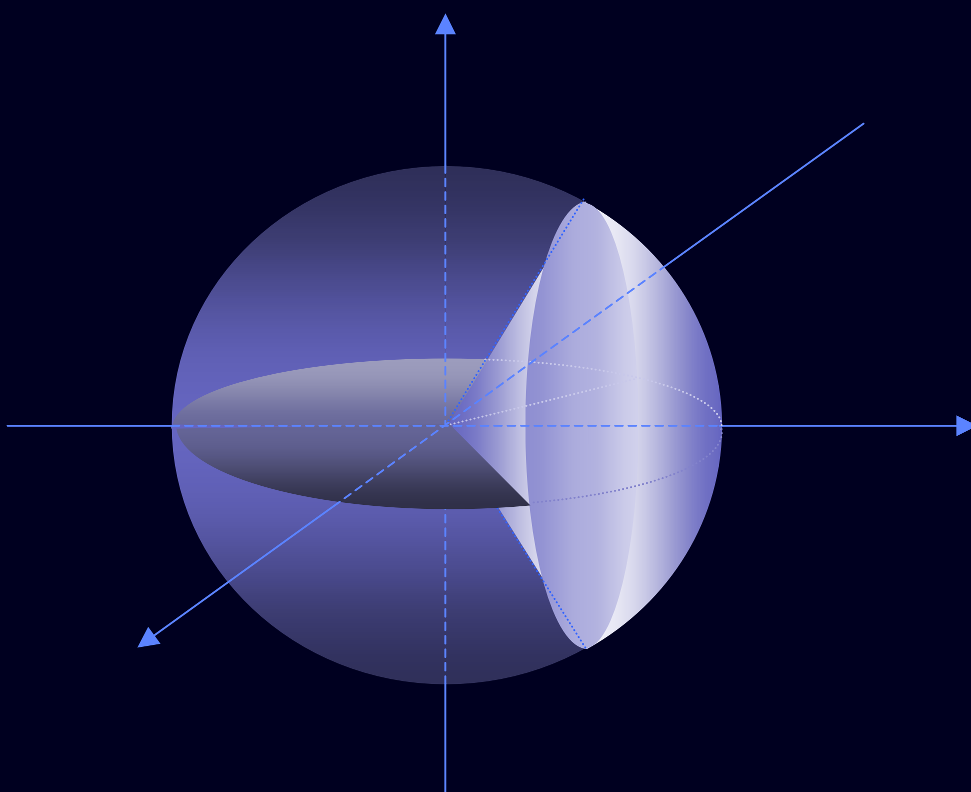
거듭제곱과 로그

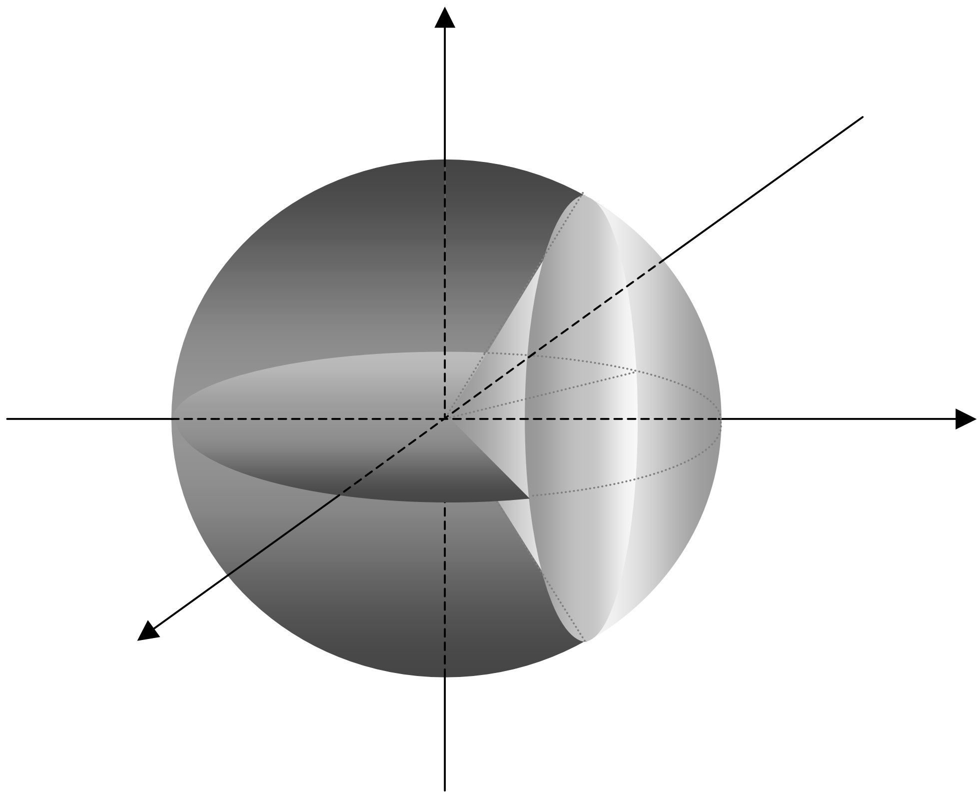
Powers and Logarithms



김성렬

Seong R. Kim

Powers and Logarithms



Seong R. KIM

Dear students,

Students need the best teacher, so you need examples, because examples are the best teacher. This is thus, a book of examples. All the examples here are fully worked, and explain **how** the basic but essential tools in math are made, together with **what** they are, **how** they work, and **how** to work with them. Such tools include numbers, formulas, identities, equations, laws, etc.

Doing math, we work with ideas and run ideas, because every thing in math is an idea. What idea though?

It's a concept. So we need to understand it if we want to use it. A number is an idea, for instance, and the same is true for a line or circle, too. And putting ideas together, we can build another, which becomes the base or an element of another, and each is connected. And that's the way your math grows.

So you get to build a circuit, and sometimes, need to fill the gap or repair the circuit so that you can get the sense of it. Your calculation thus, runs properly, and gets to the solution.

Getting the concept doing good examples, and gaining the fluency doing right examples, you can build the circuit strong.

This book is however, nothing but a bunch of examples until you get it powered. How then to get it powered, and make it run and work for you?

Read it closely at your pace, and then, do each example yourself. It is quite important to note that you do it in **your** own writing. Just watching someone doing it, you just only feel that you can do it, too, but can hardly do it when you need it. You do it, you get it. And you've got to do it over with different examples. It's a cliché, of course, but is always true that knowing is one thing and doing is another.

I've been helping students get conceptual understanding and fluency so that they can grow, take care of, and run their own math. The area covers the basics in algebra and geometry, and includes equations for unknowns and those for curves, functions and their graphs, and other math tools, which are the basic elements in calculus, which's been the core of my interest from my early age in high school.

Of my students, some are quite poor in math, and thus, are afraid of or even hate math, some require special education because of exceptional intelligence, some are smart enough, some are naïve and diligent, and many are clever but lazy. All the students are badly after though, one thing in common.

A Strong Foundation in Algebra

It is of course, the prime objective of my work, and I'm always happy and eager to help them achieve it. I can only help though. Your math grows as much as you grow and take care of it yourself. So does mine. What math then do students most often do or use in high schools or colleges?

Algebra and geometry. What algebra then?

Elementary algebra, of course. Doing the algebra, you should be able to not only work with, but manipulate, that is, change, alter, and convert math expressions the way you want, so for instance, you can break apart and factor polynomials in many kinds. Also, you can handle properly ratios and rates, trigonometry, factors or divisors, constants and variables, powers and logarithms or exponents, equations and functions, along with their graphs, equalities and inequalities, identities and formulas, laws and rules, etc.

And if you want to handle them properly, you want to know their natures and how they mingle with each other.

So studying those stated above, you want to know **what** they are, **how** they work, and **how** to work with them or **what** to do with them. What then about the geometry?

Basically, the geometry is about shapes, and has much to do with positions and angles in algebra.   Why algebra?

You often need to compute lengths, areas, or volumes in many situations. The shapes begin with points, lines, triangles, and circles, and move on to rectangles, squares, parallelograms or rhombuses, trapezoids, tetragons, other polygons as pentagons, polyhedrons as hexahedrons, etc.

Doing the geometry, too, we need to do the algebra stated above such as trigonometry. So it is analytic geometry, often called coordinate geometry, too. And doing it, we can specify positions using coordinates. So in that geometry, basically, we work with graphs. Putting a math idea in a graph, we can not only effectively think about it, but actually see it, too, and therefore, can not just easily, but efficiently work with it, too. What idea then is it?

The idea begins with a point, a line, a parabola, a circle, an ellipse, and a hyperbola, called a conic section or basic curve, and then, moves on to other curves, planes, surfaces, volumes, and other objects in various dimensional spaces, together with vectors. And using an angle, we can specify and work with an amount of turning or change in direction.

So learning, using, or applying those ideas or tools in math, we get to solve problems. And this book can help. It can help you build and run your own math engine in your brain, and thus, can help achieve your solid math skill.

It is however, not a magic book giving you a math skill of high caliber overnight. And it’s just a book of examples, and can have many mistakes. I’m a human, too. There’s no mojo making a math genius, and math is science, and is full of facts and ideas. And it is after all, not me, not your teacher, and not any book but you who actually make you put together some of those facts and ideas, and understand it.

Putting facts and ideas together, understanding it, and taking care of what you have learned doing examples, you grow your math. And this book is ready to help and can help you help yourself.

Solving a problem, we put it many different ways. For instance, while expanding or reducing it, or modifying or converting it, we keep searching for the solution, keep approaching the solution, and eventually, can get there.

It is often the case a problem itself is the solution. We can put a problem in many different ways, and eventually, can end up with the solution. How come then is the solution no other than the problem?

For instance, the solution to 3232 ÷ 101 is 32. And we can put it this way:

$3232 \div 101 = \frac{3232}{101} = \frac{32 \text{ x } 101}{101} = \frac{32}{1} = 32 \Rightarrow 3232 \div 101 = 32.$

And we can get this, too: 32 = 3232 ÷ 101. How?

$32 = \frac{32}{1} = \frac{32 \text{ x } 101}{101} = \frac{3232}{101} =$ 3232/101 = 3232 ÷ 101.

Too easy?

For another instance, the solution to $\boldsymbol{ax^2 + bx + c = 0}$ is

$\boldsymbol{x} = \frac{-\boldsymbol{b} \pm \sqrt{\boldsymbol{b}^2 - 4\boldsymbol{ac}}}{2\boldsymbol{a}}$, which is called the quadratic formula.

How come then is the solution no other than the problem?

We can put it this way:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow 2ax = -b \pm \sqrt{b^2 - 4ac}$$

$$\Rightarrow 2ax + b = \pm\sqrt{b^2 - 4ac} \Rightarrow (2ax + b)^2 = b^2 - 4ac$$

$$\Rightarrow 4a^2x^2 + 4abx + b^2 = b^2 - 4ac \Rightarrow 4a^2x^2 + 4abx = -4ac$$

$$\Rightarrow ax^2 + bx = -c \Rightarrow ax^2 + bx + c = 0.$$

And we can get this, too:

$$ax^2 + bx + c = 0 \quad \Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{How?}$$

$$ax^2 + bx + c = a(x^2 + \frac{b}{a}x) + c = a(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}) + c$$

$$= a(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}) - \frac{b^2}{4a} + c = a(x + \frac{b}{2a})^2 - \frac{b^2 - 4ac}{4a} = 0$$

$$\Rightarrow a(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a} \Rightarrow (x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\Rightarrow x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \Rightarrow x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

So when solving a problem, we expand or reduce it, modify or convert it, and keep searching for and approaching the solution. Then eventually, we can get there. What then do we do when doing the set of processes above?

We do algebra. You need a strong foundation in algebra. So if a problem is well defined, that is, if it makes sense, we should be able to get it solved the way as follows.

A problem $\Rightarrow$ **…** $\Rightarrow$ **…** $\Rightarrow$ **the solution**,

and thus, **the problem** $\Rightarrow$ **the solution**.

So solving a problem, we put it many different ways so that we can get to the solution. And that's the way math runs.

Many more examples are available for free, and you can get them downloaded at **www.runmath.com**. So visit the website, get them downloaded, and do the examples. The website is getting updated weekly or monthly.

May your math run very well.

Seong R. Kim
B.S. Math. Michigan Tech. Univ.   M.S. Math. Rensselaer Polytechnic Institute

Contents

0. What is a power?

It is a math tool, and in fact, is a power tool, has quite a bit of power, and gets powered by numbers, together with your reasoning.
Being a power user of such a tool, you can grab and handle readily many values, excessively big as 5233476330273605372135 11521 or unreasonably small as 0.0000000000000022876792454961.
Also, taking care of ordinary values as 64, 243, 625, 1024, and such, you can effectively handle them, too, using the tool called a power, of course. What then, is the tool?

When adding together many of the same numbers, we don't actually do additions so many times, do we? We just multiply, of course, one of those many by the number of those, and get the product, which is the sum.

So for instance, adding together five **3**s, we don't normally do **3 + 3 + 3 + 3 + 3**, but just do **3 · 5**, that is, **3 × 5**, and get **15**, which is the sum, as well as the product of 3 and 5.

Next, what if we want to multiply 1 by 3 many times, and specify the product?

If we just want to multiply 1 by 3 twice, we just do 1·3·3, and get 9, which is the product.
What if however, we want to multiply 1 by 3 five hundred times, and specify the product?

Let alone specifying such a product, we are probably not going to actually do multiplications so many times, are we?

Suppose however, we actually did do multiplications so many times, and somehow did find the product. Then, we would get a big number, which is a long number.

The number is the product of five hundred **3**s, and has a huge number of digits. In fact, it has a whopping array of 239 digits. So it's an unreasonably long number with a huge value. What if however, we've got to specify such a huge value anyway?

We don't actually specify such a long number digit by digit. We can quickly put such an enormous value in a short and simple manner. Using a special notation, we can effectively specify a humongous value or a number with a gigantic value.

A number indicates a value.
And we can put a value in various forms. We have a variety of ways or methods, where we can indicate the same value. Depending on the purpose, we can show a value in a particular form. For instance, we can put 8 in such many ways as follows.

1×8, $8{\cdot}1$, $4{\cdot}2$, $4{\cdot}(3-1)$, $1+2+3+2$, $3+1+4$, $1+3-3+7$, $9-1$, $\frac{16}{2}$, $\frac{3\cdot 8}{3}$, etc.

Quite often, we need to show values with unduly large or extremely small magnitudes, but want to express them in a succinct manner. If for instance, we have to put in writing values like 100000000000000000 and 0.000000000001, they not only take up too much space, but are hard to read, too. What then do we do?

In such a case, we can use a tool, called a *power*. And the tool takes two numbers, one is called an *exponent*, and the other is called a *base*. So we get the tool powered by two numbers called an *exponent* and a *base*.

Suppose for instance, a value is a product we get multiplying 1 by 7 a particular number of times. Then, we can specify the value using a power. Then, the particular number is called the *exponent*, and 7 is called the *base*.

So for instance, multiplying 1 by 9 twenty eight times, and expressing the product using a power, we use 9 as the base, and use 28 as the exponent.

So expressing $1{\cdot}9$ which is 523347633027360537213511521, we can use a power, where the base is 9, and the exponent is 28. How then can we specify a power?

Specifying a power, we put it in terms of two numbers, one is called a base, and the other is called an exponent.

A power is therefore, a structured number composed of two parts, one is a base, and the other is an exponent.

And in this book, such a structured number is said to be in *power notation*.
Using power notation, we put an exponent at the upper right hand corner of a base.
In other words, we use an exponent as a superscript.

So for instance, using power notation, we can put 8 in 2^3, and put 1,000,000,000 in 10^9. And 2^3 is called a power, 2 is called the base, and 3 is called the exponent. The same is true for the power 10^9, too. So 10 is the base of the power 10^9, and 9 is the exponent.

Putting thus, a value or number in power notation, we make a power, which is a number made of a base and an exponent.

Also, calling a power with a particular base, we call it a power of the particular base.

So for instance, 10^9 can be called a power of 10, and specifically, it is called the *ninth power* of 10. And also, we can call it differently, too, and will cover shortly how to do so.

And for another instance, we can simply put 523347633027360537213511521 in 9^{28}, which is a power of 9, and specifically, is called the twenty eighth power of 9. And thus, we can put such a huge value in a succinct manner.

What then about small numbers as 0.0000001?

In such a case, we can use a *negative* number as the *exponent*.

A value can be a reciprocal of a large number, so such a value is very small, and we can get such a number dividing 1 by a particular number many times. For instance, it can be 1/23/23/23/23/23/23, which equals $\frac{1}{148035889}$, which we get dividing 1 by 23 six times.

And expressing a number we get doing divisions by the same number repeatedly, we can use power notation, too. Then, the base is the divisor, which is 23 in the case above, but the exponent is a negative integer, because a division is opposite of a multiplication. So –6 is the exponent, because the division by 23 happens 6 times.

Using thus, power notation, we put $\frac{1}{148035889}$ in 23^{-6}, which is a power of 23, and specifically, is called the *negative sixth* power of 23.

And we put 0.0000001 in 10^{-7}, because 0.0000001 = 1/10/10/10/10/10/10/10. And 10^{-7} is a power of 10, and is called the negative seventh power of 10.

And multiplying for instance, 1 by $\boldsymbol{b}$ twice, and specifying the product, we can do this: $\mathbf{1}\cdot\boldsymbol{b}\cdot\boldsymbol{b}$, and then, can put the product this way: $\boldsymbol{bb}$. Normally though, we put it this way: $\boldsymbol{b}^{\mathbf{2}}$. And dividing 1 by $\boldsymbol{b}$ twice, we put the result this way: $\boldsymbol{b}^{\mathbf{-2}}$. What if however, we want to indicate a value we get multiplying (or dividing) 1 by $\boldsymbol{b}$ an arbitrary number of times?

We can indicate the value by <u>$\boldsymbol{b}^{\boldsymbol{n}}$ where $\boldsymbol{n}$ is an integer</u>. So the superscript $\boldsymbol{n}$ indicates the number of the multiplications made. And the same is true for divisions, too, of course.

So in the power $\boldsymbol{b}^{\boldsymbol{n}}$ where $\boldsymbol{n}$ is an integer, $\boldsymbol{n}$ indicates the number of times we multiply or divide 1 by $\boldsymbol{b}$. For instance, $\mathbf{1}\cdot\boldsymbol{b}\cdot\boldsymbol{b}\cdot\boldsymbol{b}\cdot\boldsymbol{b}\cdot\boldsymbol{b} = \mathbf{1}\boldsymbol{bbbbb} = \boldsymbol{b}^{\mathbf{5}}$, and $\mathbf{1}/\boldsymbol{b}/\boldsymbol{b}/\boldsymbol{b}/\boldsymbol{b} = \boldsymbol{b}^{\mathbf{-4}}$. And of course, in cases of divisions, the base $\boldsymbol{b}$ is not 0.

What do we get though, if we put 0 into $\boldsymbol{n}$ in the power $\boldsymbol{b}^{\boldsymbol{n}}$?

Then, we do neither multiply nor divide 1 by the base $\boldsymbol{b}$. That is, we do nothing to 1. In other words, we leave 1 alone. So we get $\boldsymbol{b}^{\mathbf{0}} = \mathbf{1}$, and for instance, $2^0 = 3^0 = 4^0 = 1$.

What if $\boldsymbol{n}$ is 1 in $\boldsymbol{b}^{\boldsymbol{n}}$?

Then, we multiply 1 by $\boldsymbol{b}$ once, and therefore, we get a $\boldsymbol{b}$. So we get $\boldsymbol{b}^{\mathbf{1}} = \boldsymbol{b}$.

Thus, we can put a value in a form of a power, which takes a form of $\boldsymbol{b}^{\boldsymbol{n}}$, where $\boldsymbol{b}$ is called a base, and $\boldsymbol{n}$ is called an exponent. How can we read a power though?

We can read $\boldsymbol{b}^{\boldsymbol{n}}$ as '$\boldsymbol{b}$ raised to the $\boldsymbol{n}^{\mathbf{th}}$ power' or 'the $\boldsymbol{n}^{\mathbf{th}}$ power of $\boldsymbol{b}$'. And briefly, we can read it as '$\boldsymbol{b}$ to the $\boldsymbol{n}^{\mathbf{th}}$'. Even more briefly, we can just read it as '$\boldsymbol{b}$ to the $\boldsymbol{n}$'.

For instance, we can read $\boldsymbol{b}^{\mathbf{7}}$ as '$\boldsymbol{b}$ raised to the seventh power', 'the seventh power of $\boldsymbol{b}$', or '$\boldsymbol{b}$ to the seventh'. And more briefly, we can read it as '$\boldsymbol{b}$ to the seven', too.

Usually though, we read $\boldsymbol{b}^2$ differently, and often read it as '***b*** squared', instead of reading it as '***b*** raised to the second power', 'the second power of ***b***', or '***b*** to the second'.

And the same is true for $\boldsymbol{b}^3$, too. So $\boldsymbol{b}^3$ is often read as '***b*** cubed'. And of course, we can read it as '***b*** to the third', too.   How then about $\boldsymbol{b}^{-3}$?

It can be read as '***b*** to the negative (minus) third', 'the negative third power of ***b***', or just can be read briefly as '***b*** to the negative three'.
And in fact, we can use any real number as an exponent if the base is positive.

So for instance, we can read

$\boldsymbol{b}^{0.3}$ as '***b*** to the 0.3', $\boldsymbol{b}^{\pi}$ as '***b*** to the pi', and $\boldsymbol{b}^{\sqrt{2}}$ as '***b*** to the square root of 2'.

