and $\boldsymbol{x + y}$.

Next, we find the smallest of all exponents used for each of the prime factors. Then,

3 is the smallest exponent used for the prime factor 2.
2 is the smallest exponent used for the prime factor 3.
1 is the smallest exponent used for the prime factor $\boldsymbol{x + 1}$.
2 is the smallest exponent used for the prime factor $\boldsymbol{x + y}$.

Next, we get the powers common to all the polynomials $\boldsymbol{M_k}$.

Then, we get four powers $\boldsymbol{2^3}$, $\boldsymbol{3^2}$, $\boldsymbol{(x + 1)^1 = x + 1}$, and $\boldsymbol{(x + y)^2}$, each of which is a factor common to all the polynomials. In the power $\boldsymbol{(x + 1)^1}$, the base is $\boldsymbol{x + 1}$, which is one of the prime factors common to all the polynomials, and the exponent is 1, which is the smallest of all used for the prime factor $\boldsymbol{x + 1}$.

Next, we get the product of the four factors, which are the four powers above.

Then, we simply get $\boldsymbol{2^3 3^2 (x + 1)(x + y)^2}$, which is the GCD of all the polynomials $\boldsymbol{M_k}$.

We have some negatives among $\boldsymbol{M_k}$ though, so why ***not*** GCD = $\boldsymbol{-2^3 3^2 (x + 1)(x + y)^2}$?

That's because we normally take a positive integer for a divisor.

The same is true for divisors in constant, monomial, and polynomial, too.
Of course, we can put the GCD above this way, too: $\boldsymbol{72(x + 1)(x + y)^2}$.

Usually though, finding the GCD, we do it in such a way as follows.

To begin with, we factorize all the polynomials we want to find GCD of.
Using the example above, we get

$\boldsymbol{2^4 3^2 (x + 1)^2 (x + y)^2 (y + 1)}$, $\boldsymbol{-2^3 3^3 (x + 1)(x + y)^3 (y + 2)}$, $\boldsymbol{2^3 3^2 5 (x + 1)(x + y)^2 (y + 3)^2}$,

$\boldsymbol{2^3 3^2 7 (x + 1)(x + y)^2 (y^2 + y + 4)}$, and $\boldsymbol{-2^3 3^2 11 (x + 1)^2 (x + y)^3 (2y + 5x)}$.

Next, we find all prime factors common to all the polynomials.
Then, we get $\mathbf{2}$, $\mathbf{3}$, $\boldsymbol{x}+\mathbf{1}$, and $\boldsymbol{x}+\boldsymbol{y}$.

Next, we take the product of all those factors above.
Then, we get $\mathbf{2\cdot 3}(\boldsymbol{x}+\mathbf{1})(\boldsymbol{x}+\boldsymbol{y})$.

Next, we apply the smallest of all exponents used for each of the prime factors above.
Then, we get $\mathbf{2^3 3^2}(\boldsymbol{x}+\mathbf{1})(\boldsymbol{x}+\boldsymbol{y})^2$, which is the GCD.

Now, let's for another example, find the GCD of polynomials as follows.

$$\boldsymbol{M_0} = \mathbf{144}\boldsymbol{a^3b^2xy}(\boldsymbol{x}+\mathbf{1})^2(\boldsymbol{x}+\boldsymbol{y})^2(\boldsymbol{y}+\mathbf{1}) = \mathbf{2^4 3^2}\boldsymbol{a^3b^2xy}(\boldsymbol{x}+\mathbf{1})^2(\boldsymbol{x}+\boldsymbol{y})^2(\boldsymbol{y}+\mathbf{1})$$
$$\boldsymbol{M_1} = \mathbf{-216}\boldsymbol{a^2by^2}(\boldsymbol{x}+\mathbf{1})(\boldsymbol{x}+\boldsymbol{y})^3(\boldsymbol{y}+\mathbf{2}) = \mathbf{-2^3 3^3}\boldsymbol{a^2by^2}(\boldsymbol{x}+\mathbf{1})(\boldsymbol{x}+\boldsymbol{y})^3(\boldsymbol{y}+\mathbf{2})$$
$$\boldsymbol{M_2} = \mathbf{360}\boldsymbol{a^2bxy}(\boldsymbol{x}+\mathbf{1})(\boldsymbol{x}+\boldsymbol{y})^2(\boldsymbol{y}+\mathbf{3})^2 = \mathbf{2^3 3^2 5}\boldsymbol{a^2bxy}(\boldsymbol{x}+\mathbf{1})(\boldsymbol{x}+\boldsymbol{y})^2(\boldsymbol{y}+\mathbf{3})^2$$
$$\boldsymbol{M_3} = \mathbf{504}\boldsymbol{a^3b^2cy^3}(\boldsymbol{x}+\mathbf{1})(\boldsymbol{x}+\boldsymbol{y})^2(\boldsymbol{y}^2+\boldsymbol{y}+\mathbf{4}) = \mathbf{2^3 3^2 7}\boldsymbol{a^3b^2cy^3}(\boldsymbol{x}+\mathbf{1})(\boldsymbol{x}+\boldsymbol{y})^2(\boldsymbol{y}^2+\boldsymbol{y}+\mathbf{4})$$
$$\boldsymbol{M_4} = \mathbf{-792}\boldsymbol{a^4bc^2y}(\boldsymbol{x}+\mathbf{1})^2(\boldsymbol{x}+\boldsymbol{y})^3(\mathbf{2}\boldsymbol{y}+\mathbf{5}\boldsymbol{x}) = \mathbf{-2^3 3^2 11}\boldsymbol{a^4bc^2y}(\boldsymbol{x}+\mathbf{1})^2(\boldsymbol{x}+\boldsymbol{y})^3(\mathbf{2}\boldsymbol{y}+\mathbf{5}\boldsymbol{x}).$$