$\mathbf{0.3}^3$ as 0.3 cubed, and of course, we can read it as 0.3 to the third, too.

$\boldsymbol{b}^{\frac{1}{n}}$ as '***b*** to the 1 over ***n***', $\boldsymbol{b}^{-\frac{y}{x}}$ as '***b*** to the negative ***y*** over ***x***', and $\boldsymbol{b}^{\frac{3}{\pi}}$ as '***b*** to the 3 over pi'.

$\boldsymbol{b}^{-\frac{2}{3}}$ as '***b*** to the negative 2 over 3', and also, as '***b*** to the negative two thirds'.

$\boldsymbol{b}^{f(2)}$ as '***b*** to the ***f*(1)**', and $\boldsymbol{b}^{g(x)}$ as '***b*** to the ***g*(*x*)**'.

$\boldsymbol{b}^{x^2}$ as '***b*** to the ***x*** squared' or '***b*** to the ***x*** to the second', and $\boldsymbol{b}^{x^{\frac{1}{3}}}$ as '***b*** to the ***x*** to the third'.

1. Exponential Identities 1

Doing math or taking courses in math, we can't do much without doing algebra. And doing algebra, we get to work with many tools as formulas, identities, theorems, etc. Or rather, whatever math we do, we have to do algebra, and always need to use such tools.

And also, doing algebra, we often get to work with powers. Or rather, we always have to work with powers. And doing algebra with powers, we do exponential algebra, and get to work with exponents and bases, of which the powers are made.

And doing exponential algebra, we can hardly do much without using some essential tools, called exponential identities, which are therefore, crucial. And thus, we want to know them very well. What do we mean by though, knowing them very well?

Knowing them very well, we know how they work, and more importantly, know how to work with them. So let's see now how they work and how we can work with them.

To begin with, keep in mind that all the letters used in the math expressions here take real numbers unless specified otherwise. Also, note in particular that we do not rule out cases where the letters can be 0 or negative unless specified otherwise.

Now, suppose that both $\boldsymbol{a}$ and $\boldsymbol{b \neq 0}$, and that both $\boldsymbol{m}$ and $\boldsymbol{n}$ are integers.
Then, the tools below are called exponential identities, and *always* work.

0. $a^m a^n = a^{m+n}$

1. $a^m / a^n = \dfrac{a^m}{a^n} = a^{m-n}$

2. $(a^m)^n = a^{mn}$

3. $(ab)^n = a^n b^n$

4. $(a/b)^n = \left(\dfrac{a}{b}\right)^n = a^n / b^n = \dfrac{a^n}{b^n}$

Let's see now, what's going on in each of the identities listed above.

0. $a^m a^n = a^{m+n}$, where $a \neq 0$, and both m and n are integers.

To begin with, we know: multiplying 1 by a, $(m + n)$ times, we get a power a^{m+n}.

What then, do we get if multiplying a power a^m by another power a^n?

We get a new power a^{m+n}. It's because a^m is the product of m of as, and a^n is the product of n of as, so $a^m a^n$ is the product of $(m + n)$ of as.

Taking thus, the product of powers sharing the same base, we get a new power, where the base remains the same, and the exponent is the sum of the exponents. For instance:

$3^4 3^5 = (3{\cdot}3{\cdot}3{\cdot}3)(3{\cdot}3{\cdot}3{\cdot}3{\cdot}3) = 3{\cdot}3{\cdot}3{\cdot}3{\cdot}3{\cdot}3{\cdot}3{\cdot}3{\cdot}3 = 3^9$, so we get $3^4 3^5 = 3^{4+5} = 3^9$.

$3^3 3^4 3^2 = (3{\cdot}3{\cdot}3)(3{\cdot}3{\cdot}3{\cdot}3)(3{\cdot}3) = 3{\cdot}3{\cdot}3{\cdot}3{\cdot}3{\cdot}3{\cdot}3{\cdot}3{\cdot}3 = 3^9$, so we get $3^3 3^4 3^2 = 3^{3+4+2} = 3^9$.

$(-3)^2(-3)^3 = \{(-3){\cdot}(-3)\}\{(-3){\cdot}(-3){\cdot}(-3)\} = (-3){\cdot}(-3){\cdot}(-3){\cdot}(-3){\cdot}(-3) = (-3)^5$, so we get

$(-3)^2(-3)^3 = (-3)^{2+3} = (-3)^5$, which equals -3^5, of course.

What if m and n are negative?

Then, we have $m < 0$, and $n < 0$, so we get $-(m + n) > 0$. That is, $-(m + n)$ is a positive integer. So dividing 1 by a, $-(m + n)$ times, we get $\frac{1}{a^{-(m+n)}}$.

And the denominator is a power, which is the product of $-(m + n)$ of as.

And thus, putting $\frac{1}{a^{-(m+n)}}$ in the power notation, we get a^{m+n}, where $m + n < 0$.

So for instance, dividing 1 by a, 3 times, we get $1/a/a/a = 1/(aaa) = \frac{1}{aaa} = \frac{1}{a^3} = a^{-3}$.

What then do we get if multiplying a power a^m by another power a^n where m and n are negative?

We have $-m > 0$, and $-n > 0$. So we can set $a^m = \frac{1}{a^{-m}}$, and $a^n = \frac{1}{a^{-n}}$, where a^{-m} is the product of $-m$ of as, and a^{-n} is the product of $-n$ of as.

Thus, we get $a^m a^n = \frac{1}{a^{-m}} \cdot \frac{1}{a^{-n}} = \frac{1}{a^{-(m+n)}} = a^{m+n}$.

For instance, we can get $a^{-2}a^{-3} = \frac{1}{a^2} \cdot \frac{1}{a^3} = \frac{1}{a^{2+3}} = a^{-(2+3)} = a^{-5}$.

And assuming $m = n = 0$, we get $a^m a^n = a^0 a^0 = 1 \cdot 1 = 1$. And we have $a^{0+0} = a^0 = 1$.

So we get $a^m a^n = a^{m+n}$ where $a \neq 0$, and both m and n are integers. What if $a = 0$?

Then, we need to have $m > 0$, and $n > 0$.

For instance, assuming $m = -1$, we get $a^m = 0^{-1} = \frac{1}{0}$, which is not possible.

What if $m = 0$, though?

Then, we get $a^m = 0^0 = 1$. That's because multiplying 1 by 0, no times, we get 1.

Thus, putting threads together, we get $a^m a^n = a^{m+n}$, where $a \neq 0$, and both m and n are integers.

1. $a^m / a^n = \frac{a^m}{a^n} = a^{m-n}$, where $a \neq 0$, and both m and n are integers.

We are going to look at two cases. In one, a power a^m gets divided by a, n times. And in the other, the power a^m gets divided by another power a^n.

So let's first, divide the power a^m by a, n times.

Then, assuming first, $m \geq n$, and dividing the power a^m by a, n times, we get a new power a^{m-n}, where $m - n \geq 0$. How?

In the power a^m, n of as get canceled due to n divisions by a. So for instance, doing 3 divisions by a to a power a^5, we get $\frac{a^5}{a^3} = \frac{a \cdot a \cdot a \cdot a \cdot a}{a \cdot a \cdot a} = a \cdot a = a^2 = a^{5-3}$

• Suppose next, $m < n$.

Then, though dividing the power a^m by a, n times, we still get the power $a^{m\text{-}n}$, in which however, $m - n < 0$. How?

First, if $m < n$, we can say that a^n has ($n - m$) more as than a^m has.

For instance, since 2 < 5, a^5 has ($5 - 2 = 3$) more as than a^2 has.

So if $m < n$, and the power a^m gets divided by a, n times, what happens first is that all of the as in the power a^m get canceled due to the first m divisions by a, and we get 1, and then, 1 gets divided by a, ($n - m$) more times.

Thus, we get $\frac{1}{a^{n-m}}$. So we get $a^{m\text{-}n}$.

So for instance, if a power a^2 gets divided by a, 5 times, what happens first is that all of the two as in the power a^2 get canceled due to the first two divisions by a, and we get 1, and then, 1 gets divided by a, (5 – 2 = 3) more times.

Thus, we get $\frac{1}{a^{5-2}}$. So we get $a^{5\text{-}2}$.

Thus, dividing 1 by the same number, many times, we get a negative exponent, and take the same number as the base in the power we get.

So assuming $m < n$, and dividing a^m by a, n times, we get $a^{m\text{-}n}$, too. Thus, for instance, dividing a^3 by a, five times, we get $\frac{a^3}{a^5} = \frac{a \cdot a \cdot a}{a \cdot a \cdot a \cdot a \cdot a} = \frac{1}{a \cdot a} = a^{-2} = a^{3-5}$.

So either way, dividing a^m by a, n times, we get the power, $a^{m\text{-}n}$.

• Suppose this time, we divide the power a^m by another power a^n. Then, we get $\frac{a^m}{a^n}$.

So assuming first, $\boldsymbol{m \geq n}$, we get a new power, $\boldsymbol{a^{m-n}}$. How?

Initially, the numerator $\boldsymbol{a^m}$ has $\boldsymbol{m}$ of $\boldsymbol{a}$s, and the denominator $\boldsymbol{a^n}$ has $\boldsymbol{n}$ of $\boldsymbol{a}$s.

During the course of the division of $\boldsymbol{a^m}$ by $\boldsymbol{a^n}$, $\boldsymbol{n}$ of $\boldsymbol{a}$s each in both the numerator and denominator get canceled, so the resultant numerator is left with $(\boldsymbol{m - n})$ of $\boldsymbol{a}$s, and the resultant denominator is 1.

Thus, we get $\frac{\boldsymbol{a^{m-n}}}{\boldsymbol{1}}$, which is $\boldsymbol{a^{m-n}}$.

So for instance, we get: $\frac{3^5}{3^3} = \frac{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3}{3 \cdot 3 \cdot 3} = \frac{3 \cdot 3}{1} = \mathbf{3^2} \Rightarrow \frac{3^5}{3^3} = \mathbf{3^{5-3}} = \mathbf{3^2}$.

And we get: $\frac{(\frac{1}{3})^5}{(\frac{1}{3})^3} = \frac{\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3}}{\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3}} = \frac{\frac{1}{3} \cdot \frac{1}{3}}{\mathbf{1}} = (\tfrac{1}{3})^2 \Rightarrow \frac{(\frac{1}{3})^5}{(\frac{1}{3})^3} = (\tfrac{1}{3})^{5-3} = (\tfrac{1}{3})^2$.

And assuming next, $\boldsymbol{m < n}$, we still get the power indicated by $\boldsymbol{a^{m-n}}$.
This time though, we get $\boldsymbol{m - n < 0}$. How?

Initially, the numerator $\boldsymbol{a^m}$ has $\boldsymbol{m}$ of $\boldsymbol{a}$s, and the denominator $\boldsymbol{a^n}$ has $\boldsymbol{n}$ of $\boldsymbol{a}$s.

During the course of the division, $\boldsymbol{m}$ of $\boldsymbol{a}$s each in both the numerator and denominator get canceled, so the numerator ends up with 1, and the denominator is left with $(\boldsymbol{n - m})$ of $\boldsymbol{a}$s.

Thus, we get $\frac{\mathbf{1}}{\boldsymbol{a^{n-m}}}$, which is the result we get if 1 gets divided by $\boldsymbol{a}$, $(\boldsymbol{n - m})$ times.

And we know $\frac{\mathbf{1}}{\boldsymbol{a^{n-m}}} = \boldsymbol{a^{-(n-m)}} = \boldsymbol{a^{m-n}}$.

Thus either way, dividing a power $\boldsymbol{a^m}$ by another power $\boldsymbol{a^n}$, we get a new power $\boldsymbol{a^{m-n}}$.

For instance, we get: $\frac{3^3}{3^5} = \frac{3 \cdot 3 \cdot 3}{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3} = \frac{1}{3 \cdot 3} = \frac{1}{3^2} = \mathbf{3^{-2}} \Rightarrow \frac{3^3}{3^5} = \mathbf{3^{3-5}} = \mathbf{3^{-2}} = \frac{1}{3^2}$.

And we get $\frac{(\frac{1}{3})^3}{(\frac{1}{3})^5} = \frac{\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3}}{\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3}} = \frac{\mathbf{1}}{\frac{1}{3} \cdot \frac{1}{3}} = \frac{\mathbf{1}}{(\frac{1}{3})^2} = (\tfrac{1}{3})^{-2} \Rightarrow \frac{(\frac{1}{3})^3}{(\frac{1}{3})^5} = (\tfrac{1}{3})^{3-5} = (\tfrac{1}{3})^{-2}$.

So doing a division with two powers with the same base $\boldsymbol{a}$, we get a new power with the same base $\boldsymbol{a}$, and the magnitude of the new exponent is the difference between the two exponents of the two powers.

Thus, we get $a^m / a^n = \dfrac{a^m}{a^n} = a^{m-n}$, where $a \neq 0$, and both m and n are integers.

2. $(a^m)^n = a^{mn}$, where $a \neq 0$, and both m and n are integers.

Multiplying 1 by a, m times, we get a power a^m.

So multiplying 1 by the power a^m, n times, we get a new power $(a^m)^n$.

And let's see now what each multiplication can produce.

First, multiplying 1 by the power a^m, we just get: a^m, which is the product of m of as.

Next, multiplying a^m by a^m, that is, multiplying 1 by a^m, 2 times, we get $a^m \cdot a^m = (a^m)^2$.

We know a^m is the product of m of as. So $a^m \cdot a^m$ is the product of $2m$ of as.

Thus, $(a^m)^2$ is the product of $2m$ of as, too. So we get $(a^m)^2 = a^{2m}$.

Next, multiplying a^{2m} by a^m, that is, multiplying 1 by a^m, 3 times, we get $a^m \cdot a^m \cdot a^m = (a^m)^3$.

Also, we know a^m is the product of m of as. So $a^m \cdot a^m \cdot a^m$ is the product of $3m$ of as.

Thus, $(a^m)^3$ is the product of $3m$ of as, too. So we get $(a^m)^3 = a^{3m}$.

Next, multiplying a^{2m} by a^{2m}, that is, multiplying 1 by a^m, 4 times, we get $a^{2m} \cdot a^{2m} = (a^{2m})^2$.

We know a^{2m} is the product of $2m$ of as. So $(a^{2m} \cdot a^{2m})$ is the product of $4m$ of as.

Thus, $(a^{2m})^2$ is the product of $4m$ of as, too. So we get $(a^{2m})^2 = a^{4m}$.

And we have $a^m a^n = a^m \cdot a^n = a^{m+n}$. So we get $a^{4m} = a^m \cdot a^m \cdot a^m \cdot a^m = (a^m)^4$.

Thus, we get $a^{4m} = (a^m)^4$. Now, what then do we get multiplying 1 by a^m, n times?

We get $(a^m)^n = a^{mn}$. So for instance, $(5^3)^2 = (5{\cdot}5{\cdot}5){\cdot}(5{\cdot}5{\cdot}5) = 5{\cdot}5{\cdot}5{\cdot}5{\cdot}5{\cdot}5 = 5^{3\cdot 2} = 5^6$.

Also, dividing 1 by a twice, we get a^{-2}, which is $\frac{1}{a^2}$, which is $1/a/a$, too.

So dividing 1 by a^3 twice, we can get $(a^3)^{-2} = 1/(a^3)/(a^3) = 1/(a^3 \cdot a^3) = 1/a^6 = a^{-6} = a^{3(-2)}$.

And multiplying 1 by a^{-3} twice, we can put it this way: $(a^{-3})^2$, and we get $(a^{-3})^2 = (1/a^3)(1/a^3) = 1/(a^3 \cdot a^3) = 1/a^6 = a^{-6} = a^{3(-2)}$, too. So we get $(a^3)^{-2} = (a^{-3})^2 = a^{-6}$.

And dividing 1 by a^{-3} twice, we get $(a^{-3})^{-2} = 1/(a^{-3})/(a^{-3}) = 1/(a^{-3} \cdot a^{-3}) = 1/a^{-6} = a^6$.
And we know $a^6 = a^{(-3)(-2)}$.

- So we get $(a^m)^n = a^{mn}$, where $a \neq 0$, and m and n are integers.

So if a power gets raised to another power, keep the base, and take the product of the exponents.

3. $(ab)^n = a^n b^n$, where both a and $b \neq 0$, and both m and n are integers.

Multiplying 1 by a, n times, we get a power a^n.

So multiplying 1 by a product ab, n times, we get a power $(ab)^n$.

And let's see now what each multiplication can produce.

First, multiplying 1 by the product ab, we just get: ab, which is the product of 1 of as and 1 of bs.
Next, multiplying ab by ab, that is, multiplying 1 by ab, 2 times, we get $(ab)^2 = ab \cdot ab = aabb$, because multiplications are commutative, that is, $xy = yx$.

So we get: $(ab)^2 = a^2b^2$.

Next, multiplying 1 by ab, 3 times, we get

$(ab)^3 = ab \cdot ab \cdot ab = aaabbb$, since multiplications are commutative.

So we get $(ab)^3 = a^3b^3$.

Now, what then do we get multiplying 1 by ab, n times?

We get $(ab)^n = a^nb^n$. So for instance, $(3{\cdot}5)^3 = (3{\cdot}5){\cdot}(3{\cdot}5)(3{\cdot}5) = 3{\cdot}5{\cdot}3{\cdot}5{\cdot}3{\cdot}5 = 3{\cdot}3{\cdot}3{\cdot}5{\cdot}5{\cdot}5$ $= 3^35^3$, and $(12{\cdot}7{\cdot}5)^2 = 12^27^25^2$.

Also, dividing 1 by ab twice, we get $(ab)^{-2}$, which is $1/(ab)/(ab)$, which is $1/(ab)^2$, too.

So we get $(ab)^{-2} = 1/(ab)^2 = 1/(a^2b^2) = (1/a^2)(1/b^2) = a^{-2}b^{-2}$.
Thus, we get $(ab)^{-2} = a^{-2}b^{-2}$.

Besides, we have $(ab)^0 = 1$, since multiplying 1 by ab no times, we just get 1.

And we have $a^0 = 1$, and $b^0 = 1$, so we get $a^0b^0 = 1$. Thus, we get $(ab)^0 = a^0b^0$.

So we get $(ab)^n = a^nb^n$, where both a and $b \neq 0$, and both m and n are integers.

Thus, if the base is a product, take a product of powers.

4. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$, where both a and $b \neq 0$, and both m and n are integers.

Multiplying 1 by a, n times, we get a power a^n.

So multiplying 1 by a fraction $\dfrac{a}{b}$, n times, we get a power $\left(\dfrac{a}{b}\right)^n$.

And let's see now what each multiplication can produce.

First, multiplying 1 by the fraction $\frac{a}{b}$, we just get $\frac{a}{b}$.

Next, multiplying $\frac{a}{b}$ by $\frac{a}{b}$, that is, multiplying 1 by $\frac{a}{b}$, 2 times, we get

$\left(\frac{a}{b}\right)^2 = \frac{a}{b}\cdot\frac{a}{b} = \frac{aa}{bb}$. So we get $\left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}$.

Next, multiplying 1 by $\frac{a}{b}$, 3 times, we get

$\left(\frac{a}{b}\right)^3 = \frac{a}{b}\cdot\frac{a}{b}\cdot\frac{a}{b} = \frac{aaa}{bbb}$. So we get $\left(\frac{a}{b}\right)^3 = \frac{a^3}{b^3}$.

Now, what then do we get multiplying 1 by $\frac{a}{b}$, $\boldsymbol{n}$ times?

We get: $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$. So for instance, $\mathbf{(3/5)^3 = (3/5)(3/5)(3/5) = 3{\cdot}3{\cdot}3/(5{\cdot}5{\cdot}5) = 3^3/5^3}$,

and $\mathbf{(12/7/5)^2 = 12^2/7^2/5^2}$. By the way, we have $12/7/5 = 12/(7\cdot5) = 12/35$.

Also, dividing 1 by $\frac{a}{b}$ twice, we get $\boldsymbol{(a/b)^{-2}}$, which is $\boldsymbol{1/(a/b)/(a/b)}$, which is $\boldsymbol{1/(a/b)^2}$, too.

So we get $\boldsymbol{(a/b)^{-2} = 1/(a/b)^2 = 1/(a^2/b^2) = b^2/a^2}$.

That is, we get $\left(\frac{a}{b}\right)^{-2} = \frac{1}{\left(\frac{a}{b}\right)^2} = \frac{1}{\frac{a^2}{b^2}}$.

And multiplying by $\boldsymbol{b^2}$, the numerator and the denominator above, that is, multiplying the numerator 1 by $\boldsymbol{b^2}$, and multiplying the denominator $\frac{a^2}{b^2}$ by $\boldsymbol{b^2}$, we get $\frac{b^2}{a^2}$.

So we get $\left(\dfrac{a}{b}\right)^{-2} = \dfrac{1}{\left(\dfrac{a}{b}\right)^2} = \dfrac{1}{\dfrac{a^2}{b^2}} = \dfrac{b^2}{a^2}$. That is, we get $\left(\dfrac{a}{b}\right)^{-2} = \dfrac{b^2}{a^2}$.

And we have $b^2 = \dfrac{1}{b^{-2}}$, because $\dfrac{1}{b^{-2}} = b^{-(-2)} = b^2$.

Also, we have $\dfrac{1}{a^2} = a^{-2}$.

So we get $\dfrac{b^2}{a^2} = b^2\left(\dfrac{1}{a^2}\right) = \dfrac{1}{b^{-2}} \cdot a^{-2} = \dfrac{a^{-2}}{b^{-2}}$. Thus, we get $\dfrac{b^2}{a^2} = \dfrac{a^{-2}}{b^{-2}}$.

And we have $\left(\dfrac{a}{b}\right)^{-2} = \dfrac{b^2}{a^2}$, too.

Besides, we have $\left(\dfrac{a}{b}\right)^{0} = 1$, since multiplying 1 by $\dfrac{a}{b}$ no times, we just get 1.

And we have: $\boldsymbol{a^0 = 1}$, and $\boldsymbol{b^0 = 1}$, so we get: $\boldsymbol{a^0/b^0 = 1}$. Thus, we get: $\boldsymbol{(a/b)^0 = a^0/b^0}$.

So in sum, we get: $\boldsymbol{(a/b)^n = a^n/b^n}$, where both $\boldsymbol{a}$ and $\boldsymbol{b \neq 0}$, and both $\boldsymbol{m}$ and $\boldsymbol{n}$ are integers.

For instance, we can get $\left(\dfrac{3}{5}\right)^3 = \left(\dfrac{3}{5}\right)\left(\dfrac{3}{5}\right)\left(\dfrac{3}{5}\right) = \dfrac{3 \cdot 3 \cdot 3}{5 \cdot 5 \cdot 5} = \dfrac{3^3}{5^3}$.

And we can get $\left(\dfrac{3}{5}\right)^{-3} = \dfrac{1}{\left(\dfrac{3}{5}\right)\left(\dfrac{3}{5}\right)\left(\dfrac{3}{5}\right)} = \dfrac{1}{\dfrac{3 \cdot 3 \cdot 3}{5 \cdot 5 \cdot 5}} = \dfrac{1}{\dfrac{3^3}{5^3}} = \dfrac{5^3}{3^3} = 5^3 \cdot \dfrac{1}{3^3} = \dfrac{1}{5^{-3}} \cdot 3^{-3} = \dfrac{3^{-3}}{5^{-3}}$.

Too easy? So boring? Not quite?

Right after the next section, we will get the extended versions of the identities **2**, **3**, and **4** listed above. Thus, we will get some more identities, which can give us more power when we do exponential algebra.

They are basically the same though, as the ones listed above.
The exponents ***m*** and ***n*** used in the identities above are integers, but the exponents in the extended versions are rational numbers. So the actual extensions are done on exponents.

(In fact, the exponents can be irrational, too, and thus, can be real. It is not the case though, the exponents can be any real numbers for any bases. And we will see how it is not the case in the section, **Problem Exponents**.)

Prior to the extended version, that is, in the next section, we will get familiar with another important tool, called an n^{th} root, which is a solution to an equation of degree ***n***, and is in fact, another form of a power.
And we are going to use the idea of such a root constructing the version extended.

2. What is an N^{th} Root?

Multiplying 1 by a number a particular number of times, we get a power made of a base and an exponent. For instance, multiplying 1 by 5, seven times, we get 5^7, where 5 is the base, and 7 is the exponent. In general, multiplying 1 by $\boldsymbol{b}$, $\boldsymbol{n}$ times, we get $\boldsymbol{b^n}$, where $\boldsymbol{b}$ is the base, and $\boldsymbol{n}$ is the exponent. And the same idea applies to divisions, too.

Dividing thus, 1 by a number a particular number of times, we get a power, too. For instance, dividing 1 by 3, eight times, we get 3^{-8}, where 3 is the base, and -8 is the exponent. In general, dividing 1 by $\boldsymbol{b}$, $\boldsymbol{n}$ times, we get $\boldsymbol{b^{-n}}$, where $\boldsymbol{b}$ is the base, and $\boldsymbol{-n}$ is the exponent. So we get a power, also, doing divisions. That is because a division can be taken for a multiplication by the reciprocal.

What if however, we want to express a value, which has the converse nature of a power? In other words, assuming we get a value if multiplying 1 by a certain value a particular number of times, we need to find the certain value.

- That is, given the value of a power and an exponent, we have to find the base.

Suppose for instance, multiplying 1 by $\boldsymbol{x}$ seven times, we get 9.
What then is $\boldsymbol{x}$? That is, by what value do we have to multiply 1, seven times to get 9?
So we want to find $\boldsymbol{x}$, by which we multiply 1 seven times to get 9.

We call it a seventh root. What do we mean by though, such a root?

Saying a root in math, we can mean the solution to an equation. So finding the root, we can set up the equation describing the situation we have. And the situation is as follows:

- Multiplying 1 by an unknown value, 7 times, we get 9.

What equation then can we set up?

Assuming $\boldsymbol{x}$ is the unknown, we can set $\boldsymbol{x^7 = 9}$. Then, the value of $\boldsymbol{x}$, that is, the solution to the equation above is called a 7th root. What then is an $\boldsymbol{n}$th root?

It is the solution to an equation describing a situation as follows.

- Multiplying 1 by an unknown value, $\boldsymbol{n}$ times, we get a value called $\boldsymbol{A}$.

So assuming again, $\boldsymbol{x}$ is the unknown, we can set up the equation the way as follows.

$\boldsymbol{x^n = A}$ where $\boldsymbol{A > 0}$, and $\boldsymbol{n}$ is a ***positive integer***.

And we call the solution an $\boldsymbol{n}$th root. How then can we get the solution, the $\boldsymbol{n}$th root?

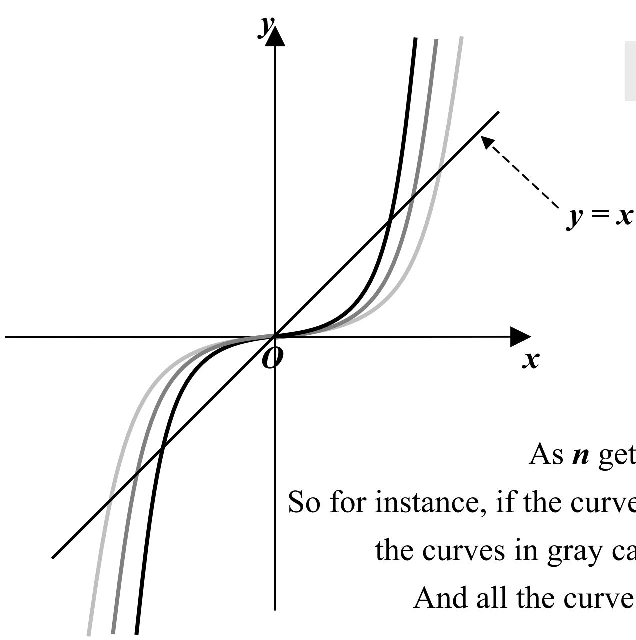
Let's first, take a look at some curves of an equation $\boldsymbol{y = x^n}$ for some values of $\boldsymbol{n \geq 1}$.

If $\boldsymbol{n = 1}$, we just get a line passing through the origin and making 45° against the $\boldsymbol{x}$-axis.

Next, if $\boldsymbol{n}$ is odd, and $\geq \boldsymbol{3}$, we can put in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane, the curves the way as follows.

Fig. 2.0

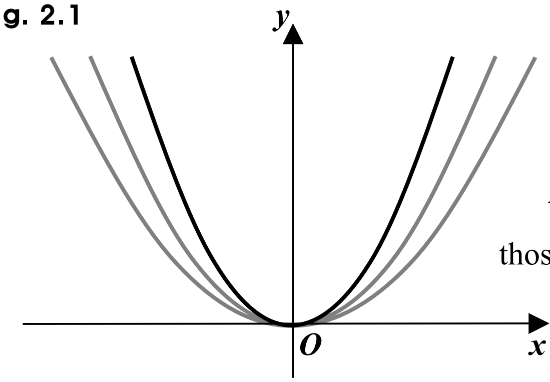
$\boldsymbol{y = x^n}$ for $\boldsymbol{n}$ odd, and $\geq \boldsymbol{1}$



As $\boldsymbol{n}$ gets larger, the curve gets steeper.
So for instance, if the curve in black is the curve of $\boldsymbol{y = x^7}$,
the curves in gray can be those of $\boldsymbol{y = x^5}$ and $\boldsymbol{y = x^3}$.
And all the curve are passing through the origin.

And next, if $\boldsymbol{n}$ is even, and $\geq \boldsymbol{2}$, we can put in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane, the curves the way below.

Fig. 2.1



$y = x^n$ for n even, and ≥ 2

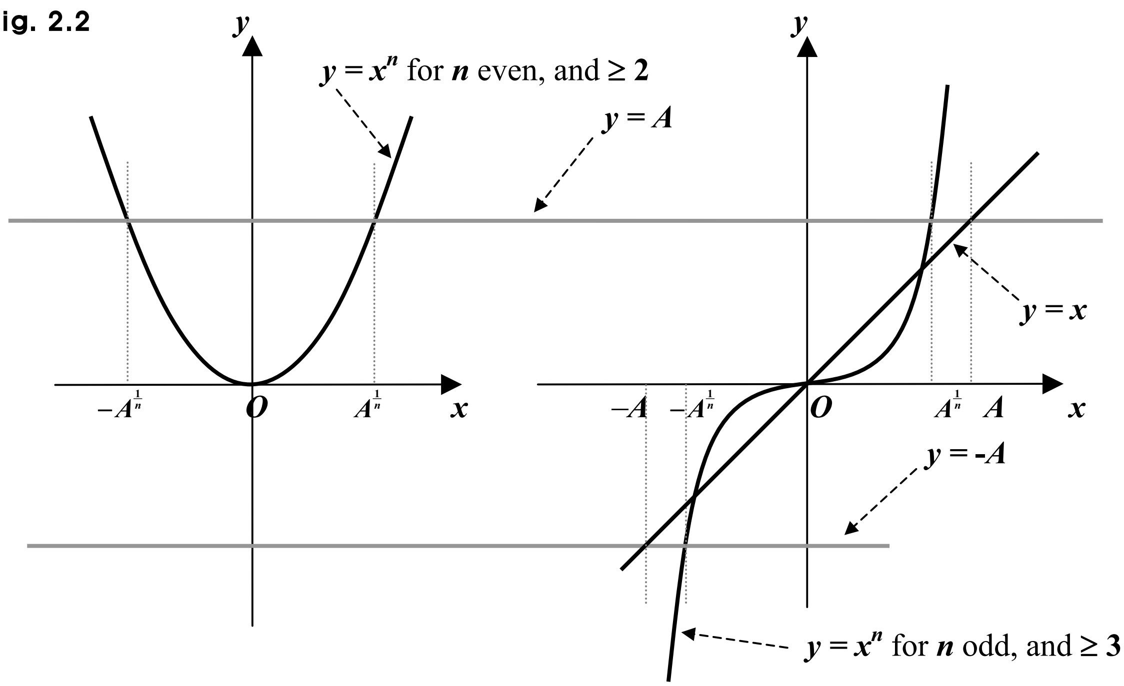
As n gets larger, the curve gets steeper. So for instance, if the curve in black is the curve of $y = x^6$, the curves in gray can be those of $y = x^4$ and $y = x^2$. And all the curves are passing through the origin.

Let's now get back to the equation $x^n = A$ where $A > 0$, and n is a ***positive integer***.

Then, considering the identity $(a^m)^n = a^{mn}$, where $a \neq 0$, and both m and n are integers, we can notice that we can put an n^{th} root this way: $x = A^{\frac{1}{n}}$.

And putting schematically in the x-y plane, the curves of $y = x^n$, $y = A$, and $y = -A$, we can put them the way below.

Fig. 2.2



Then, we can see that the solution can be either one n^{th} root or two n^{th} roots.

It depends on the value of $\boldsymbol{n}$. So let's see now how it depends.

To begin with, if $\boldsymbol{n} = \mathbf{1}$ or $\boldsymbol{A} = \mathbf{0}$, there isn't much point of calling the solution an $\boldsymbol{n}^{\text{th}}$ root. That's because if $\boldsymbol{n} = \mathbf{1}$, we just get $\boldsymbol{x} = \boldsymbol{A}$, and if $\boldsymbol{A} = \mathbf{0}$, we simply get $\boldsymbol{x} = \mathbf{0}$.

Next, when $\boldsymbol{n} = \mathbf{2}$ or $\mathbf{3}$, we normally call the solution a ***square root*** or a ***cube root*** rather than a second root or a third root.

What if $\boldsymbol{A} < \mathbf{0}$, though?

Then, only if $\boldsymbol{n}$ is odd, we can get a solution, which is an $\boldsymbol{n}^{\text{th}}$ root, too, which is negative though, as shown in the graph above. So if $\boldsymbol{n}$ is even, we get no solution.

• Thus, if we want to get an $\boldsymbol{n}^{\text{th}}$ root for ***any*** integer $\boldsymbol{n} > \mathbf{1}$, we need to have $\boldsymbol{A} > \mathbf{0}$.

As we can see in Fig. 2.2 above, if $\boldsymbol{A} > \mathbf{0}$, we can get a solution for any positive integer $\boldsymbol{n}$.

So putting threads together, we can say these:

If $\boldsymbol{n}$ is even and $\boldsymbol{A} > \mathbf{0}$, we get two $\boldsymbol{n}^{\text{th}}$ roots, which are indicated by $\boldsymbol{A}^{\frac{1}{n}}$ and $-\boldsymbol{A}^{\frac{1}{n}}$.
If $\boldsymbol{n}$ is odd, we get one $\boldsymbol{n}^{\text{th}}$ root, which is indicated by $\boldsymbol{A}^{\frac{1}{n}}$.
Either way, if $\boldsymbol{A} > \mathbf{0}$, we get the $\boldsymbol{n}^{\text{th}}$ root $\boldsymbol{A}^{\frac{1}{n}}$, which is positive.

• If however, $\boldsymbol{A} < \mathbf{0}$ and $\boldsymbol{n}$ is even, we get no solution, that is, no root, and thus, we get no $\boldsymbol{n}^{\text{th}}$ root.

So what is an $\boldsymbol{n}^{\text{th}}$ root about?

It is about powers, and in fact, is a power, too.

Getting back to the equation, we have $x^n = A$, which is saying that multiplying 1 by an unknown value, n times, we get a value called A.

And the solution is $x = A^{\frac{1}{n}}$, which is called an n^{th} root.

How then can we call A and $\frac{1}{n}$?

We can call A the **base**, and can call $\frac{1}{n}$ the exponent.
So the n^{th} root $A^{\frac{1}{n}}$ is a power, too, and thus, is about powers.

Besides, the equation $x^n = A$ is originally from the situation below:

- Multiplying 1 by a certain value, n times, we get a value called A.

And we know now that the certain value is the n^{th} root, which is $A^{\frac{1}{n}}$.

So multiplying 1 by the n^{th} root $A^{\frac{1}{n}}$, n times, we get A. And multiplying 1 by the n^{th} root $A^{\frac{1}{n}}$, n times, we get a power $(A^{\frac{1}{n}})^n$. Thus, we get $(A^{\frac{1}{n}})^n = A$.

That is, the n^{th} root $A^{\frac{1}{n}}$ is the certain value we want, and multiplying 1 by it, n times, we get A, which is the base in the power called the n^{th} root, which is $A^{\frac{1}{n}}$.

Therefore, for instance, multiplying 1 by $9^{\frac{1}{7}}$, 7 times, we get 9, that is, we get $(9^{\frac{1}{7}})^7 = 9$.

So we can notice now that there can be exponential identities where fractions are used as exponents. And in fact, we are going to see what such identities are, and get those identities, so we'll get to see how they work, and how to use them in the next section.

3. Exponential Identities 2

The exponential identities covered here are the extended versions of those covered in the section, **Exponential Identities 1**, which therefore, can be called the original version in this section. So we are going to look at how we can get the original versions extended.

And getting the versions extended, we are going to use the idea of the $\boldsymbol{n}^{\text{th}}$ root, covered in the previous section. And getting the concept of the $\boldsymbol{n}^{\text{th}}$ root, we use the structure or the configuration of such a root. What is an $\boldsymbol{n}^{\text{th}}$ root, though?

It is a root in math, and thus, is the solution to an equation, which is of degree $\boldsymbol{n}$.
And the equation is in fact, $\boldsymbol{x^n = A}$ where $\boldsymbol{A > 0}$, and $\boldsymbol{n}$ is a *positive integer* $> \mathbf{1}$.
What then is the equation saying?

It is saying that multiplying 1 by a certain value, $\boldsymbol{n}$ times, we get $\boldsymbol{A}$.
So solving the equation, that is, getting the solution, we get the certain value.

And we call the solution an $\boldsymbol{n}^{\text{th}}$ root. More specifically:

If $\boldsymbol{n}$ is even, we get two $\boldsymbol{n}^{\text{th}}$ roots, which are $\boldsymbol{A}^{\frac{1}{n}}$ and $-\boldsymbol{A}^{\frac{1}{n}}$.
If $\boldsymbol{n}$ is odd, we get one $\boldsymbol{n}^{\text{th}}$ root only, which is $\boldsymbol{A}^{\frac{1}{n}}$.
Either way, if $\boldsymbol{A} > \mathbf{0}$, we get the $\boldsymbol{n}^{\text{th}}$ root $\boldsymbol{A}^{\frac{1}{n}}$, which is positive.

So multiplying 1 by the $\boldsymbol{n}^{\text{th}}$ root $\boldsymbol{A}^{\frac{1}{n}}$, $\boldsymbol{n}$ times, we get $\boldsymbol{A}$.

Also, multiplying 1 by the $\boldsymbol{n}^{\text{th}}$ root $\boldsymbol{A}^{\frac{1}{n}}$, $\boldsymbol{n}$ times, we get a power $(\boldsymbol{A}^{\frac{1}{n}})^{\boldsymbol{n}}$, which is $\boldsymbol{A}$.

So we get $\boldsymbol{A} = (\boldsymbol{A}^{\frac{1}{n}})^{\boldsymbol{n}}$, and thus, can expect that $(\boldsymbol{A}^{\frac{1}{n}})^{\boldsymbol{n}} = \boldsymbol{A}^{\frac{1}{n}\cdot n} = \boldsymbol{A}$, which is in fact, the case, and is similar to an identity $(\boldsymbol{a}^{\boldsymbol{m}})^{\boldsymbol{n}} = \boldsymbol{a}^{\boldsymbol{mn}}$, covered in **Exponential Identities 1**.

So to begin with, setting $\boldsymbol{x}^{\boldsymbol{n}} = \boldsymbol{A}$, and $\boldsymbol{x} > \mathbf{0}$, we get
$\boldsymbol{A} > \mathbf{0}$ and $\boldsymbol{x} = \boldsymbol{A}^{\frac{1}{n}}$, which is an $\boldsymbol{n}^{\text{th}}$ root positive, and thus, we get $(\boldsymbol{A}^{\frac{1}{n}})^{\boldsymbol{n}} = \boldsymbol{A}$.

Next, getting back to the list of the original versions, and thus, assuming that both $\boldsymbol{a}$ and $\boldsymbol{b}$ are not $\mathbf{0}$, and that $\boldsymbol{m}$ and $\boldsymbol{n}$ both are integers, we get these:

0. $a^m a^n = a^{m+n}$

1. $a^m / a^n = \dfrac{a^m}{a^n} = a^{m-n}$

2. $(a^m)^n = a^{mn}$

3. $(ab)^n = a^n b^n$

4. $(a/b)^n = \left(\dfrac{a}{b}\right)^n = a^n / b^n = \dfrac{a^n}{b^n}$

And next, assuming that $\boldsymbol{a}$ and $\boldsymbol{b}$ both are positive, and that $\boldsymbol{m}$ and $\boldsymbol{n}$ both are integers positive, we can put the list of the extended versions the way as follows.

3.1. $a^{\frac{1}{n}} b^{\frac{1}{n}} = (ab)^{\frac{1}{n}}.$

4.1. $\dfrac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \left(\dfrac{a}{b}\right)^{\frac{1}{n}}.$

2.1. $(a^{\frac{1}{n}})^m = (a^m)^{\frac{1}{n}}.$

2.2. $(a^{\frac{1}{n}})^{\frac{1}{m}} = a^{\frac{1}{mn}} = (a^{\frac{1}{m}})^{\frac{1}{n}}.$

2.3. $(a^{mp})^{\frac{1}{np}} = (a^m)^{\frac{1}{n}}$, where $\boldsymbol{p}$ is a positive integer.

And let see now, how we can get the extended versions above. That is, the proofs.

3.1. $\boldsymbol{a^{\frac{1}{n}}b^{\frac{1}{n}} = (ab)^{\frac{1}{n}}}$.

To begin with, we have $\boldsymbol{a}$ and $\boldsymbol{b > 0}$, so we get $\boldsymbol{ab > 0}$, and thus, we get $\boldsymbol{a^{\frac{1}{n}}b^{\frac{1}{n}} > 0}$.

Next, from the original identities 3 and 2 above, we can get this:

$$\boldsymbol{(a^{\frac{1}{n}}b^{\frac{1}{n}})^n = (a^{\frac{1}{n}})^n(b^{\frac{1}{n}})^n = a^{\frac{1}{n}\cdot n} \cdot b^{\frac{1}{n}\cdot n} = ab}.$$

So we get $\boldsymbol{(a^{\frac{1}{n}}b^{\frac{1}{n}})^n = ab}$. Assuming thus, $\boldsymbol{x = a^{\frac{1}{n}}b^{\frac{1}{n}}}$, and $\boldsymbol{A = ab}$, we can set $\boldsymbol{x^n = A}$.