To begin with, we factorize all the polynomials we want to find GCD of.

All the polynomials $\boldsymbol{M_k}$ are all fully factorized already, so we are ready to get the GCD.

Next, finding all prime factors common to all $\boldsymbol{M_k}$, we get $\mathbf{2}$, $\mathbf{3}$, $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{y}$, $\boldsymbol{x}+\mathbf{1}$, and $\boldsymbol{x}+\boldsymbol{y}$.

Next, taking the product of all those factors above, we get $\mathbf{2\cdot 3}\boldsymbol{aby}(\boldsymbol{x}+\mathbf{1})(\boldsymbol{x}+\boldsymbol{y})$.

Next, we apply the smallest of all exponents used for each of the prime factors above.

Then, we get $\mathbf{2^3 3^2}\boldsymbol{a^2by}(\boldsymbol{x}+\mathbf{1})(\boldsymbol{x}+\boldsymbol{y})^2$, which is the GCD.

So in short, finding GCD, what do we do?

Put in a form of a product all the prim factors common, and then, apply to each of the factors the smallest exponent used.

Now, let's move on to LCM of polynomials.

LCM has quite the opposite sense when it is compared to GCD.

Whereas GCD is a divisor the greatest, LCM is a multiple the smallest.

Both share the same nature though. Both are common to many integers or are common to many polynomials.

So GCD is the common divisor the greatest, LCM is the common multiple the smallest.

Now, suppose first, ***B*** and ***C*** are monomials different from each other.
Suppose next, ***A*** is a multiple of ***B***, and also, is a multiple of ***C***.

Then, ***A*** is a multiple common to ***B*** and ***C***. So do we have to have $A = BC$?

Not necessarily, of course. Why not though?

Suppose first, ***B*** and ***C*** are prime to each other.
That is, ***B*** does not have any prime factor that ***C*** has. For instance, $B = uv$, and $C = st$.

Then, ***A*** has to have as prime factors ***u***, ***v***, ***s***, and ***t*** at least.
So ***A*** can have more prime factors, too.

That's because ***A*** is a multiple common to ***B*** and ***C***, so ***B*** can divide ***A***, and so can ***C***.

For instance, ***A*** can be $uvst = BC$, $2uv^2st = 2vBC$, $3cuvst = 3cBC$, $uvstxy = BCxy$, etc.

Suppose next, ***B*** has some prime factors that ***C*** has.

Then, it can rather be the case where ***BC*** can be a multiple of ***A***. How?

Suppose for instance, $B = uv$, and $C = vw$.

Then, ***A*** has to have ***u***, ***v***, and ***w*** as prime factors at least, since ***B*** and ***C*** both divide ***A***, because ***A*** is a multiple common to ***B*** and ***C***.

So ***A*** can be ***uvw***, because ***B*** divides ***A***, and so does ***C***, since ***A*** is a multiple common to ***B*** and ***C***.

And we have $\boldsymbol{BC = uv^2w}$. So if $\boldsymbol{A = uvw}$, we get $\boldsymbol{BC = vA}$, so $\boldsymbol{BC}$ is a multiple of $\boldsymbol{A}$.

And of course, in this case, too, $\boldsymbol{A}$ can have more prime factors other than $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$, because $\boldsymbol{A}$ is a multiple common to $\boldsymbol{B}$ and $\boldsymbol{C}$.

For instance, $\boldsymbol{A}$ can be $\boldsymbol{uvw}$, $\boldsymbol{2uvw}$, $\boldsymbol{au^2vw}$, etc. Then, $\boldsymbol{B}$ divides $\boldsymbol{A}$, and so does $\boldsymbol{C}$.
So there can be infinitely many monomials that can be $\boldsymbol{A}$, a multiple common to $\boldsymbol{B}$ and $\boldsymbol{C}$.