So we get $\boldsymbol{x = A^{\frac{1}{n}}}$, which is an $\boldsymbol{n}^{\text{th}}$ root. Thus, we can get $\boldsymbol{a^{\frac{1}{n}}b^{\frac{1}{n}} = (ab)^{\frac{1}{n}}}$.

4.1. $\boldsymbol{\dfrac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \left(\dfrac{a}{b}\right)^{\frac{1}{n}}}$.

To begin with, we have $\boldsymbol{a}$ and $\boldsymbol{b > 0}$, so we get $\boldsymbol{\dfrac{a}{b} > 0}$. Thus, we get $\boldsymbol{\dfrac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} > 0}$.

Next, from the original identities 4 and 2 above, we can get this:

$$\boldsymbol{\left(\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}\right)^n = \frac{(a^{\frac{1}{n}})^n}{(b^{\frac{1}{n}})^n} = \frac{a^{\frac{1}{n}\cdot n}}{b^{\frac{1}{n}\cdot n}} = \frac{a}{b}}.$$

So we get $\boldsymbol{\left(\dfrac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}\right)^n = \dfrac{a}{b}}$. And assuming $\boldsymbol{x = \dfrac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}}$, and $\boldsymbol{A = \dfrac{a}{b}}$, we can set $\boldsymbol{x^n = A}$.

So we get $\boldsymbol{x = A^{\frac{1}{n}}}$, which is an $\boldsymbol{n}^{\text{th}}$ root. Thus, we can get $\boldsymbol{\dfrac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \left(\dfrac{a}{b}\right)^{\frac{1}{n}}}$.

2.1. $(a^{\frac{1}{n}})^m = (a^m)^{\frac{1}{n}}$.

To begin with, we have $a > 0$, so we get $a^{\frac{1}{n}} > 0$, and thus, we get $(a^{\frac{1}{n}})^m > 0$.

And since we have $a > 0$, we get $a^m > 0$, too.

Next, from the original identity 2 above, we can get

$$\{(a^{\frac{1}{n}})^m\}^n = (a^{\frac{1}{n}})^{mn} = (a^{\frac{1}{n}})^{nm} = \{(a^{\frac{1}{n}})^n\}^m = (a^{\frac{1}{n}\cdot n})^m = a^m.$$

Thus, we get $\{(a^{\frac{1}{n}})^m\}^n = a^m$. And assuming $x = (a^{\frac{1}{n}})^m$, and $A = a^m$, we can set $x^n = A$.

So we get $x = A^{\frac{1}{n}}$, which is an n^{th} root. Thus, we can get $(a^{\frac{1}{n}})^m = (a^m)^{\frac{1}{n}}$.

And of course, from the original identity 2 above, we can get the same this way, too:

$$(a^m)^{\frac{1}{n}} = a^{\frac{m}{n}} = (a^{\frac{1}{n}})^m.$$

2.2. $(a^{\frac{1}{n}})^{\frac{1}{m}} = a^{\frac{1}{mn}} = (a^{\frac{1}{m}})^{\frac{1}{n}}$.

To begin with, we have $a > 0$, so we get $(a^{\frac{1}{n}})^{\frac{1}{m}} > 0$, and $(a^{\frac{1}{m}})^{\frac{1}{n}} > 0$.

Next, from the original identity 2 above, we can get

$\{(a^{\frac{1}{n}})^{\frac{1}{m}}\}^{mn} = [\{(a^{\frac{1}{n}})^{\frac{1}{m}}\}^m]^n = \{(a^{\frac{1}{n}})^{\frac{m}{m}}\}^n = (a^{\frac{1}{n}})^n = a$, and also:

$$\{(a^{\frac{1}{m}})^{\frac{1}{n}}\}^{mn} = \{(a^{\frac{1}{m}})^{\frac{1}{n}}\}^{nm} = [\{(a^{\frac{1}{m}})^{\frac{1}{n}}\}^n]^m = \{(a^{\frac{1}{m}})^{\frac{n}{n}}\}^m = (a^{\frac{1}{m}})^m = a.$$

So we get $\{(a^{\frac{1}{n}})^{\frac{1}{m}}\}^{mn} = a$, and $\{(a^{\frac{1}{m}})^{\frac{1}{n}}\}^{mn} = a$.

Now, the product $\boldsymbol{mn}$ is another integer positive, since $\boldsymbol{m}$ and $\boldsymbol{n}$ both are integers positive.

And assuming $\boldsymbol{x} = (\boldsymbol{a}^{\frac{1}{n}})^{\frac{1}{m}}$, and $\boldsymbol{A} = \boldsymbol{a}$, we can set $\boldsymbol{x}^{mn} = \boldsymbol{A}$.

So we get $\boldsymbol{x} = \boldsymbol{A}^{\frac{1}{mn}}$, which is an $\boldsymbol{mn}^{\text{th}}$ root. Thus, we can get $(\boldsymbol{a}^{\frac{1}{n}})^{\frac{1}{m}} = \boldsymbol{a}^{\frac{1}{mn}}$.

Also, assuming $\boldsymbol{x} = (\boldsymbol{a}^{\frac{1}{m}})^{\frac{1}{n}}$, and $\boldsymbol{A} = \boldsymbol{a}$, we can set again $\boldsymbol{x}^{mn} = \boldsymbol{A}$.

So we get $\boldsymbol{x} = \boldsymbol{A}^{\frac{1}{mn}}$, too, which is an $\boldsymbol{mn}^{\text{th}}$ root, too. Thus, we can get $(\boldsymbol{a}^{\frac{1}{m}})^{\frac{1}{n}} = \boldsymbol{a}^{\frac{1}{mn}}$, too.

Thus, putting threads together, we get $(\boldsymbol{a}^{\frac{1}{n}})^{\frac{1}{m}} = \boldsymbol{a}^{\frac{1}{mn}} = (\boldsymbol{a}^{\frac{1}{m}})^{\frac{1}{n}}$.

2.3. $(\boldsymbol{a}^{mp})^{\frac{1}{np}} = (\boldsymbol{a}^{m})^{\frac{1}{n}}$, where $\boldsymbol{p}$ is a positive integer.

To begin with, we have $\boldsymbol{a} > \boldsymbol{0}$, so we get $\boldsymbol{a}^{m} > \boldsymbol{0}$, and thus, we can get $(\boldsymbol{a}^{mp})^{\frac{1}{np}} > \boldsymbol{0}$.

Next, from the original identity 2 above, we can get

$$\{(\boldsymbol{a}^{mp})^{\frac{1}{np}}\}^{n} = [\{(\boldsymbol{a}^{mp})^{\frac{1}{p}}\}^{\frac{1}{n}}]^{n} = \{(\boldsymbol{a}^{mp})^{\frac{1}{p}}\}^{\frac{n}{n}} = (\boldsymbol{a}^{mp})^{\frac{1}{p}} = \{(\boldsymbol{a}^{m})^{p}\}^{\frac{1}{p}} = (\boldsymbol{a}^{m})^{\frac{p}{p}} = \boldsymbol{a}^{m}.$$

So we get $\{(\boldsymbol{a}^{mp})^{\frac{1}{np}}\}^{n} = \boldsymbol{a}^{m}$. And assuming $\boldsymbol{x} = (\boldsymbol{a}^{mp})^{\frac{1}{np}}$, and $\boldsymbol{A} = \boldsymbol{a}^{m}$, we can set $\boldsymbol{x}^{n} = \boldsymbol{A}$.

So we get $\boldsymbol{x} = \boldsymbol{A}^{\frac{1}{n}}$, which is an $\boldsymbol{n}^{\text{th}}$ root. Thus, we can get $(\boldsymbol{a}^{mp})^{\frac{1}{np}} = (\boldsymbol{a}^{m})^{\frac{1}{n}}$.

What if though, $\boldsymbol{m}$ and $\boldsymbol{n}$ both are integers negative?

Suppose first, $\boldsymbol{x}^{k} = \frac{1}{A}$, where $\boldsymbol{A} > \boldsymbol{0}$, and $\boldsymbol{k}$ is an integer positive.

Then, we get $\dfrac{\boldsymbol{1}}{\boldsymbol{A}} > \boldsymbol{0}$, so we can get $\boldsymbol{x} = \left(\dfrac{\boldsymbol{1}}{\boldsymbol{A}}\right)^{\frac{1}{k}}$, which is a $\boldsymbol{k}^{\text{th}}$ (that is, $\boldsymbol{n}^{\text{th}}$) root.

Next, we have $\dfrac{\boldsymbol{1}}{\boldsymbol{A}} = \boldsymbol{A}^{-1}$. So we get $\boldsymbol{x} = \left(\dfrac{\boldsymbol{1}}{\boldsymbol{A}}\right)^{\frac{1}{k}} = (\boldsymbol{A}^{-1})^{\frac{1}{k}} = \boldsymbol{A}^{-\frac{1}{k}} = \boldsymbol{A}^{\frac{1}{-k}} \Rightarrow \boldsymbol{x} = \boldsymbol{A}^{\frac{1}{-k}}$.

And we have $\boldsymbol{x} = \left(\frac{\boldsymbol{1}}{\boldsymbol{A}}\right)^{\frac{1}{k}}$.

So we get $\boldsymbol{A}^{\frac{1}{-k}} = \left(\frac{\boldsymbol{1}}{\boldsymbol{A}}\right)^{\frac{1}{k}}$, which is a $\boldsymbol{k}^{\text{th}}$ root, and thus, $\boldsymbol{A}^{\frac{1}{-k}}$ is a $\boldsymbol{k}^{\text{th}}$ root, too.

Now, $\boldsymbol{k}$ is an integer positive, so $\boldsymbol{-k}$ is an integer negative.

Also, $\boldsymbol{n}$ in '$\boldsymbol{n}^{\text{th}}$ root' is an integer, too.
So we can see that the *idea* of an $\boldsymbol{n}^{\text{th}}$ root works for $\boldsymbol{n < 0}$, too, in the extended versions.

And getting back to the list of the extended versions, and assuming that $\boldsymbol{a}$ and $\boldsymbol{b}$ both are positive, and that $\boldsymbol{m}$ and $\boldsymbol{n}$ both are integers positive, we get these:

3.1. $a^{\frac{1}{n}}b^{\frac{1}{n}} = (ab)^{\frac{1}{n}}$. **4.1.** $\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \left(\frac{a}{b}\right)^{\frac{1}{n}}$. **2.1.** $(a^{\frac{1}{n}})^m = (a^m)^{\frac{1}{n}}$.

2.2. $(a^{\frac{1}{n}})^{\frac{1}{m}} = a^{\frac{1}{mn}} = (a^{\frac{1}{m}})^{\frac{1}{n}}$. **2.3.** $(a^{mp})^{\frac{1}{np}} = (a^m)^{\frac{1}{n}}$, where $\boldsymbol{p}$ is a positive integer.

• Suppose now, we replace $\boldsymbol{a}$ with $\frac{1}{a}$, and $\boldsymbol{b}$ with $\frac{1}{b}$ in the extended versions.

Then, we'll get to see these: $\frac{1}{-m}, \frac{1}{-n}$, and $\boldsymbol{-m}$ instead of these: $\frac{1}{m}, \frac{1}{n}$, and $\boldsymbol{m}$.

For instance, in $a^{\frac{1}{n}}b^{\frac{1}{n}} = (ab)^{\frac{1}{n}}$, replacing $\boldsymbol{a}$ and $\boldsymbol{b}$ with $\frac{1}{a}$ and $\frac{1}{b}$ respectively, we get

$$a^{\frac{1}{n}}b^{\frac{1}{n}} = (ab)^{\frac{1}{n}} \Rightarrow (\tfrac{1}{a})^{\frac{1}{n}}(\tfrac{1}{b})^{\frac{1}{n}} = (\tfrac{1}{a}\cdot\tfrac{1}{b})^{\frac{1}{n}}.$$

Meanwhile,

$$(\tfrac{1}{a})^{\frac{1}{n}}(\tfrac{1}{b})^{\frac{1}{n}} = (a^{-1})^{\frac{1}{n}}(b^{-1})^{\frac{1}{n}} = a^{-\frac{1}{n}}b^{-\frac{1}{n}} = a^{\frac{1}{-n}}b^{\frac{1}{-n}}.$$

$$(\tfrac{1}{a}\cdot\tfrac{1}{b})^{\frac{1}{n}} = (a^{-1}b^{-1})^{\frac{1}{n}} = \{(ab)^{-1}\}^{\frac{1}{n}} = (ab)^{-\frac{1}{n}} = (ab)^{\frac{1}{-n}}.$$

So we can get $a^{\frac{1}{n}}b^{\frac{1}{n}} = (ab)^{\frac{1}{n}} \Rightarrow a^{\frac{1}{-n}}b^{\frac{1}{-n}} = (ab)^{\frac{1}{-n}}$.

And by the same token, doing the same replacement as above, we will get to see these:

4.1. $\dfrac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \left(\dfrac{a}{b}\right)^{\frac{1}{n}} \Rightarrow \dfrac{a^{\frac{1}{-n}}}{b^{\frac{1}{-n}}} = \left(\dfrac{a}{b}\right)^{\frac{1}{-n}}$.

2.1. $(a^{\frac{1}{n}})^m = (a^m)^{\frac{1}{n}} \Rightarrow (a^{\frac{1}{-n}})^m = (a^m)^{\frac{1}{-n}}$.

2.2. $(a^{\frac{1}{n}})^{\frac{1}{m}} = a^{\frac{1}{mn}} = (a^{\frac{1}{m}})^{\frac{1}{n}} \Rightarrow (a^{\frac{1}{-n}})^{\frac{1}{-m}} = a^{\frac{1}{mn}} = (a^{\frac{1}{-m}})^{\frac{1}{-n}}$.

2.3. $(a^{mp})^{\frac{1}{np}} = (a^m)^{\frac{1}{n}} \Rightarrow (a^{-mp})^{\frac{1}{np}} = (a^{-m})^{\frac{1}{n}}$, where p is a positive integer.

Now, we have $a > 0$, and $b > 0$. So we get $\frac{1}{a} > 0$, and $\frac{1}{b} > 0$.
And thus, setting $c > 0$ and $d > 0$, and replacing $\frac{1}{a}$ and $\frac{1}{b}$ with c and d, we can make another set of versions separate from the extended versions.

We don't have to bother doing that, though. The extended versions use a and b, which cover all numbers positive. So we can just use the extended versions for that purpose, too. What then about m and n?

We know that in $\frac{1}{-m}$ and $\frac{1}{-n}$, $-m$ and $-n$ are negative integers.

So we just let m and n in the extended versions take integers negative, too.
In other words, we can just replace 'positive' with 'nonzero' in the extended versions.
So we can put the extended versions the way as follows.

Assuming that a and $b > 0$, and that m, n, and p are *integers nonzero*, we get

3.1. $a^{\frac{1}{n}}b^{\frac{1}{n}} = (ab)^{\frac{1}{n}}$. **4.1.** $\dfrac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \left(\dfrac{a}{b}\right)^{\frac{1}{n}}$. **2.1.** $(a^{\frac{1}{n}})^m = (a^m)^{\frac{1}{n}}$.

2.2. $(a^{\frac{1}{n}})^{\frac{1}{m}} = a^{\frac{1}{mn}} = (a^{\frac{1}{m}})^{\frac{1}{n}}$. **2.3.** $(a^{mp})^{\frac{1}{np}} = (a^m)^{\frac{1}{n}}$.

What if $\boldsymbol{a} < \mathbf{0}$ or $\boldsymbol{b} < \mathbf{0}$, though?

Then, the extended versions might not work.

In fact, if in a power, the base is negative and the exponent is a fraction where the denominator is even, the power is very likely to be in trouble.   Why?

The same issue will be raised in the next section, too, and the detailed explanations will be presented in the sections for **Problem Bases**.

4. What are Radical Signs?

Expressing an $\boldsymbol{n}^{\text{th}}$ root, we can put it in power notation as in $\boldsymbol{A}^{\frac{1}{n}}$. It's not the only way though. We can indicate it by means of a radical sign, too. What then is a radical?

It is just another word for a *root* in math, and thus, can be called a solution to an equation as $\boldsymbol{x^2 - x - 2 = 0}$. So an $\boldsymbol{n}^{\text{th}}$ root can be called an $\boldsymbol{n}^{\text{th}}$ radical, too.
Thus, if a radical sign is used to indicate an $\boldsymbol{n}^{\text{th}}$ root, it is called an $\boldsymbol{n}^{\text{th}}$ root sign or an $\boldsymbol{n}^{\text{th}}$ radical sign, and has such a simple syntax as follows.

$\sqrt[n]{}$, where $\boldsymbol{n}$ is a positive integer $\geq \mathbf{2}$, and indicates the degree of the radical.

So we can set $\boldsymbol{A}^{\frac{1}{n}} = \sqrt[n]{\boldsymbol{A}}$, and we get $(\sqrt[n]{\boldsymbol{A}})^{\boldsymbol{n}} = \boldsymbol{A}$.

Thus, we call $\sqrt[n]{\boldsymbol{A}}$ an $\boldsymbol{n}^{\text{th}}$ root of $\boldsymbol{A}$ or an $\boldsymbol{n}^{\text{th}}$ radical of $\boldsymbol{A}$.

For instance, we can put $\mathbf{9}^{\frac{1}{7}}$ in $\sqrt[7]{\mathbf{9}}$, which can be called a seventh root (radical) of 9.

So $\mathbf{9}^{\frac{1}{7}}$ (9 to the one seventh) = $\sqrt[7]{\mathbf{9}}$ (a seventh root of 9), and we get $(\sqrt[7]{\mathbf{9}})^{7} = \mathbf{9}$.

For another instance, $\sqrt[4]{\mathbf{16}}$ can be called a fourth root of 16, and we get $(\sqrt[4]{\mathbf{16}})^{\mathbf{4}} = \mathbf{16}$.

In addition, we have $\mathbf{16 = 2^4}$ and $\sqrt[4]{\mathbf{16}} = \mathbf{16}^{\frac{1}{4}}$, so we get $\sqrt[4]{\mathbf{16}} = \mathbf{16}^{\frac{1}{4}} = (\mathbf{2}^{4})^{\frac{1}{4}} = \mathbf{2}^{\frac{4}{4}} = \mathbf{2}$.

In particular, if $\boldsymbol{n} = \mathbf{2}$ in $\sqrt[n]{A}$, we usually omit 2 in the radical sign $\sqrt[2]{\ }$, so we normally put it this way: $\sqrt{A}$, and call $\sqrt{\ }$ a square root sign. So $\sqrt{A}$ can be read as a square root of $\boldsymbol{A}$.

And of course, the sign, $\sqrt{\ }$ can be called a second radical sign or a second root sign, too. So $\sqrt{A}$ can be read as a second radical of $\boldsymbol{A}$ or a second root of $\boldsymbol{A}$, too.

So for instance, we can set $\sqrt[2]{3} = \sqrt{3}$, which is usually called a square root of 3, and can be called a second radical of 3 or a second root of 3, too.

Besides, $\sqrt[3]{\ }$ is often called a cube root sign, so for instance, $\sqrt[3]{5}$ can be read as a cube root of 5. And of course, it can be called a third radical of 5 or a third root of 5, too.

And if $\boldsymbol{n} \neq \mathbf{2}$ in $\sqrt[n]{A}$, we have to specify $\boldsymbol{n}$, which indicates the degree of the radical. So for instance, we can set $7^{\frac{1}{3}} = \sqrt[3]{7}$, which is a third root (radical) of 7, and can set $3^{\frac{1}{7}} = \sqrt[7]{3}$, which is a seventh root (radical) of 3, etc.   What then about $5^{\frac{2}{3}}$ or $7^{-\frac{2}{3}}$?

Let's first, express an $\boldsymbol{n}^{\text{th}}$ root of a number in some more general manner.

Suppose that $A = b^m$, where $b \geq 0$, and $\boldsymbol{m}$ is an integer, and that $\boldsymbol{n}$ is an integer $\geq \mathbf{2}$.

Then, we can get $A = b^m \Rightarrow \sqrt[n]{A} = \sqrt[n]{b^m}$, which is an $\boldsymbol{n}^{\text{th}}$ root in more general form.

And we can put it this way, too: $A = b^m \Rightarrow A^{\frac{1}{n}} = \sqrt[n]{A} = (b^m)^{\frac{1}{n}} = b^{\frac{m}{n}} \Rightarrow \sqrt[n]{A} = b^{\frac{m}{n}}$.

So we can put the $\boldsymbol{n}^{\text{th}}$ root either of the two ways as follows: $\sqrt[n]{A} = \sqrt[n]{b^m}$, and $\sqrt[n]{A} = b^{\frac{m}{n}}$.

Thus, we can get $b^{\frac{m}{n}} = \sqrt[n]{b^m}$, which can be read as 'an $\boldsymbol{n}^{\text{th}}$ root of $\boldsymbol{b}$ to the $\boldsymbol{m}$'.

So for instance, we can set $\mathbf{5^{\frac{2}{3}}} = \sqrt[3]{\mathbf{5^2}}$, which is usually read as 'a cube root of five squared', and of course, can be read as 'a third root of five to the second', too.

Also, we can set $\mathbf{7^{-\frac{2}{3}}} = \sqrt[3]{\mathbf{7^{-2}}} = \sqrt[3]{\frac{\mathbf{1}}{\mathbf{7^2}}} = \mathbf{...}$, or can set $\mathbf{7^{-\frac{2}{3}}} = \frac{\mathbf{1}}{\mathbf{7^{\frac{2}{3}}}} = \frac{\mathbf{1}}{\sqrt[3]{\mathbf{7^2}}} = \mathbf{...}$

And for more examples, using the exponential identities, we can get these:

$$\mathbf{8^{\frac{2}{3}}} = \sqrt[3]{\mathbf{8^2}} = \sqrt[3]{\mathbf{(2^3)^2}} = \sqrt[3]{\mathbf{(2^3)^2}} = \sqrt[3]{\mathbf{2^6}} = \mathbf{2^{\frac{6}{3}}} = \mathbf{2^2} = \mathbf{4}.$$

$$\mathbf{8^{-\frac{2}{3}}} = \sqrt[3]{\mathbf{8^{-2}}} = \sqrt[3]{\mathbf{(8^{-1})^2}} = \sqrt[3]{\mathbf{(\tfrac{1}{8})^2}} = \sqrt[3]{\mathbf{(\tfrac{1}{2^3})^2}} = \sqrt[3]{\mathbf{(\tfrac{1}{2})^6}} = \mathbf{\left(\tfrac{1}{2}\right)^{\frac{6}{3}}} = \mathbf{2^{-2}} = \mathbf{\tfrac{1}{4}}.$$

So we can put the same radical many different ways.

Let's next, take a look at some cases where the bases are fractions.

To begin with, we have an exponential identity $\left(\frac{x}{y}\right)^z = \frac{x^z}{y^z}$, where both $\boldsymbol{x}$ and $\boldsymbol{y} > \mathbf{0}$.
So assuming both $\boldsymbol{a}$ and $\boldsymbol{b} > \mathbf{0}$, we can have

$$\left(\frac{a}{b}\right)^{\frac{m}{n}} = \frac{a^{\frac{m}{n}}}{b^{\frac{m}{n}}} = \frac{\sqrt[n]{a^m}}{\sqrt[n]{b^m}} = \sqrt[n]{\frac{a^m}{b^m}} = \sqrt[n]{\left(\frac{a}{b}\right)^m}.$$ How can we get $\frac{\sqrt[n]{a^m}}{\sqrt[n]{b^m}} = \sqrt[n]{\frac{a^m}{b^m}}$, though?

We can have $\frac{a^{\frac{m}{n}}}{b^{\frac{m}{n}}} = \left(\frac{a^m}{b^m}\right)^{\frac{1}{n}} = \sqrt[n]{\frac{a^m}{b^m}}$, and $\frac{a^{\frac{m}{n}}}{b^{\frac{m}{n}}} = \frac{\sqrt[n]{a^m}}{\sqrt[n]{b^m}}$. So we can get $\frac{\sqrt[n]{a^m}}{\sqrt[n]{b^m}} = \sqrt[n]{\frac{a^m}{b^m}}$.

Now, what if $\boldsymbol{b} < \mathbf{0}$ in $\sqrt[n]{b^m}$?

Then, $\sqrt[n]{b^m}$ might not be a real number.

For instance, $\sqrt[4]{(\textbf{-16})^2}$ is a real number, but $\sqrt[4]{(\textbf{-16})^3}$ is not a real number. Why not?

The number $\sqrt[4]{(\textbf{-16})^3}$ means that multiplying 1 by $\sqrt[4]{(\textbf{-16})^3}$ four times, we get $(-16)^3$.

However, $\sqrt[4]{(-\textbf{16})^3}$ does not exist in the real number space, that is, it is not a real number.

That's because $(-16)^3$ is negative, and there is no real number by which we can multiply 1, four times to get $(-16)^3$.

In fact, no matter what real number it may be, multiplying 1 by it, even number of times, we get no negative number.

Let's take a closer look at this radical: $\sqrt[4]{(\textbf{-16})^3}$.

We have $\mathbf{(-16)^3 = (-2^4)^3 = -2^{12}}$, so we get $\sqrt[4]{(\textbf{-16})^3} = \sqrt[4]{\textbf{-2}^{12}}$.

However, we have $\sqrt[4]{\textbf{-2}^{12}} \neq \textbf{-2}^{\frac{12}{4}} = \textbf{-2}^3 = \textbf{-8}$. So we get $\sqrt[4]{(\textbf{-16})^3} \neq \textbf{-8}$.

That is to say that we get $\sqrt[4]{(-\mathbf{16})^3} = \mathbf{(-16)^{\frac{3}{4}} = (-2^4)^{\frac{3}{4}} \neq -2^{4\cdot\frac{3}{4}} = -2^3 = -8}$.

• It's because processing a radical, we have to process what's inside the radical sign first.

So we cannot have $\sqrt[4]{(\textbf{-16})^3} = \textbf{-2}^{\frac{12}{4}}$, but we can have $\sqrt[4]{(\textbf{-16})^3} = \sqrt[4]{(\textbf{-2}^4)^3} = \sqrt[4]{\textbf{-2}^{12}}$, which is not however, a real number, because -2^{12} is negative, and no matter what real number it may be, multiplying 1 by it, even number of times, we cannot get a negative number.

Thus, in $\sqrt[n]{\boldsymbol{b}^{\boldsymbol{m}}}$, if the base $\boldsymbol{b} < \mathbf{0}$, $\boldsymbol{n}$ is even, and $\boldsymbol{m}$ is odd, we get $\boldsymbol{b}^{\boldsymbol{m}} < \mathbf{0}$, so $\sqrt[n]{\boldsymbol{b}^{\boldsymbol{m}}}$ cannot exist in the real number space, and thus, is not a real number.

In the real number space, no matter what number we may multiply 1 by, even number of times, we get no negative number, that is, we get a number $\geq \mathbf{0}$.

However, if $\boldsymbol{n}$ is odd, $\sqrt[n]{\boldsymbol{b}^{\boldsymbol{m}}}$ is a real number even if the base $\boldsymbol{b} < \mathbf{0}$.

That is to say that if $\boldsymbol{n}$ is odd, $\sqrt[n]{\boldsymbol{b}^{\boldsymbol{m}}}$ is a real number no matter what real number the base $\boldsymbol{b}$ may be. How?

To begin with, if $\boldsymbol{b} = \mathbf{0}$, we just get $\sqrt[n]{\boldsymbol{b}^{\boldsymbol{m}}} = \mathbf{0}$, which is of course, a real number.

Next, if $\boldsymbol{b} > \mathbf{0}$, we simply get $\boldsymbol{b}^{\boldsymbol{m}} > \mathbf{0}$, so $\sqrt[n]{\boldsymbol{b}^{\boldsymbol{m}}}$ is a real number regardless of $\boldsymbol{n}$.

And next, if $\boldsymbol{b} < \mathbf{0}$, we get $\boldsymbol{b}^{\boldsymbol{m}} < \mathbf{0}$ if $\boldsymbol{m}$ is odd, so even if $\boldsymbol{n}$ is odd, $\sqrt[n]{\boldsymbol{b}^{\boldsymbol{m}}}$ is a real number, too. How?

There is a particular real number by which we can multiply 1, an odd number of times to get a number negative, and the particular real number is negative, too. For instance,

We have $\sqrt[5]{\mathbf{-32}} = (\mathbf{-32})^{\frac{1}{5}}$, which is a real number. It's because

A radical $\sqrt[5]{\mathbf{-32}}$ means that multiplying 1 by the radical $\sqrt[5]{\mathbf{-32}}$ itself, 5 times, we get –32.

And we have $\sqrt[5]{\mathbf{-32}} = \sqrt[5]{(\mathbf{-2})^{\mathbf{5}}} = (\mathbf{-2})^{\frac{5}{5}} = (\mathbf{-2})^{\mathbf{1}} = \mathbf{-2}$. So we get $\sqrt[5]{\mathbf{-32}} = \mathbf{-2}$.

Thus, multiplying 1 by $\sqrt[5]{\mathbf{-32}}$, 5 times, we multiply 1 by -2, 5 times,
and then, we get $(\mathbf{-2})^{\mathbf{5}} = \mathbf{-32}$.

And for another instance, we have $\sqrt[5]{(-32)^3} = (-32)^{\frac{3}{5}}$, which is real. It's because

$\sqrt[5]{(-32)^3}$ means that multiplying 1 by the radical $\sqrt[5]{(-32)^3}$ itself, 5 times, we get $(-32)^3$.

And we have $\sqrt[5]{(-32)^3} = \sqrt[5]{\{(-2)^5\}^3} = \sqrt[5]{(-2)^{15}} = (-2)^{\frac{15}{5}} = (-2)^3 = -8$.

So we get $\sqrt[5]{(-32)^3} = -8$.

Thus, multiplying 1 by $\sqrt[5]{(-32)^3}$, 5 times, we multiply 1 by -8, 5 times,
and then we get $(-8)^5 = \{(-2)^3\}^5 = (-2)^{15} = \{(-2)^5\}^3 = (-32)^3$.

So even if b is negative, $\sqrt[n]{b^m}$, that is, $b^{\frac{m}{n}}$ is a real number if m and n both are odd.

And note that we have $-b^n \neq (-b)^n$, and that if $b = -2$ in $b^{\frac{m}{n}}$, we get $(-2)^{\frac{m}{n}}$, and not $-2^{\frac{m}{n}}$.

Also, if n and m both are even, $\sqrt[n]{b^m}$ is a real number even if the base $b < 0$ since $b^m > 0$.

For instance, we have $\sqrt[5]{(-2)^2} = (-2)^{\frac{2}{5}} = \{(-2)^2\}^{\frac{1}{5}} = 4^{\frac{1}{5}}$, and $\sqrt[5]{(-2)^2} = \sqrt[5]{4}$, which is real.

Besides, when the *base* is *negative*, there can be a significant difference between using a radical sign and using power notation. And we will cover in the next section how it is the case. What in the world is the real number space, though?

It is a conceptual place where all real numbers exist, and real numbers only can exist. So the space of real numbers is in fact, a set of all real numbers, and therefore, if x does not exist in the real number space, x is not a real number.

So we can use the word, '*space*' for expressing an area or region where all the items in a particular category exist, and such items only can exist. And in such a space, the number of items is normally infinite. A space can mean therefore, an infinite set.

So for instance,

The space of integers or the integer space is a conceptual place or region where all integers exist, and integers only can exist. It is therefore, a set of all integers. So if $\boldsymbol{A}$ does not belong to an integer space, $\boldsymbol{A}$ is not an integer.

The space of rectangles is a set of all rectangles, and is a conceptual place where all rectangles exist, and rectangles only can exist. So if $\boldsymbol{B}$ does not belong to a space of rectangles, $\boldsymbol{B}$ is not a rectangle.

The space of functions is a conceptual place where all functions exist, and functions only can exist. So if $\boldsymbol{C}$ does not belong to the space of functions, $\boldsymbol{C}$ is not a function.

The space of sequences is a conceptual place where all sequences exist, and sequences only can exist. So if $\boldsymbol{D}$ does not exist in the space of sequences, $\boldsymbol{D}$ is not a sequence.

And note that an $\boldsymbol{n}^{\text{th}}$ root $\boldsymbol{A}^{\frac{1}{n}}$ is a power, too, because we can take $\boldsymbol{A}$ as the base, and can take $\frac{1}{n}$ as the exponent. So $\boldsymbol{A}$ in a radical $\sqrt[n]{\boldsymbol{A}}$ is called the base of the radical, too.

That is to say that <u>what's inside a radical sign</u> is called <u>the base of the radical</u>.

So can the base of a radical be a power?

Yes, it can be. We can have a radical like this $\sqrt[n]{\boldsymbol{b}^{m}}$, and in this case, the base of the radical is $\boldsymbol{b}^{m}$, which is a power, so the base of a radical can be a power, too.
For instance, in a radical $\sqrt[4]{\boldsymbol{2}^{3}}$, 2^3 is the base, and is a power of 2.

And next taking some samples on arithmetic in radicals, we can have these:

$$2^2\sqrt{2}=4\sqrt{2}=\sqrt{16}\sqrt{2}=\sqrt{16\cdot 2}=\sqrt{2^4 2}=\sqrt{2^5}=(2^5)^{\frac{1}{2}}=2^{\frac{5}{2}}$$

$$2^2 2^{\frac{1}{2}}=2^{2+\frac{1}{2}}=2^{\frac{5}{2}}$$

$$2^2\div\sqrt{2}=\frac{2^2}{\sqrt{2}}=\frac{4}{\sqrt{2}}=\frac{\sqrt{16}}{\sqrt{2}}=\sqrt{\frac{16}{2}}=\sqrt{8}=\sqrt{2^3}=(2^3)^{\frac{1}{2}}=2^{\frac{3}{2}}=2^{1+\frac{1}{2}}=2^1 2^{\frac{1}{2}}=2\sqrt{2}$$

$$2^2\div 2^{\frac{1}{2}}=2^{2-\frac{1}{2}}=2^{\frac{3}{2}}$$

$$\frac{2^{10.2}}{2^{0.8}}=2^{10.2-0.8}=2^{9.4}=2^{\frac{94}{100}}=2^{\frac{47}{50}}=\sqrt[50]{2^{47}}$$, and also, $$\frac{2^{10.2}}{2^{0.8}}=\frac{1}{2^{0.8-10.2}}=\frac{1}{2^{-9.4}}=2^{9.4}$$

$(2^2)^{\frac{1}{2}}=2^{2\cdot\frac{1}{2}}=2^1=2$, and also, $\sqrt{2^2}=(2^2)^{\frac{1}{2}}=2$.

$6^{0.8}=6^{\frac{8}{10}}=6^{\frac{4}{5}}=\sqrt[5]{6^4}$, and also, $6^{0.8}=(2\cdot 3)^{0.8}=2^{0.8}3^{0.8}$.

$$\left(\frac{2}{3}\right)^{\frac{1}{2}}=\sqrt{\frac{2}{3}}=\frac{\sqrt{2}}{\sqrt{3}}=\frac{2^{\frac{1}{2}}}{3^{\frac{1}{2}}},\ \left(\frac{2}{3}\right)^{\frac{2}{3}}=\sqrt[3]{\left(\frac{2}{3}\right)^2}=\sqrt[3]{\frac{2^2}{3^2}}=\frac{\sqrt[3]{2^2}}{\sqrt[3]{3^2}}=\frac{2^{\frac{2}{3}}}{3^{\frac{2}{3}}},$$

$$\left(\frac{2}{3}\right)^{-\frac{5}{3}}=\sqrt[3]{\left(\frac{2}{3}\right)^{-5}}=\sqrt[3]{\left(\frac{3}{2}\right)^{5}}=\sqrt[3]{\frac{3^5}{2^5}}=\frac{\sqrt[3]{3^5}}{\sqrt[3]{2^5}}=\frac{\sqrt[3]{3^{3+2}}}{\sqrt[3]{2^{3+2}}}=\frac{\sqrt[3]{3^3 3^2}}{\sqrt[3]{2^3 2^2}}=\frac{\sqrt[3]{3^3}\sqrt[3]{3^2}}{\sqrt[3]{2^3}\sqrt[3]{2^2}}=\frac{3\sqrt[3]{3^2}}{2\sqrt[3]{2^2}}=\frac{3^{\frac{5}{3}}}{2^{\frac{5}{3}}}$$

$$\left(\frac{2}{3}\right)^{-0.1}=\frac{2^{-0.1}}{3^{-0.1}}=\frac{\sqrt[10]{2^{-1}}}{\sqrt[10]{3^{-1}}}=\frac{\sqrt[10]{\frac{1}{2}}}{\sqrt[10]{\frac{1}{3}}}=\sqrt[10]{\frac{\frac{1}{2}}{\frac{1}{3}}}=\sqrt[10]{\frac{3}{2}}$$

$$\left(\frac{2}{3}\right)^{-0.1}=\left(\left(\frac{2}{3}\right)^{-1}\right)^{0.1}=\left(\frac{3}{2}\right)^{0.1}=\frac{3^{0.1}}{2^{0.1}}=\frac{\sqrt[10]{3}}{\sqrt[10]{2}}=\sqrt[10]{\frac{3}{2}}$$

And let's look at more closely, the case where the base $\boldsymbol{b}$ is 0 in a power $\boldsymbol{b^p}$.

• Suppose first, the base $\boldsymbol{b}$, and exponent $\boldsymbol{p}$ both are 0. Then, we get 0^0, which is technically a power, too, but cannot be one particular number. Why not?

We can get this: $\mathbf{0^0 = 1}$, because multiplying 1 by 0 no times, we get 1, and we can get this, too: $\mathbf{0^0 = 0^{1-1} = \frac{0^1}{0^1} = \frac{0}{0}}$. So 0^0 can be any number. How?

Setting $\boldsymbol{x} = \mathbf{0^0}$, we can get $\boldsymbol{x} = \frac{0}{0}$, because $\frac{0}{0} = \mathbf{0^{1-1} = 0^0}$.

So we can get $\mathbf{0}\cdot\boldsymbol{x} = \mathbf{0}$, since $\boldsymbol{x} = \frac{0}{0}$.

Therefore, $\boldsymbol{x}$ can be any number, because $\mathbf{0}\cdot\boldsymbol{x} = \mathbf{0}$ can hold for any value of $\boldsymbol{x}$.
So $\boldsymbol{x}$, that is, the power 0^0 cannot be one particular number.

Thus, the exponent $\boldsymbol{p}$ cannot be 0 if the base $\boldsymbol{b} = \mathbf{0}$ in the power $\boldsymbol{b^p}$.

• Suppose next, the base $\boldsymbol{b}$ is 0, and the exponent $\boldsymbol{p}$ is negative.

Then, we get such weird things as $\mathbf{0^{-1}}$, $\mathbf{0^{-3}}$, $\mathbf{0^{-0.2}}$, etc., which are thus, all garbage in math. How?

We know $\mathbf{0^{-1} = \frac{1}{0}}$, $\mathbf{0^{-3} = \frac{1}{0^3} = \frac{1}{0}}$, and $\mathbf{0^{-0.2} = \frac{1}{0^{0.2}} = \frac{1}{0}}$, but no division by 0 is allowed.

Consequently therefore, if the base $\boldsymbol{b}$ is 0 in a power $\boldsymbol{b^p}$, the exponent $\boldsymbol{p}$ cannot be $\leq \mathbf{0}$, that is, $\boldsymbol{p}$ has to be $> \mathbf{0}$ only.

So if we want to use all real numbers as the exponent $\boldsymbol{p}$, we don't want to use 0 as the base $\boldsymbol{b}$. And the same is true for the case where the base $\boldsymbol{b}$ is negative, too. Why?

We will cover it in the next three sections.

5.0. Problem Bases 1

Normally, using numbers, we use real numbers, often just called real.
So unless specified otherwise, the numbers we use are real.

And using a number, we can use it in a form of a power as 2^3, which is a number, too.
So unless specified otherwise, the powers we use are real, too.

A power is though, not simply a number, but is made of two numbers called a base and an exponent. And making a power, we use real numbers as the base and the exponent.
So if we use real numbers as the base and the exponent, is the power a real number, too?
In other words, if the base and the exponent are real, is the power real, too?

It's not always the case. So it can be the case where the power is not real even if the base and the exponent both are real. It all depends on the number used as the base.
What number then, do we have to use as the base?

What matters is the sign of the number. So the sign of the base matters.
If the base is negative or 0, the power can be in trouble. What then, is the trouble?

The power cannot be a real number. If the base is not positive, it can be the case where the power is not real. And thus, bases not positive are problematic.
What's wrong with such a base though?

The base itself is in fact, not a problem, because it cannot cause trouble alone. It depends on the exponent. If the base is not positive, we cannot use some numbers as the exponent.

• That is to say that for some exponents, we cannot use as bases numbers negative or 0.

So we want to be careful with using numbers as the exponents or the bases.
Why do we do this though?

This book is in fact, for your algebra, and is in particular, for your calculation skill, which matters when you do problems as well as when you learn things in math.
Taking care of powers or radicals when doing algebra, we do exponential algebra.
It is quite often the case, doing problems, you want to know how to work with radicals or powers, that is, how to do calculations with such numbers or expressions.
So all the material in this book is for your algebra. Knowing it very well, you can grow algebra skill fast and make it robust.

So let's take a close look at what we need to be careful with when we work with radicals or powers. We are going to cover situations where radicals or powers can be in trouble.

To begin with, if the base is 0, we cannot use a negative number as the exponent.
It's because no division by 0 is allowed. So for instance, $\mathbf{0^{-2} = \frac{1}{0^2} = \frac{1}{0}}$ is not allowed.

And next, if the exponent is a fraction, and the denominator is even, the power can get into trouble. That is to say that when such a fractional exponent gets applied to a negative base, the power can get messed up. So a <u>negative base</u> is <u>problematic</u>, and it is particularly the case if a fractional exponent has an <u>even denominator</u>.

And thus, we want to <u>be careful with</u> situations where powers have <u>bases negative</u>.

Exactly what fractions then, cannot be used as the exponent if the base is negative?

Not all fractional exponents with even denominators can mess up the power.
Fractions as $\frac{3}{2}$, $\frac{1}{2}$, $-\frac{5}{4}$, $\frac{7}{4}$, -1.5, -0.5, -1.75, 0.3, 1.25, and such are not allowed to be the exponent if the base is negative. What kind of fractions then, are they?

In such a fraction as above, the denominator is even, but the numerator is odd. How then, can the negative base mess up the power when the exponent is such a fraction?

We have in fact, already covered it in the previous section.
Let's now have one more look at though, how the problem can occur.
So to begin with, what do we mean by the problem or trouble?