Thus, there does not exist the greatest multiple common if no other condition is applied.
There does exist though, the least common multiple, called LCM.

So let's see now, how it can be made, that is, the principle behind the LCM.
In the case of the examples $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ above, we have found a fact as follows.

A multiple common to a set of monomials has to have ***at least*** all the prime factors all the monomials have, and the exponent to be used for each prime factor is ***at least*** the largest of all used for the prime factor.

So let's now, for instance, consider the case of a multiple common to $\boldsymbol{uv^2ws}$ and $\boldsymbol{vw^2t}$.

First, all the prime factors the two monomials $\boldsymbol{uv^2ws}$ and $\boldsymbol{vw^2t}$ have are $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$, $\boldsymbol{s}$, and $\boldsymbol{t}$.
Next, 1 is the largest exponent used for $\boldsymbol{u}$, 2 is the one for each of $\boldsymbol{v}$ and $\boldsymbol{w}$, and 1 is the one for each of $\boldsymbol{s}$ and $\boldsymbol{t}$.

So a multiple common to $\boldsymbol{uv^2ws}$ and $\boldsymbol{vw^2t}$ has to have as prime factors ***at least*** $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$, $\boldsymbol{s}$, and $\boldsymbol{t}$, and the exponents to be used for the prime factors are ***at least*** 1, 2, 2, 1, and 1 respectively in the order of $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$, $\boldsymbol{s}$, and $\boldsymbol{t}$.

Therefore, the multiple the ***least*** and common to the two monomials has *all the prime factors* all the two monomials have, and the *exponent* to be used for each prime factor is *the largest* of all used for the prime factor. Thus, the LCM of $\boldsymbol{uv^2ws}$ and $\boldsymbol{vw^2t}$ is $\boldsymbol{uv^2w^2st}$.

And the same is true for integers and polynomials, too.
So in the case of polynomials, too, the multiple the ***least*** and common to a set of polynomials, that is, the LCM has *all the prime factors* all the polynomials have, and the *exponent* to be used for each prime factor is *the largest* of all used for the prime factor.

Let's now, for instance find the LCM of polynomials as follows.

$\boldsymbol{P_0 = (x + y)(y + 1)}$, $\boldsymbol{P_1 = (x + y)^2(y + 1)}$,

$\boldsymbol{P_2 = (x + 2y)(y^2 + y + 4)}$, and $\boldsymbol{P_3 = (x + 1)(x + y)(x + 3y)}$.

Then, all the polynomials are fully factorized already, so we are ready to find the LCM.

So next, we just take the product of ***all the prime factors*** that all the polynomials have.

Then, we get $\boldsymbol{(x + y)(y + 1)(x + 2y)(y^2 + y + 4)(x + 1)(x + 3y)}$.

Next, apply to each prime factor above ***the largest exponent*** of all used for the prime factor.

Then, 2 is the largest used for the factor $\boldsymbol{(x + y)}$, and 1 is the largest used for each of the factors $\boldsymbol{(y + 1)}$, $\boldsymbol{(x + 2y)}$, $\boldsymbol{(y^2 + y + 4)}$, $\boldsymbol{(x + 1)}$, and $\boldsymbol{(x + 3y)}$.

Therefore, the LCM is $\boldsymbol{(x + y)^2(y + 1)(x + 2y)(y^2 + y + 4)(x + 1)(x + 3y)}$.

Suppose this time, we want to find the LCM of polynomials below as follows.

$\boldsymbol{P_0 = (x + 1)(x + y)(y + 1)}$

$\boldsymbol{P_1 = (x + 1)(x + y)^2(y + 1)}$

$\boldsymbol{P_2 = (x + 1)(x + 2y)(y^2 + y + 4)}$

$\boldsymbol{P_3 = (x + 1)^2(x + y)(x + 3y)}$

Then again, all the polynomials are factorized already, so we are ready to find the LCM.

Thus next, we just take the product of ***all the prime factors*** all the polynomials have.

Then, we get $\boldsymbol{(x + 1)(x + y)(y + 1)(x + 2y)(y^2 + y + 4)(x + 3y)}$.

That's because the LCM has all the prime factors all the polynomials have so that each and every polynomial can divide the LCM.

Next, apply to each prime factor above ***the largest exponent*** of all used for the prime factor. Then, 2 is the largest for each of $(x + 1)$ and $(x + y)$, and 1 is the largest used for each of $(y + 1)$, $(x + 2y)$, $(y^2 + y + 4)$, and $(x + 3y)$.

Therefore, the LCM is $(x + 1)^2(x + y)^2(y + 1)(x + 2y)(y^2 + y + 4)(x + 3y)$.