The problem is that such a power troubled cannot be a real number. That is, in the real number space, which is a set of all real numbers, we do not have a power where the base is negative, and the exponent is such a fraction. What fraction?

- It is a fraction where the denominator is even, but the numerator is odd as $\frac{1}{2}$ or $\frac{3}{4}$.

Suppose $\boldsymbol{n}$ is even, $\boldsymbol{m}$ is odd, and $\boldsymbol{b}$ is negative in a power $\boldsymbol{b}^{\frac{m}{n}}$.
Then, by an exponential identity where $\boldsymbol{(x^u)^v = x^{uv}}$, we can get: $\boldsymbol{b^{\frac{m}{n}} = (b^m)^{\frac{1}{n}}}$. So multiplying 1 by $\boldsymbol{b}^{\frac{m}{n}}$, $\boldsymbol{n}$ times, we should be able to get $\boldsymbol{b^m}$, but we can't get it. Why not?

That's because $\boldsymbol{n}$ is even, and $\boldsymbol{b^m < 0}$, since $\boldsymbol{b < 0}$ and $\boldsymbol{m}$ is odd.
In the real number space, no matter what number it may be, multiplying 1 by it, even number of times, we get no negative number. We can only get a number positive or 0.

Therefore, $\boldsymbol{b}^{\frac{m}{n}}$ does not exist in the real number space if $\boldsymbol{b < 0}$, $\boldsymbol{m}$ is odd, and $\boldsymbol{n}$ is even.

That is, $\boldsymbol{b}^{\frac{m}{n}}$ is not a real number, which is the very problem we have been talking about.

And thus, if a negative base is used, we cannot use a fractional exponent where the numerator is odd, but the denominator is even.

So let's now, take a closer look at how a power can get messed up if we apply such a fractional exponent to a base negative.

To begin with, it is *not always* the case where we can put a radical in the power form. That is to say that it is *not always* the case where we get: $\sqrt[n]{\boldsymbol{b}^m} = \boldsymbol{b}^{\frac{m}{n}}$. Why not?

First, if $\boldsymbol{b} < \mathbf{0}$, and $\boldsymbol{m}$ is *odd*, but $\boldsymbol{n}$ is *even*, we cannot have a power $\boldsymbol{b}^{\frac{m}{n}}$.

That is to say that in the real number space, we cannot raise a number negative to a fractional exponent where the numerator is odd, but the denominator is even.

And the same is true for $\sqrt[n]{\boldsymbol{b}^m}$, too, which is an $\boldsymbol{n}^{\mathbf{th}}$ radical, called an $\boldsymbol{n}^{\text{th}}$ root, too.

So in the real number space, we cannot have a radical $\sqrt[n]{\boldsymbol{b}^m}$ if $\boldsymbol{b} < \mathbf{0}$, and $\boldsymbol{m}$ is odd, but $\boldsymbol{n}$ is even. That is to say that there is no real number by which we can multiply 1, even number of times to get a number negative. In other words, if in a radical, the base is negative, and the degree is even, the radical is not real.

(Note that in a radical $\sqrt[n]{\boldsymbol{b}^m}$, the base is $\boldsymbol{b}^m$, and the degree is $\boldsymbol{n}$.)

So for instance, if $\boldsymbol{b} = \mathbf{-1}$, $\boldsymbol{m} = \mathbf{1}$, and $\boldsymbol{n} = \mathbf{2}$, we get $\sqrt{-\mathbf{1}}$, which is however, not real.

That is, we do not have a real number by which we can multiply 1 twice to get -1.

What then, do we mean by $\sqrt{-\mathbf{1}}$?

It's a radical, of course, but is not real. And putting it in the power form, we can put it this way: $(-1)^{\frac{1}{2}}$, too. So we can do get –1 multiplying 1 by the power $(-1)^{\frac{1}{2}}$, twice or by the radical $\sqrt{-1}$, twice. Such a power or radical is however, not a real number.

It is called an *imaginary* number, and exists in a number space called the imaginary number space, which is a part of the entire number space called the complex number space.

And thus, in the imaginary number space, we have $(-1)^{\frac{1}{2}}$, which is a power not real but imaginary, which equals $\sqrt{-1}$, which is a radical not real but imaginary.

So we can put the idea above this way, too: there is no real number b for which $b^2 = -1$.

In fact, there is no real number b for which $b^p = c$ where $c < 0$, and p is an even integer.

Assuming for instance, $b^4 = -27$, we get $b = \sqrt[4]{-27}$, which is however, not a real number.

That's because there is no real number by which we can multiply 1 four times to get –27.

Anyway, we can set $\sqrt[4]{-27} = \sqrt[4]{(-3)^3} = (-3)^{\frac{3}{4}}$, which is not a real number though.

So in the real number space, we do not have $\sqrt[4]{(-3)^3}$ and $(-3)^{\frac{3}{4}}$, because the base is < 0, and the exponent is a fraction where the numerator is odd, and the denominator is even.

In fact, we can't have in general, $b^{\frac{m}{n}}$ and $\sqrt[n]{b^m}$ if $b < 0$, and m is odd or irrational, but n is an even integer, in the real number space, of course. Why?

Let's suppose $\boldsymbol{b}^{\frac{m}{n}}$ is real even if $\boldsymbol{b} < \mathbf{0}$, and $\boldsymbol{m}$ is odd or irrational, but $\boldsymbol{n}$ is even.
Then, setting $\boldsymbol{A} = \boldsymbol{b}^{\frac{m}{n}}$, we can say that $\boldsymbol{A}$ is real, and we get $\boldsymbol{A} = \boldsymbol{b}^{\frac{m}{n}} \Rightarrow \boldsymbol{A}^{n} = (\boldsymbol{b}^{\frac{m}{n}})^{n} = \boldsymbol{b}^{m}$.

So we get $\boldsymbol{A}^{n} = \boldsymbol{b}^{m}$.

And we have $\boldsymbol{b} < \mathbf{0}$, and $\boldsymbol{m}$ is odd or irrational. So we get $\boldsymbol{b}^{m} < \mathbf{0}$, because $\boldsymbol{m}$ is not even.

However, we have $\boldsymbol{A}^{n} \geq \mathbf{0}$ no matter what real number $\boldsymbol{A}$ may be, because $\boldsymbol{n}$ is even.

Then, since we have $\boldsymbol{A}^{n} \geq \mathbf{0}$ and $\boldsymbol{b}^{m} < \mathbf{0}$, we get $\boldsymbol{A}^{n} \neq \boldsymbol{b}^{m}$, which however, contradicts the equality $\boldsymbol{A}^{n} = \boldsymbol{b}^{m}$ stated above, which in turn, contradicts the statement that $\boldsymbol{A}$ is real.

So $\boldsymbol{A}$ is not real, which means, $\boldsymbol{b}^{\frac{m}{n}}$ is not real, because we have set $\boldsymbol{A} = \boldsymbol{b}^{\frac{m}{n}}$.
That is to say that the supposition stated in the first place is false.

Thus, $\boldsymbol{b}^{\frac{m}{n}}$ is not real if $\boldsymbol{b} < \mathbf{0}$, $\boldsymbol{m}$ is odd or irrational, and $\boldsymbol{n}$ is even.
For instance, in the real number space, we don't have $(-4)^{\frac{3}{2}}$, $(-1)^{\frac{1}{4}}$, $(-16)^{-\frac{1}{4}}$, and $(-9)^{-\frac{\pi}{2}}$.

Next, we have another situation where powers can get messed up when *fractional exponents* are applied to *negative bases*. So let's take a look at what the situation is.

Suppose $\boldsymbol{b} < \mathbf{0}$, and $\boldsymbol{m}$ and $\boldsymbol{n}$ *both* are *even*. Then, we can have one of two cases below:

- Though we can have a radical $\sqrt[n]{\boldsymbol{b}^{m}}$, we might *not* be able to have a power $\boldsymbol{b}^{\frac{m}{n}}$.
- We might get $\sqrt[n]{\boldsymbol{b}^{m}} \neq \boldsymbol{b}^{\frac{m}{n}}$ even if we can have the power $\boldsymbol{b}^{\frac{m}{n}}$.

So we can say that it's *not always* the case where $\boldsymbol{A} = \sqrt[n]{\boldsymbol{b}^{m}} \Rightarrow \boldsymbol{A} = \boldsymbol{b}^{\frac{m}{n}}$.

We always have this, though: $\boldsymbol{A} = \boldsymbol{b}^{\frac{m}{n}} \Rightarrow \boldsymbol{A} = \sqrt[n]{\boldsymbol{b}^{m}}$.

So anyway we *cannot* have $\boldsymbol{A} = \boldsymbol{b}^{\frac{m}{n}} \Leftrightarrow \boldsymbol{A} = \sqrt[n]{\boldsymbol{b}^{m}}$.

It's because it is *not always* the case where $\sqrt[n]{\boldsymbol{b}^{m}} = \boldsymbol{b}^{\frac{m}{n}}$.
We'll cover why not in the next section.

5.1. Problem Bases 2

In the previous section, we had a situation where powers can get messed up when *fractional exponents* are applied to *negative bases*. And the situation is as follows.

Assuming $\boldsymbol{b} < \mathbf{0}$, and $\boldsymbol{m}$ and $\boldsymbol{n}$ *both* are *even*, we can have one of two cases below.

• Though we can have a radical $\sqrt[n]{b^m}$, we might *not* be able to have a power $b^{\frac{m}{n}}$.

• We might get $\sqrt[n]{b^m} \neq b^{\frac{m}{n}}$ even if we can have the power $b^{\frac{m}{n}}$.

Now, let's take a close look at the two cases sated above, and begin with the first case. The first case is saying that if $\boldsymbol{b} < \mathbf{0}$, and $\boldsymbol{m}$ and $\boldsymbol{n}$ *both* are *even*, though we can have a radical $\sqrt[n]{b^m}$, we might *not* be able to have a power $b^{\frac{m}{n}}$. That is to say that if $\boldsymbol{b} < \mathbf{0}$, and $\boldsymbol{m}$ and $\boldsymbol{n}$ *both* are *even*, though a radical $\sqrt[n]{b^m}$ is real, a power $b^{\frac{m}{n}}$ *might not*. Why not?

Can we say that $\sqrt[4]{(-7)^2} = (-7)^{\frac{2}{4}}$?

We have $\frac{2}{4} = \frac{1}{2}$, so we get $\sqrt[4]{(-7)^2} = \sqrt[4]{7^2} = 7^{\frac{2}{4}} = 7^{\frac{1}{2}} = \sqrt{7}$, which is real.

We have this, though: $(-7)^{\frac{2}{4}} = (-7)^{\frac{1}{2}} = \sqrt{-7}$, which is not real.

So we get $\sqrt[4]{(-7)^2} \neq (-7)^{\frac{2}{4}}$.

Thus, if the base $\boldsymbol{b<0}$, $\boldsymbol{b^{\frac{m}{n}}}$ might not be a real number even if $\boldsymbol{m}$ and $\boldsymbol{n}$ both are even.

That's because $\boldsymbol{m}$ is the numerator, $\boldsymbol{n}$ is the denominator in a fraction, and both are even, so simplification of the fractional exponent $\boldsymbol{\frac{m}{n}}$ could end up with a fraction where the numerator is odd, but the denominator is even. For instance, $\boldsymbol{\frac{2}{6}=\frac{1}{3}}$, $\boldsymbol{\frac{4}{6}=\frac{2}{3}}$, but $\boldsymbol{\frac{6}{4}=\frac{3}{2}}$.

So for instance, $\boldsymbol{(-3)^{\frac{2}{6}}=(-3)^{\frac{1}{3}}=\sqrt[3]{-3}=-\sqrt[3]{3}<0}$, and $\boldsymbol{(-3)^{\frac{4}{6}}=(-3)^{\frac{2}{3}}=\sqrt[3]{(-3)^2}=\sqrt[3]{9}>0}$.
However, $\boldsymbol{(-3)^{\frac{6}{4}}=(-3)^{\frac{3}{2}}=\sqrt[2]{(-3)^3}=\sqrt{(-3)^3}=\sqrt{-3^3}}$, which is not real.

So we get $\boldsymbol{\sqrt{-3^3}\neq-\sqrt{3^3}=-3^{\frac{3}{2}}}$, simply because $\boldsymbol{-3^{\frac{3}{2}}}$ is real, but $\boldsymbol{\sqrt{-3^3}}$ is not.
Thus, even if $\boldsymbol{m}$ and $\boldsymbol{n}$ *both* are *even*, if $\boldsymbol{b<0}$, it can be the case where $\boldsymbol{b^{\frac{m}{n}}}$ is not real.

So putting threads together, in the real number space, we can say that

if $\boldsymbol{b<0}$, and $\boldsymbol{m}$ and $\boldsymbol{n}$ *both* are *even*, though we can have a radical $\boldsymbol{\sqrt[n]{b^m}}$, we might *not* be able to have a power $\boldsymbol{b^{\frac{m}{n}}}$. So it is not always the case where $\boldsymbol{A=\sqrt[n]{b^m}\Rightarrow A=b^{\frac{m}{n}}}$.

Next, let's move on to the second case as follows.

If $\boldsymbol{b<0}$, and $\boldsymbol{m}$ and $\boldsymbol{n}$ *both* are *even*. we might get $\boldsymbol{\sqrt[n]{b^m}\neq b^{\frac{m}{n}}}$ even if we can have a power $\boldsymbol{b^{\frac{m}{n}}}$, in the real number space, of course.

We show the truth of the statement above if we can come up with at least one example satisfying each of the two cases as follows: $\boldsymbol{\sqrt[n]{b^m}\neq b^{\frac{m}{n}}}$, and $\boldsymbol{\sqrt[n]{b^m}=b^{\frac{m}{n}}}$.

So first, can we say that $\boldsymbol{\sqrt[6]{(-8)^2}=(-8)^{\frac{2}{6}}}$?

We have $\sqrt[6]{(-8)^2} = \sqrt[6]{8^2} = 8^{\frac{2}{6}} = 8^{\frac{1}{3}} = (2^3)^{\frac{1}{3}} = 2$, but $(-8)^{\frac{2}{6}} = (-8)^{\frac{1}{3}} = \{(-2)^3\}^{\frac{1}{3}} = -2$.

So we get $\sqrt[6]{(-8)^2} \neq (-8)^{\frac{2}{6}}$.

Therefore, if $b < 0$, and m and n both are even, we can get $\sqrt[n]{b^m} \neq b^{\frac{m}{n}}$ even if $b^{\frac{m}{n}}$ is a real number if the exponent $\dfrac{m}{n}$ gets simplified to be a fraction where both the numerator and the denominator are odd.

Well then, can we say that $\sqrt[6]{(-8)^4} = (-8)^{\frac{4}{6}}$?

Yes, we can. We can have $\sqrt[6]{(-8)^4} = \sqrt[6]{8^4} = 8^{\frac{4}{6}} = (2^3)^{\frac{2}{3}} = 2^2$, and also,
$(-8)^{\frac{4}{6}} = (-8)^{\frac{2}{3}} = \{(-2)^3\}^{\frac{2}{3}} = (-2)^2 = 2^2$. So we get $\sqrt[6]{(-8)^4} = (-8)^{\frac{4}{6}}$.

Thus, even if $b < 0$, if m and n both are even, we can get $\sqrt[n]{b^m} = b^{\frac{m}{n}}$ if the exponent $\dfrac{m}{n}$ gets simplified to be a fraction where the numerator is *even*, but the denominator is *odd*.

So we can say that if $b < 0$, and m and n *both* are *even*, we might get $\sqrt[n]{b^m} \neq b^{\frac{m}{n}}$ even if we can have a power $b^{\frac{m}{n}}$.

Thus, taking care of radicals or powers, we want to keep in mind the fact as follows.

If $b < 0$, and m and n *both* are *even*, we can get one of two cases as follows.

- Though a radical $\sqrt[n]{b^m}$ is real, the power $b^{\frac{m}{n}}$ might not be real.
- We might get $\sqrt[n]{b^m} \neq b^{\frac{m}{n}}$ even if $b^{\frac{m}{n}}$ is real.

Now, can we say that if $m = n$, we get $\sqrt[n]{b^m} = b$? In other words, can we get $\sqrt[n]{b^n} = b$?

Yes, we can if $b \geq 0$. If $b < 0$ however, we can't, and yet we can still get something, which is not simple, though. What then is it?

To begin with, if $b < 0$, and m and n both are *equal* and *even*, we get $\sqrt[n]{b^m} = -b$.
That's because we have $\sqrt[n]{b^m} > 0$, and $b < 0$.
More specifically, if m is *even*, and $b < 0$, we get $b^m > 0$. So $\sqrt[n]{b^m}$ is real, and is positive.
Thus, we get $\sqrt[n]{b^m} = -b$, because we have $b < 0$, and $m = n \Rightarrow \dfrac{m}{n} = 1$.

Next, if $b < 0$, and m and n both are *equal* and *odd*, we get $\sqrt[n]{b^m} = b$, because $\sqrt[n]{b^m} < 0$.
More specifically.
if m is *odd*, and $b < 0$, we get $b^m < 0$, so if n is also odd, $\sqrt[n]{b^m}$ is real, and is negative.
Thus, we get $\sqrt[n]{b^m} = b$, because we have $b < 0$, and $m = n \Rightarrow \dfrac{m}{n} = 1$.
Thus, even if $m = n$, we don't want to just assume that $\sqrt[n]{b^m} = b$, because we could get $\sqrt[n]{b^m} = -b$. It all depends on all of m, n, and b.

So we may want to put it the way as follows.

If n is even, we get $\sqrt[n]{b^n} = |b|$. If n is odd, we get $\sqrt[n]{b^n} = b$.
For instance, $\sqrt[4]{(-2)^4}$ is not $(-2)^{\frac{4}{4}} = (-2)^1 = -2$,
but $\sqrt[4]{(-1 \cdot 2)^4} = \sqrt[4]{(-1)^4 \cdot 2^4} = \sqrt[4]{1 \cdot 2^4} = \sqrt[4]{2^4} = 2^{\frac{4}{4}} = 2$, and thus, $\sqrt[4]{(-2)^4}$ is not –2 but 2.
And we can put it this way: |-2|.
On the other hand, we have $\sqrt[3]{(-2)^3} = (-2)^{\frac{3}{3}} = (-2)^1 = -2$, and thus, $\sqrt[3]{(-2)^3} = -2$.

And for the same reason as above, we want to set
$\sqrt[n]{b^{-n}} = \left|\dfrac{1}{b}\right|$ if n is even, and $\sqrt[n]{b^{-n}} = \dfrac{1}{b}$ if n is odd.

Note:
On a radical, operations begin with what's inside the radical sign.
That is, in a radical, operations begin with the base.
And it is particularly so if such a base is a power that has a base *negative*.
So for instance, $\sqrt[2]{(-3)^6} = \sqrt[2]{3^6} = 3^{\frac{6}{2}} = 3^3$, and $\sqrt[2]{(-3)^6} \neq (-3)^{\frac{6}{2}} = (-3)^3 = -3^3 = -27$.

Now, the bottom line is, "What's inside the radical sign $\sqrt[n]{\ }$ has to be $\geq \mathbf{0}$ if $\boldsymbol{n}$ is even."
Why?

First, assuming $\boldsymbol{a} = \sqrt[n]{\boldsymbol{c}}$, where $\boldsymbol{n}$ is even, we get $\boldsymbol{a}^n = (\sqrt[n]{\boldsymbol{c}})^n = \boldsymbol{c} \Rightarrow \boldsymbol{a}^n = \boldsymbol{c}$.

Next, we have $\boldsymbol{a}^n \geq \mathbf{0}$ if $\boldsymbol{n}$ is even whatever the real number $\boldsymbol{a}$ may be.

(e.g. $\mathbf{(-3)^2 = 3^2 > 0}$, and $\mathbf{0^2 = 0}$.)

So we need to have $\boldsymbol{c} \geq \mathbf{0}$, and $\boldsymbol{c}$ is what's inside the radical sign $\sqrt[n]{\ }$ where $\boldsymbol{n}$ is even.

Thus, the statement, "What's inside the radical sign $\sqrt[n]{\ }$ has to be $\geq \mathbf{0}$ if $\boldsymbol{n}$ is even." can be taken as a basic rule for an $\boldsymbol{n}^{\text{th}}$ radical where $\boldsymbol{n}$ is even.

So for instance, we can have these:

$$\sqrt{x^2 - 2x - 3} \Rightarrow x^2 - 2x - 3 = (x-3)(x+1) \geq 0 \Rightarrow x \geq 3 \text{ or } x \leq -1.$$

$$\sqrt{-(x^2 - 2x - 3)} \Rightarrow x^2 - 2x - 3 = (x-3)(x+1) \leq 0 \Rightarrow -1 \leq x \leq 3.$$

$$\sqrt{\frac{1}{1-x}} \Rightarrow 1 - x > 0 \Rightarrow x < 1. \qquad \sqrt[4]{\frac{-1}{1-x}} \Rightarrow 1 - x < 0 \Rightarrow x > 1.$$

$$\sqrt{\frac{1}{-(x^2 - 2x - 3)}} \Rightarrow x^2 - 2x - 3 = (x-3)(x+1) < 0 \Rightarrow -1 < x < 3.$$

What then about $\sqrt[3]{\dfrac{1}{1-x}}$?

"*What's inside* the radical sign $\sqrt[n]{\ }$ can be *any real* number if $\boldsymbol{n}$ is *odd*."

Therefore, $\boldsymbol{x}$ can be any real number except 1. Why not 1 though?

We know in the expression above, $(\mathbf{1} - \boldsymbol{x})$ is the denominator.
So $(\mathbf{1} - \boldsymbol{x})$ can be any real number other than 0, since no division by 0 is allowed.

Thus, $\boldsymbol{x}$ can be any real number other than 1.

Note that $\dfrac{\mathbf{1}}{\mathbf{1}-\boldsymbol{x}}$ cannot be 0 for any value of $\boldsymbol{x}$.

And let's now, practice some more algebra. This book is in fact, about algebra.
So next, for another instance, we can put the same fraction many ways as follows.

$$\sqrt[3]{\frac{\mathbf{1}}{\boldsymbol{x}^2-\mathbf{2}\boldsymbol{x}-\mathbf{3}}}=\frac{\mathbf{1}}{\sqrt[3]{\boldsymbol{x}^2-\mathbf{2}\boldsymbol{x}-\mathbf{3}}}=\frac{\mathbf{1}}{(\boldsymbol{x}^2-\mathbf{2}\boldsymbol{x}-\mathbf{3})^{\frac{1}{3}}}=(\boldsymbol{x}^2-\mathbf{2}\boldsymbol{x}-\mathbf{3})^{-\frac{1}{3}}.$$

And we can get $\sqrt[3]{\dfrac{\mathbf{1}}{\boldsymbol{x}^2-\mathbf{2}\boldsymbol{x}-\mathbf{3}}} \Rightarrow \boldsymbol{x}^2-\mathbf{2}\boldsymbol{x}-\mathbf{3}=(\boldsymbol{x}-\mathbf{3})(\boldsymbol{x}+\mathbf{1})\neq\mathbf{0}\Rightarrow\boldsymbol{x}\neq\mathbf{3}$ and $\boldsymbol{x}\neq\mathbf{-1}$.

That is to say that $\boldsymbol{x}$ can be any number other than 3 and –1.   Why?

We have this: "*What's inside* the radical sign $\sqrt[n]{\ }$ can be *any real* number if $\boldsymbol{n}$ is *odd*."
So in the radical above, what's inside the radical sign can be any real, and thus, can be positive, 0, or negative.

However, $(\boldsymbol{x}^2-\mathbf{2}\boldsymbol{x}-\mathbf{3})$ is the denominator, and thus, cannot be 0.

So $(\boldsymbol{x}^2-\mathbf{2}\boldsymbol{x}-\mathbf{3})$ can be any real number other than 0, (that is, positive or negative only).

Meanwhile, we have $\boldsymbol{x}^2-\mathbf{2}\boldsymbol{x}-\mathbf{3}=(\boldsymbol{x}-\mathbf{3})(\boldsymbol{x}+\mathbf{1})$.

So if $\boldsymbol{x}^2-\mathbf{2}\boldsymbol{x}-\mathbf{3}=\mathbf{0}$, we get $(\boldsymbol{x}-\mathbf{3})(\boldsymbol{x}+\mathbf{1})=\mathbf{0}\Rightarrow\boldsymbol{x}=\mathbf{3}$ or $\boldsymbol{x}=\mathbf{-1}$.

That is to say that if $\boldsymbol{x}^2-\mathbf{2}\boldsymbol{x}-\mathbf{3}\neq\mathbf{0}$, we get $(\boldsymbol{x}-\mathbf{3})(\boldsymbol{x}+\mathbf{1})\neq\mathbf{0}\Rightarrow\boldsymbol{x}\neq\mathbf{3}$ and $\boldsymbol{x}\neq\mathbf{-1}$.

Thus, if $\boldsymbol{x}^2-\mathbf{2}\boldsymbol{x}-\mathbf{3}$ can be any real number other than 0, $\boldsymbol{x}$ can be any number other than 3 and –1.
Let's now, take another example.

$\sqrt{\dfrac{\boldsymbol{x}}{\mathbf{1}-\boldsymbol{x}}}\Rightarrow\dfrac{\boldsymbol{x}}{\mathbf{1}-\boldsymbol{x}}\geq\mathbf{0}$. So we get $\boldsymbol{x}(\mathbf{1}-\boldsymbol{x})>\mathbf{0}$ or $\boldsymbol{x}=\mathbf{0}$, but $\mathbf{1}-\boldsymbol{x}\neq\mathbf{0}$.   Why?

We have this: "What's inside the radical sign $\sqrt[n]{\ }$ has to be $\geq \mathbf{0}$ if $\boldsymbol{n}$ is even."

So we get $\dfrac{\boldsymbol{x}}{\mathbf{1}-\boldsymbol{x}} \geq \mathbf{0}$, because in this case we have $\boldsymbol{n} = \mathbf{2}$, which is even.

Then, first, we can notice that $(\mathbf{1} - \boldsymbol{x})$ is a denominator, so we get $\mathbf{1} - \boldsymbol{x} \neq \mathbf{0}$.

Next, we can put the expression above, this way, too: $\dfrac{\boldsymbol{x}}{\mathbf{1}-\boldsymbol{x}} > \mathbf{0}$, or $\dfrac{\boldsymbol{x}}{\mathbf{1}-\boldsymbol{x}} = \mathbf{0}$.

Thus, taking care of $\dfrac{\boldsymbol{x}}{\mathbf{1}-\boldsymbol{x}} = \mathbf{0}$ first, we get $\boldsymbol{x} = \mathbf{0}$.

Next, moving on to $\dfrac{\boldsymbol{x}}{\mathbf{1}-\boldsymbol{x}} > \mathbf{0}$, we can notice that $\boldsymbol{x}$ and $(\mathbf{1} - \boldsymbol{x})$ both have the same sign.

The negative over the negative is positive, and so is the positive over the positive.
And the product of two negatives is positive, and so is the product of two positives.
So we get $\boldsymbol{x}(\mathbf{1} - \boldsymbol{x}) > \mathbf{0}$.

Thus, putting threads together, we have $\dfrac{\boldsymbol{x}}{\mathbf{1}-\boldsymbol{x}} \geq \mathbf{0} \Rightarrow \mathbf{1} - \boldsymbol{x} \neq \mathbf{0}$, $\boldsymbol{x} = \mathbf{0}$, or $\boldsymbol{x}(\mathbf{1} - \boldsymbol{x}) > \mathbf{0}$.

Then, beginning again with $\boldsymbol{x}(\mathbf{1} - \boldsymbol{x}) > \mathbf{0}$, we get $\boldsymbol{x}(\boldsymbol{x} - \mathbf{1}) < \mathbf{0} \Rightarrow \mathbf{0} < \boldsymbol{x} < \mathbf{1}$.

Next, moving on to $\mathbf{1} - \boldsymbol{x} \neq \mathbf{0}$, we get $\boldsymbol{x} \neq \mathbf{1}$. And we have $\boldsymbol{x} = \mathbf{0}$, too.

So now, putting threads together again, we have $\mathbf{0} < \boldsymbol{x} < \mathbf{1}$, $\boldsymbol{x} \neq \mathbf{1}$, or $\boldsymbol{x} = \mathbf{0}$.

In other words, we have $\mathbf{0} \leq \boldsymbol{x} < \mathbf{1}$. That is to say that $\sqrt{\dfrac{\boldsymbol{x}}{\mathbf{1}-\boldsymbol{x}}} \Rightarrow \mathbf{0} \leq \boldsymbol{x} < \mathbf{1}$.

Therefore, if we want $\sqrt{\dfrac{\boldsymbol{x}}{\mathbf{1}-\boldsymbol{x}}}$ to be real, we have to have $\mathbf{0} \leq \boldsymbol{x} < \mathbf{1}$.

Next, let's see now how $\sqrt[n]{\boldsymbol{b}^{m}}$ can be real even if $\boldsymbol{b} < \mathbf{0}$, if $\boldsymbol{m}$ and $\boldsymbol{n}$ both are even.

If $\boldsymbol{n}$ is even, and we want $\sqrt[n]{\boldsymbol{b}^{m}}$ to be real, we need to have $\boldsymbol{b}^{m} \geq \mathbf{0}$ due to the basic rule explained earlier. So in this case, if $\boldsymbol{b} < \mathbf{0}$, $\boldsymbol{m}$ has to be even.

Thus, even if $\boldsymbol{b} < \mathbf{0}$, the radical $\sqrt[n]{\boldsymbol{b}^{m}}$ is real if $\boldsymbol{m}$ and $\boldsymbol{n}$ *both* are *even*.

For instance, we can have $\sqrt[4]{(-7)^2} = \sqrt[4]{7^2}$, and $\sqrt[4]{(-7)^{-2}} = \sqrt[4]{\frac{1}{(-7)^2}} = \sqrt[4]{\frac{1}{7^2}}$, but *cannot* have

$\sqrt{-6}$, $\sqrt[4]{-6}$, and $\sqrt[4]{(-2)^{-3}} = \sqrt[4]{\frac{1}{(-2)^3}} = \sqrt[4]{-\frac{1}{8}}$, since they are not real, of course.

And if $\boldsymbol{n}$ is odd, $\sqrt[n]{b^m}$ is real for any $\boldsymbol{m}$ even if $\boldsymbol{b < 0}$.

That's because we can get a real number by which we can multiply 1, an odd number of times to get a negative number, and the real number we get is negative, too, of course.

In fact, we can have $\sqrt[n]{b^m}$ for any $\boldsymbol{b}$ nonzero and any $\boldsymbol{m}$ if $\boldsymbol{n}$ is odd, in the real number space, of course. For instance, we can have these:

$\sqrt[5]{7^{0.3}} = 7^{\frac{0.3}{5}} = 7^{\frac{3}{50}} = \sqrt[50]{7^3}$, and $\sqrt[5]{(-7)^{-3}} = (-7)^{-\frac{3}{5}} = -7^{-\frac{3}{5}} = -\dfrac{1}{7^{\frac{3}{5}}} = -\dfrac{1}{\sqrt[5]{7^3}}$.

And taking more instances, we can have these:

$\sqrt[3]{\left(-\dfrac{1}{2}\right)^{-\sqrt{5}}} = \sqrt[3]{(-2^{-1})^{-\sqrt{5}}} = \sqrt[3]{-2^{\sqrt{5}}} = (-2)^{\frac{\sqrt{5}}{3}} = -2^{\frac{\sqrt{5}}{3}} = -\sqrt[3]{2^{\sqrt{5}}}$,

$\sqrt[3]{-7}$, $\sqrt[5]{(-7)^{-3}}$, $\sqrt[3]{(-2)^9} = -2^{\frac{9}{3}} = -2^3$, and $\sqrt[3]{(-2)^{-9}} = (-2)^{-3} = -\dfrac{1}{2^3} = -\dfrac{1}{8}$.

So note that the radical is negative if the base is negative and the degree is odd.
That is, we can get this: $\sqrt[n]{c} < 0$ if $\boldsymbol{c < 0}$, and $\boldsymbol{n}$ is odd.
And of course, we can get this, too: $c^{\frac{1}{n}} < 0$ if $\boldsymbol{c < 0}$, and $\boldsymbol{n}$ is odd.
How though, for instance, are $\sqrt[3]{-3}$ and $\sqrt[3]{(-3)^5}$ negative real numbers?

First, if we get a negative real number multiplying 1 by a particular number, an odd number of times, the particular number is negative and real.

For instance, we have $\mathbf{1\cdot(-2)\cdot(-2)\cdot(-2) = (-2)^3 = -8}$.

So next, multiplying 1 by $\sqrt[3]{-3}$, three times, we get –3, which is negative and real.
Thus, $\sqrt[3]{-3}$ is a real number negative.

What if however, we get a negative real number multiplying 1 by a particular number, an even number of times?

And next, what do we mean by $\sqrt[n]{a}$?

We mean that multiplying 1 by $\sqrt[n]{\boldsymbol{a}}$, $\boldsymbol{n}$ times, we get $\boldsymbol{a}$.

So for instance, multiplying 1 by $\sqrt[3]{\mathbf{-3}}$, three times, we get –3.

Thus, $\sqrt[3]{\mathbf{-3}}$ is a negative real number, and is of course, equal to $\mathbf{-3^{\frac{1}{3}}}$.

So next, what do we mean by $\sqrt[n]{\boldsymbol{b^m}}$?

We mean that multiplying 1 by $\sqrt[n]{\boldsymbol{b^m}}$, $\boldsymbol{n}$ times, we get $\boldsymbol{b^m}$.

So for instance, multiplying 1 by $\sqrt[3]{\mathbf{(-3)^5}}$, three times, we get $(-3)^5$, which is negative.

Thus, $\sqrt[3]{\mathbf{(-3)^5}}$ is a negative real number, and is of course, equal to $\mathbf{-\sqrt[3]{3^5} = -3^{\frac{5}{3}}}$.

So doing algebra with powers and radicals, we may want to be careful with the base and the exponent.

5.2. Problem Bases 3

Radical signs can be nested. Then, operations begin with what's inside the inner most radical sign. So for instance,

$$\sqrt[p]{\sqrt[q]{\sqrt[r]{\sqrt[s]{b^t}}}} = \sqrt[p]{\sqrt[q]{\sqrt[r]{\sqrt[s]{c}}}} = \sqrt[p]{\sqrt[q]{\sqrt[r]{d}}} = \sqrt[p]{\sqrt[q]{e}} = \sqrt[p]{f} = g,$$

where $c = b^t$, $d = \sqrt[s]{c}$, $e = \sqrt[r]{d}$, and $f = \sqrt[q]{e}$.

And for specific instance, $\sqrt[4]{\sqrt[3]{\sqrt[5]{\sqrt[2]{2^{120}}}}} = \sqrt[4]{\sqrt[3]{\sqrt[5]{2^{60}}}} = \sqrt[4]{\sqrt[3]{2^{12}}} = \sqrt[4]{2^4} = 2$.

What then about $\sqrt[4]{\sqrt[2]{\sqrt[5]{\sqrt[3]{-2^{120}}}}}$?

Such a radical is not real, that is, it's not a real number. Why not though?

To begin with, we can have $\sqrt[3]{-2^{120}} = -2^{\frac{120}{3}} = -2^{40}$, so we get $\sqrt[4]{\sqrt[2]{\sqrt[5]{\sqrt[3]{-2^{120}}}}} = \sqrt[4]{\sqrt[2]{\sqrt[5]{-2^{40}}}}$.

Next, we can have $\sqrt[5]{-2^{40}} = -2^{\frac{40}{5}} = -2^8$, but we cannot get $\sqrt[4]{\sqrt[2]{-2^8}}$. Why not though?

We can't have $\sqrt[2]{-2^8}$, because the base is negative, but the degree is even.

So we can't get $\sqrt[4]{\sqrt[2]{-2^8}}$, and in turn, we can't have $\sqrt[4]{\sqrt[2]{\sqrt[5]{-2^{40}}}}$ and $\sqrt[4]{\sqrt[2]{\sqrt[5]{\sqrt[3]{-2^{120}}}}}$, either.

Thus, not all radicals in the form of $\sqrt[p]{\sqrt[q]{\sqrt[r]{\sqrt[s]{b^t}}}}$ where p, q, r, and s are integers are real.

In what case then such a radical as $\sqrt[p]{\sqrt[q]{\sqrt[r]{\sqrt[s]{b^t}}}}$ is real?

Suppose $A = \sqrt[p]{\sqrt[q]{\sqrt[r]{\sqrt[s]{b^t}}}}$. Then, A is real in each of the three cases as follows.

(If $b = 0$, and $t > 0$, A is 0, since $b^t = 0$. For the case where $t \le 0$, refer to the section **1**.)

- $b > 0$.
- $b < 0$, and t is even.
- $b < 0$, and the product $pqrst$ is odd (in other words, all of p, q, r, s, and t are odd.)

Can we set $\sqrt[p]{\sqrt[q]{\sqrt[r]{\sqrt[s]{b^t}}}} = b^{\frac{t}{pqrs}}$, though?

We can *simply* set it that way if $b > 0$. For instance, $\sqrt[4]{\sqrt[3]{\sqrt[5]{\sqrt[2]{2^{120}}}}} = 2^{\frac{120}{4 \cdot 3 \cdot 5 \cdot 2}} = 2^{\frac{120}{120}} = 2^1 = 2$.

What if $b \le 0$?

If $b = 0$, we can do so if $t > 0$. Then, we get $0 = 0$. (For $t \le 0$, refer to the section **1**.)

If $b < 0$, we can set it that way if the product $pqrst$ is odd.

That is, if $b < 0$, and all of p, q, r, s, and t are odd, we can do so. For instance,

$\sqrt[7]{\sqrt[3]{\sqrt[5]{\sqrt[3]{(-2)^{45}}}}} = (-2)^{\frac{45}{7\cdot3\cdot5\cdot3}} = (-2)^{\frac{1}{7}} = \sqrt[7]{-2} = -\sqrt[7]{2}$. What if the product $pqrst$ is even?

If $b < 0$, we can set it that way if t is even, and the product $pqrs$ is odd.
That is, if $b < 0$, and all of p, q, r, and s are odd, but t is even, we can do so. For instance,

$\sqrt[7]{\sqrt[3]{\sqrt[5]{\sqrt[3]{(-2)^{36}}}}} = (-2)^{\frac{36}{7\cdot3\cdot5\cdot3}} = 2^{\frac{4}{35}} = \sqrt[35]{2^4}$. What if the product $pqrs$ is even?

If $b < 0$, t is even, and the product $pqrs$ is even, we could.
Well then, what else do we need to set it that way?

In addition to the condition that $b < 0$, t is even, and $pqrs$ is even, if in the power $b^{\frac{t}{pqrs}}$, the exponent $\frac{t}{pqrs}$ gets simplified to be another fraction where the numerator is even, but the denominator is odd (e.g. $\frac{6}{9} = \frac{2}{3}$), we can.

That is, if $b < 0$, t is even, and $\dfrac{t}{pqrs}$ gets simplified to be $\dfrac{\text{even}}{\text{odd}}$, we can. For instance,

$\sqrt[7]{\sqrt[5]{\sqrt[2]{\sqrt[3]{(-2)^{28}}}}} = (-2)^{\frac{28}{7\cdot5\cdot2\cdot3}} = (-2)^{\frac{4}{5\cdot2\cdot3}} = (-2)^{\frac{2}{15}} = 2^{\frac{2}{15}}$. What other case then, can we have?

We can have one more case where $\sqrt[p]{\sqrt[q]{\sqrt[r]{\sqrt[s]{b^t}}}}$ and $b^{\frac{t}{pqrs}}$ both are real, but we cannot set

$\sqrt[p]{\sqrt[q]{\sqrt[r]{\sqrt[s]{b^t}}}} = b^{\frac{t}{pqrs}}$, that is, we get $\sqrt[p]{\sqrt[q]{\sqrt[r]{\sqrt[s]{b^t}}}} \neq b^{\frac{t}{pqrs}}$.

What then, exactly is the case, and how are they not equal?

In addition to the condition that $\boldsymbol{b} < \mathbf{0}$, $\boldsymbol{t}$ is even, and $\boldsymbol{pqrs}$ is even, if in the power $\boldsymbol{b}^{\frac{t}{pqrs}}$, the exponent $\frac{t}{pqrs}$ gets reduced to a fraction where the numerator and denominator both are odd (e.g. $\frac{6}{10} = \frac{3}{5}$), the radical and the power both are real, but are not equal.

For instance, $\sqrt[7]{\sqrt[4]{\sqrt[5]{\sqrt[3]{(-2)^{36}}}}}$ and $(-2)^{\frac{36}{7\cdot4\cdot5\cdot3}}$ both are real, but are not equal. Why not?

We have $\sqrt[7]{\sqrt[4]{\sqrt[5]{\sqrt[3]{(-2)^{36}}}}} = \sqrt[7]{\sqrt[4]{\sqrt[5]{\sqrt[3]{2^{36}}}}}$, which is positive, but we have

$(-2)^{\frac{36}{7\cdot4\cdot5\cdot3}} = (-2)^{\frac{9}{7\cdot5\cdot3}} = (-2)^{\frac{3}{35}}$, which is negative, so we get $\sqrt[7]{\sqrt[4]{\sqrt[5]{\sqrt[3]{(-2)^{36}}}}} \neq (-2)^{\frac{36}{7\cdot4\cdot5\cdot3}}$.

Now, putting threads together, we have five cases where we can set $\sqrt[p]{\sqrt[q]{\sqrt[r]{\sqrt[s]{b^t}}}} = b^{\frac{t}{pqrs}}$.

And they are as follows.

- If $\boldsymbol{b} > \mathbf{0}$, we get $\sqrt[p]{\sqrt[q]{\sqrt[r]{\sqrt[s]{b^t}}}} = b^{\frac{t}{pq\dots rs}} > 0$.
- If $\boldsymbol{b} = \mathbf{0}$, and $\boldsymbol{t} > \mathbf{0}$, we get $\mathbf{0} = \mathbf{0}$. (We can see 'why not $\boldsymbol{t} \le \mathbf{0}$' in the section **1**.)
- If $\boldsymbol{b} < \mathbf{0}$ and the product $\boldsymbol{pqrst}$ is odd, we get $\sqrt[p]{\sqrt[q]{\sqrt[r]{\sqrt[s]{b^t}}}} = b^{\frac{t}{pqrs}} < 0$.
- If $\boldsymbol{b} < \mathbf{0}$, $\boldsymbol{t}$ is even, and the product $\boldsymbol{pqrs}$ is odd, we get $\sqrt[p]{\sqrt[q]{\sqrt[r]{\sqrt[s]{b^t}}}} = b^{\frac{t}{pqrs}} > 0$.
- If $\boldsymbol{b} < \mathbf{0}$, $\boldsymbol{t}$ is even, and $\frac{t}{pqrs}$ gets reduced to $\frac{\text{even}}{\text{odd}}$, we get $\sqrt[p]{\sqrt[q]{\sqrt[r]{\sqrt[s]{b^t}}}} = b^{\frac{t}{pqrs}} > 0$.