Let's next, find the LCM of polynomials as follows.

$$P_0 = 144(x + 1)^2(x + y)^2(y + 1) = 2^4 3^2(x + 1)^2(x + y)^2(y + 1)$$

$$P_1 = -216(x + 1)(x + y)^3 = -2^3 3^3(x + 1)(x + y)^3$$

$$P_2 = 360(x + 1)(x + y)^2(y + 3)^2 = 2^3 3^2 5(x + 1)(x + y)^2(y + 3)^2$$

$$P_3 = 504(x + 1)(x + y)^2 = 2^3 3^2 7(x + 1)(x + y)^2$$

$$P_4 = -792(x + 1)^2(x + y)^3(2y + 5x) = -2^3 3^2 11(x + 1)^2(x + y)^3(2y + 5x).$$

Then, all the polynomials are fully factorized already, so we are ready to find the LCM.

So next, we want to take the product of ***all the prime factors*** all the polynomials have.

Then, we can simply get $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11(x + 1)(x + y)(y + 1)(y + 3)(2y + 5x)$.

Next, we want to apply to each prime factor above ***the largest exponent*** of all used for the prime factor.

Then, 4 is the largest used for the prime factor 2, 3 is the one used for the factor 3, 1 is the one used for the factor 5, 1 is used for 7, 1 is used for 11, 2 is used for $(x + 1)$, 3 is used for $(x + y)$, 1 is used for $(y + 1)$, 2 is used for $(y + 3)$, and 1 is used for $(2y + 5x)$.

Therefore, the LCM is $2^4 3^3 5 \cdot 7 \cdot 11(x + 1)^2(x + y)^3(y + 1)(y + 3)^2(2y + 5x)$.

We can of course, put the LCM above this way, too:

$166320(x + 1)^2(x + y)^3(y + 1)(y + 3)^2(2y + 5x)$, since $166320 = 2^4 3^3 5 \cdot 7 \cdot 11$.

Now, let's for another example, find the LCM of polynomials as follows.

$$M_0 = 144a^3b^2xy(x+1)^2(x+y)^2(y+1) = 2^4 3^2 a^3b^2xy(x+1)^2(x+y)^2(y+1)$$

$$M_1 = -216a^2by^2(x+1)(x+y)^3 = -2^3 3^3 a^2by^2(x+1)(x+y)^3$$

$$M_2 = 360a^2bxy(x+1)(x+y)^2(y+3)^2 = 2^3 3^2 5a^2bxy(x+1)(x+y)^2(y+3)^2$$

$$M_3 = 648a^3b^2cy^3(x+1)(x+y)^2 = 2^3 3^4 a^3b^2cy^3(x+1)(x+y)^2$$

$$M_4 = -72a^4bc^2y(x+1)^2(x+y)^3 = -2^3 3^2 a^4bc^2y(x+1)^2(x+y)^3.$$

Then, all the polynomials are fully factorized already, so we are ready to find the LCM.

So next, we just take the product of ***all the prime factors*** all the polynomials have.

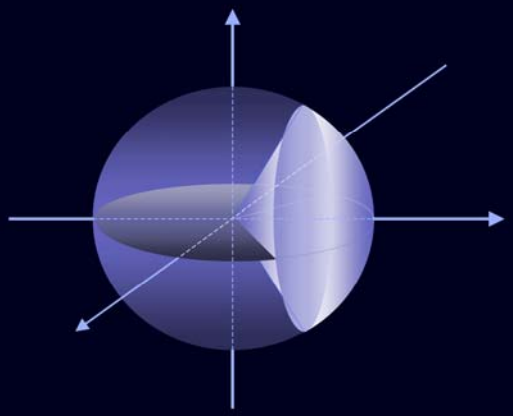
Then, we can simply get $2\cdot 3\cdot 5\cdot 7\cdot abcxy(x+1)(x+y)(y+1)(y+3)$.

Next, we want to apply to each prime factor ***the largest exponent*** of all used for the prime factor.

Then, 4 is the largest used for the prime factor 2, 4 is the one used for the factor 3, 1 is the one used for the factor 5, 4 is used for the factor a, 2 is used for b, 2 is used for c, 1 is used for x, 3 is used for y, 2 is used for $(x+1)$, 3 is used for $(x+y)$, 1 is used for $(y+1)$, and 2 is used for $(y+3)$.

Therefore, the LCM is $2^4 3^4 5a^4b^2c^2xy^3(x+1)^2(x+y)^3(y+1)(y+3)^2$.

We can put the LCM above this way, too: $6480a^4b^2c^2xy^3(x+1)^2(x+y)^3(y+1)(y+3)^2$, since $6480 = 2^4 3^4 5$.



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