The same idea applies to powers, too. So expanding a power as $\{(b^x)^y\}^z$, the operations proceed *from inside to outside*, and it is particularly the case if the base is *negative*.

If the base is positive though, the direction of operations doesn't matter.

So for instance, we have an exponential identity where $(b^x)^y = (b^y)^x = b^{xy}$, where $b > 0$.

That is, the identity above always works if the base $b > 0$.

If the base $b < 0$ however, the identity above might not work because of the direction of the operation: the operation proceeds *from inside to outside*.

For instance, we have $\{(-2)^4\}^{\frac{3}{2}} \neq \{(-2)^{\frac{3}{2}}\}^4$. How?

Expanding the power $\{(-2)^{\frac{3}{2}}\}^4$, we want to process $(-2)^{\frac{3}{2}}$ first, which however, is not real, and thus, cannot even be processed. So $\{(-2)^{\frac{3}{2}}\}^4$ is not a real number.

On the other hand, we can have $\{(-2)^4\}^{\frac{3}{4}} = (2^4)^{\frac{3}{4}} = 2^{\frac{12}{4}} = 2^3$.

So expanding $\{(-2)^4\}^{\frac{3}{4}}$, we *cannot* do this: $\{(-2)^4\}^{\frac{3}{4}} = (-2)^{\frac{12}{4}} = (-2)^3 = -8$.

Of course, we can have cases where the identity above works even if the base is negative.

We can have $\{(-2)^3\}^{\frac{1}{3}} = \sqrt[3]{(-2)^3} = \sqrt[3]{(-2)(-2)(-2)} = \sqrt[3]{-2}\cdot\sqrt[3]{-2}\cdot\sqrt[3]{-2} = \left(\sqrt[3]{-2}\right)^3 = -2$, and $\{(-2)^{\frac{1}{3}}\}^3 = \left(\sqrt[3]{-2}\right)^3 = -2$. Thus, we can have $\{(-2)^3\}^{\frac{1}{3}} = \{(-2)^{\frac{1}{3}}\}^3$.

So expanding $\{(-2)^3\}^{\frac{1}{3}}$ and $\{(-2)^{\frac{1}{3}}\}^3$, we can just do it the way below.

$\{(-2)^3\}^{\frac{1}{3}} = (-2)^{3\cdot\frac{1}{3}} = -2$, and $\{(-2)^{\frac{1}{3}}\}^3 = (-2)^{\frac{1}{3}\cdot 3} = -2$.

For another instance, we can have $\{(-2)^{-\frac{1}{3}}\}^3 = \{(-2)^3\}^{-\frac{1}{3}}$. That's because,

we have $(-2)^{-\frac{1}{3}} = \dfrac{1}{(-2)^{\frac{1}{3}}} = \dfrac{1}{\sqrt[3]{-2}} = \dfrac{1}{-\sqrt[3]{2}}$.

So we get $\{(-2)^{-\frac{1}{3}}\}^3 = (\dfrac{1}{-\sqrt[3]{2}})^3 = -\dfrac{1}{(-\sqrt[3]{2})^3} = \dfrac{1}{-2} = -\dfrac{1}{2}$.

Also, we can get $\{(-2)^3\}^{-\frac{1}{3}} = \dfrac{1}{\{(-2)^3\}^{\frac{1}{3}}} = \dfrac{1}{\sqrt[3]{(-2)^3}} = \dfrac{1}{-2} = -\dfrac{1}{2}$.

So expanding $\{(-2)^{-\frac{1}{3}}\}^3$ and $\{(-2)^3\}^{-\frac{1}{3}}$, we can just do it the way below:

$\{(-2)^{-\frac{1}{3}}\}^3 = \{(-2)^3\}^{-\frac{1}{3}} = (-2)^{-\frac{3}{3}} = (-2)^{-1} = -\dfrac{1}{2}$.

Now, another difference between $\sqrt[n]{b^m}$ and $b^{\frac{m}{n}}$ is that in the radical $\sqrt[n]{b^m}$, n can only be an integer $\geq \mathbf{2}$, but in the power $b^{\frac{m}{n}}$, n can be any nonzero real number.

So for instance, we can have $(-2)^{\frac{2}{\pi}} = \{(-2)^{\frac{1}{\pi}}\}^2 = \{(-2)^2\}^{\frac{1}{\pi}} = 4^{\frac{1}{\pi}} = \mathbf{1.55468...} > \mathbf{0}$, where π is an irrational number, and is the circular ratio, which is 3.141592…

However, we do not have such a radical as $\sqrt[\pi]{(-2)^2}$ because n in $\sqrt[n]{b^m}$ is an integer $\geq \mathbf{2}$.

In particular, if $b > 0$ in $b^{\frac{m}{n}}$, n can be any nonzero real number, and m can be any real number, too. In fact, if $b > 0$ in a power b^x, x can be any real number.

So for instance, we can have $4^{\frac{e}{\pi}} = \mathbf{3.31845\ldots}$, where e is irrational, called the Euler's number, and is 2.7182818… So a power b^x always works if $b > 0$.

• In other words, *bases negative* or 0 are problematic.

That is to say that if the base is negative, it can be the case where the radical or the power is not a number real but a number called imaginary.

What is an example of an imaginary number though?

The typical example is $(-1)^{\frac{1}{2}}$, which is $\sqrt{-1}$, which is often denoted by a letter $\boldsymbol{i}$. Where in the world then can $(-1)^{\frac{1}{2}}$ exist?

In the real number space, we do not have $(-1)^{\frac{1}{2}}$, which exists however, in the imaginary number space, which is a part of the complex number space, which has the real number space, too. What in the world is the complex number space, then?

The complex number space is a conceptual place where all numbers belong, and thus, is the entire number space.

For instance, we have $(\sqrt{-1})^2 = \{(-1)^{\frac{1}{2}}\}^2 = (-1)^1 = -1$ in the complex number space.

So we have $(\sqrt{-1})^2 = \sqrt{-1} \cdot \sqrt{-1} \neq \sqrt{(-1)(-1)} = \sqrt{(-1)^2} = \sqrt{1} = 1$.

That is, we have $\{(-1)^{\frac{1}{2}}\}^2 \neq \{(-1)^2\}^{\frac{1}{2}}$.

For another example, we cannot have $\{(-4)^{\frac{3}{4}}\}^4$ because we don't have $(-4)^{\frac{3}{4}}$ in the real number space, so we get $\{(-4)^{\frac{3}{4}}\}^4 \neq \{(-4)^4\}^{\frac{3}{4}} = (4^4)^{\frac{3}{4}} = 4^{\frac{12}{4}} = 4^3$.

Note that $\{(-4)^4\}^{\frac{3}{4}} \neq (-4)^{\frac{12}{4}} = (-4)^3 = -4^3$, and $\{(-4)^4\}^{\frac{3}{4}} = (4^4)^{\frac{3}{4}} = 4^{\frac{12}{4}} = 4^3$.

So we have been through quite a few cases and examples where a negative base can be a problem. Let's now, put them in a summary.

Normally, using numbers, we use real numbers, which are often just called real.

And using a number, we can use it in a form of a power or a radical, which is a number, too. So unless specified otherwise, the powers or radicals we use are real, too.

Now, expressing a power with a fractional exponent, we can use either of two notations.

• In power notation, we can put it this way: $\boldsymbol{b^{\frac{m}{n}}}$, which can be called the power form, too.

• In radical notation, we can put it this way: $\sqrt[n]{\boldsymbol{b^m}}$, often called a radical or an $\boldsymbol{n}^{\text{th}}$ root .

Whichever notation we may use though, we want to make sure that the power or the radical is *real*. If the base is negative or 0, the power might not be a real number. So bases not positive are problematic. What's wrong with such a base though?

First, if the base is 0, we cannot use a negative number as the exponent.
It's because no division by 0 is allowed. So for instance, $0^{-2} = \frac{1}{0^2} = \frac{1}{0}$ is not allowed.

Next, if $\boldsymbol{b < 0}$, and $\boldsymbol{m}$ is *odd*, but $\boldsymbol{n}$ is *even*, the power $\boldsymbol{b^{\frac{m}{n}}}$ is not real.

And the same is true for a radical $\sqrt[n]{\boldsymbol{b^m}}$, too.

Assuming for instance, $\boldsymbol{b^4 = -27}$, we get $\boldsymbol{b} = \sqrt[4]{\boldsymbol{-27}}$, which however, is not a real number.

And in general, $\boldsymbol{b^{\frac{m}{n}}}$ and $\sqrt[n]{\boldsymbol{b^m}}$ both are not real if $\boldsymbol{b < 0}$, and $\boldsymbol{m}$ is odd or irrational, but $\boldsymbol{n}$ is an even integer.

Next, we have another situation where powers can get messed up when fractional exponents are applied to negative bases, and the situation is as follows.

Suppose $\boldsymbol{b} < \mathbf{0}$, and $\boldsymbol{m}$ and $\boldsymbol{n}$ *both* are *even*. Then, we can have one of two cases below.

- Though a radical $\sqrt[n]{b^m}$ is real, the power $b^{\frac{m}{n}}$ might not be real.
- We might get $\sqrt[n]{b^m} \neq b^{\frac{m}{n}}$ even if the power $b^{\frac{m}{n}}$ is real.

That is to say that we do *not always* get $\sqrt[n]{b^m} = b^{\frac{m}{n}}$.

And the bottom line is

"What's inside the radical sign $\sqrt[n]{\ }$ has to be $\geq \mathbf{0}$ if $\boldsymbol{n}$ is even.", which can be therefore, taken as a basic rule for an $\boldsymbol{n}^{\text{th}}$ radical where $\boldsymbol{n}$ is even. And of course,

"*What's inside* the radical sign $\sqrt[n]{\ }$ can be *any real* number if $\boldsymbol{n}$ is *odd*."

So to begin with, why do we get the situation as follows?

If the base $\boldsymbol{b} < \mathbf{0}$, $b^{\frac{m}{n}}$ might not be a real number even if $\boldsymbol{m}$ and $\boldsymbol{n}$ both are even.

That's because $\boldsymbol{m}$ and $\boldsymbol{n}$ both are even, the simplification of the exponent $\dfrac{m}{n}$ could end up with a fraction where the numerator is odd, but the denominator is even.

So for instance, we get $\sqrt[4]{(-7)^2} \neq (-7)^{\frac{2}{4}}$.

Next, what do we mean by the situation as follows?

If $\boldsymbol{b} < \mathbf{0}$, and $\boldsymbol{m}$ and $\boldsymbol{n}$ *both* are *even*, though a radical $\sqrt[n]{b^m}$ is real, the power $b^{\frac{m}{n}}$ might not be real.

It means that we do not always get: $A = \sqrt[n]{b^m} \Rightarrow A = b^{\frac{m}{n}}$. In other words,

If $\boldsymbol{b} < \mathbf{0}$, and $\boldsymbol{m}$ and $\boldsymbol{n}$ *both* are *even*. we might get: $\sqrt[n]{b^m} \neq b^{\frac{m}{n}}$ even if the power $b^{\frac{m}{n}}$ is real.

So for instance, we can get $\sqrt[6]{(-8)^2} \neq (-8)^{\frac{2}{6}}$.

We can have this, too, though $\sqrt[6]{(-8)^4} = (-8)^{\frac{4}{6}} = (-8)^{\frac{2}{3}} = 8^{\frac{2}{3}}$, and $\sqrt[6]{(-8)^4} = \sqrt[6]{8^4} = \sqrt[3]{8^2}$.

So even if $b < 0$, if m and n both are even, we can get $\sqrt[n]{b^m} = b^{\frac{m}{n}}$ if the exponent $\dfrac{m}{n}$ gets simplified to be a fraction where the numerator is *even* but the denominator is *odd*.

And even if $m = n$, we don't want to just assume that $\sqrt[n]{b^m} = b$, because we could get: $\sqrt[n]{b^m} = -b$. It all depends on all of m, n, and b. For instance, we have

$\sqrt[4]{(-2)^4} = \sqrt[4]{2^4} = 2^{\frac{4}{4}} = 2$, and $\sqrt[3]{-2^3} = \sqrt[3]{(-2)^3} = (-2)^{\frac{3}{3}} = (-2)^1 = -2$.

So we may want to put it this way:

- If n is even, we get $\sqrt[n]{b^n} = |b|$. And if n is odd, we get $\sqrt[n]{b^n} = b$.

And for the same reason as above, we want to set

- $\sqrt[n]{b^{-n}} = \left|\frac{1}{b}\right|$ if n is even, and $\sqrt[n]{b^{-n}} = \frac{1}{b}$ if n is odd.

On a radical, operations begin with what's inside the radical sign.
That is, in a radical, operations begin with the base.
And it is particularly so if such a base is a power that has a base *negative*.

So for instance, $\sqrt[2]{(-3)^6} = \sqrt[2]{3^6} = 3^{\frac{6}{2}} = 3^3$, and $\sqrt[2]{(-3)^6} \neq (-3)^{\frac{6}{2}} = (-3)^3 = -3^3 = -27$.

Suppose next, $A = \sqrt[p]{\sqrt[q]{\sqrt[r]{\sqrt[s]{b^t}}}}$. Then, A is real in each of three cases as follows.

(If $b = 0$, and $t > 0$, A is 0 since $b^t = 0$. For the case where $t \leq 0$, refer to the section **1**.)

- $b > 0$.
- $b < 0$, and t is even.
- $b < 0$, and the product $pqrst$ is odd (in other words, all of p, q, r, s, and t are odd.)

And we have five cases where we can set $\sqrt[p]{\sqrt[q]{\sqrt[r]{\sqrt[s]{b^t}}}} = b^{\frac{t}{pqrs}}$. And they are as follows.

- If $b > 0$, we get $\sqrt[p]{\sqrt[q]{\sqrt[r]{\sqrt[s]{b^t}}}} = b^{\frac{t}{pq\cdots rs}} > 0$.
- If $b = 0$, and $t > 0$, we get $0 = 0$. (We can see 'why not $t \le 0$' in the section **1**.)
- If $b < 0$ and the product $pqrst$ is odd, we get $\sqrt[p]{\sqrt[q]{\sqrt[r]{\sqrt[s]{b^t}}}} = b^{\frac{t}{pqrs}} < 0$.
- If $b < 0$, t is even, and the product $pqrs$ is odd, we get $\sqrt[p]{\sqrt[q]{\sqrt[r]{\sqrt[s]{b^t}}}} = b^{\frac{t}{pqrs}} > 0$.
- If $b < 0$, t is even, and $\frac{t}{pqrs}$ gets reduced to $\frac{\text{even}}{\text{odd}}$, we get $\sqrt[p]{\sqrt[q]{\sqrt[r]{\sqrt[s]{b^t}}}} = b^{\frac{t}{pqrs}} > 0$.

And the same idea applies to powers, too. So expanding a power as $\{(b^x)^y\}^z$, we don't want to just put it this way: b^{xyz}.

It's because if $b < 0$, it can be the case where we get $\{(b^x)^y\}^z \neq b^{xyz}$. For instance, we cannot do this: $\{(-2)^4\}^{\frac{3}{4}} = (-2)^{\frac{12}{4}} = (-2)^3 = -8$, because we have $\{(-2)^4\}^{\frac{3}{4}} = (2^4)^{\frac{3}{4}} = 8$.

That's because the operations proceed *from inside to outside*, and it is particularly the case if the base is *negative*. For instance, $\{(-2)^4\}^{\frac{3}{2}} \neq \{(-2)^{\frac{3}{2}}\}^4$, since $(-2)^{\frac{3}{2}}$ is not real.

If the base is positive though, the direction of operations doesn't matter.

So it seems we have quite a few to keep in mind doing exponential algebra. And doing algebra, we can hardly do much without doing exponential algebra. And doing algebra exponential, we get to use exponential identities, which are convenient tools, but are not guaranteed to work in all circumstances.

The exponential identities 1 and 2 always work if the bases are positive, but might not work if bases negative or 0 are used. That is to say that if the base is negative or 0, the power can get messed up. So bases negative or 0 are problem bases.

And it is particularly the case if the exponent is a fraction where denominator is even. So using an exponential identity, we want to make sure that the base is positive. And it is particularly the case, when we work with a power or radical where a constant, variable, or expression is used as the base or the exponent.

If the base is positive, we can use all real numbers as the exponent, and all the exponential identities work nicely, too. If the base is not positive, not all real numbers can be used as the exponent.

And thus, if the base is not a particular number but an expression, we want to check to see if the base can be negative or 0, and take care of the base accordingly. We tend to assume that bases are simply positive doing problems on powers where bases are not particular numbers but variables, constants, or expressions. So such bases are favorites with many examiners, probably including your teacher, too.

And in the next section followed by a couple of example sets on exponents and powers, we will get to see another set of powerful tools.

Such a tool is called a *logarithm*, usually called a *log*, for short, and is powered by two numbers called a base and an *antilogarithm*, called an *antilog*, for short. A log is in fact, an exponent. And the base we use making a log is no other than the base we use making a power, and the log is the exponent we use making the power. So working with logarithms, we do not want them to be in trouble. That is, we do not want logarithms to be in trouble.   What do we mean by the trouble, though?

If unable to cover the set of all real numbers, logarithms are in trouble. So we want logarithms to cover the entire real number space. Thus, in logarithms, we do not want the bases to be 0 or negative.

That is, we want them to be positive only. So in logarithms, we do not let bases be 0 or negative. Besides, we don't allow the base to be 1, either. The exponent doesn't do much if the base is 1. If the base is 1, the power is 1 only no matter what the exponent may be. So if the base is 1, why bother working with logarithms? In logarithms therefore, we don't allow bases to be 1, 0, or negative. So in a logarithm, the base is always either between 0 and 1, or greater than 1.

6. What is a logarithm?

A logarithm is no other than an exponent, and is often just called a log.
What exponent then is it?

Exponents have to do with powers. So let's begin with a power.
Suppose for instance, the base of a power is $\boldsymbol{b}$, and the exponent is $\boldsymbol{x}$.
Then, we can indicate the power by $\boldsymbol{b}^{\boldsymbol{x}}$.
So expressing a power, we can put it in terms of the base and the exponent.

What if however, we want to express the exponent, to which a specific base is raised to equal a particular number, and we want to show the base and the particular number?

For instance, assuming $\boldsymbol{A = b^x}$, how can we put the exponent $\boldsymbol{x}$ in terms of the base $\boldsymbol{b}$ and $\boldsymbol{A}$, which is the value of the power $\boldsymbol{b}^{\boldsymbol{x}}$, and thus, is the particular number we get raising the base $\boldsymbol{b}$ to the exponent $\boldsymbol{x}$?
So in short, if $\boldsymbol{A = b^x}$, how can we express $\boldsymbol{x}$ in terms of $\boldsymbol{b}$ and $\boldsymbol{A}$?

Let's begin with some examples.
So for instance, what is the exponent to which a base 10 is raised to equal 1000?
Since 1000 can be said to be 10 to the third power, that is, we can set: $1000 = 10^3$, it is 3.
For another instance, what is the exponent to which a base 2 is raised to equal 16?

Since 16 can be the fourth power of 2, that is, $16 = 2^4$, so it is 4.
Besides, 16 can be put in 4^2, too, so 16 can be called the second power of 4, also.
So for another instance, what is the exponent to which 9 is raised to equal 27?

We have $27 = 3^3$, and $9 = 3^2$.

So we can set $\mathbf{3^3 = (3^2)^{\frac{3}{2}} = 9^{\frac{3}{2}}}$, and thus, we can get $\mathbf{27 = 9^{\frac{3}{2}}}$.

Therefore, the exponent is $\frac{3}{2}$, so 27 can be called 9 to the three halves power, too.

Besides, we can get 27 raising another base to some other power, too.
Such as what base and power?

We can have $27 = 3^3$, and $81 = 3^4$, too.

So we can set $\mathbf{3^3 = (3^4)^{\frac{3}{4}} = 81^{\frac{3}{4}}}$, too, and thus, can get $\mathbf{27 = 81^{\frac{3}{4}}}$, also. So the exponent can be $\frac{3}{4}$, then the base is 81, so 27 can be said to be 81 to the three quarters power, too.

Besides, 27 can still be another base to some other power, too.

That is to say that we can put 27 in terms of another base and another exponent.

In other words, expressing the value of 27, we can use another power where the base is different, and the exponent is different.

Suppose now, $\boldsymbol{x}$ is an exponent to which a base $\boldsymbol{b}$ is raised to equal a number $\boldsymbol{A}$.
Then, we can express the exponent $\boldsymbol{x}$ in terms of the number $\boldsymbol{A}$ and the base $\boldsymbol{b}$. How?

We can do so by means of a special notation, and we call it 'logarithmic notation', often simply called 'log notation'. And using the *log notation*, we can say we use the *log form*. So using the log notation, we use the log form, and can specify an exponent in terms of a particular base and the number we get raising the base to the exponent.

So for instance, if the particular base is 2, and the number we get is 8, we can show the exponent 3 putting 2 and 8 in the log form. How then do we call the number 8?

The number 8 is called an *antilogarithm*, usually just called an *antilog* for short.
So expressing an exponent in terms of a base and a number, we call the number an antilogarithm, often simply called an antilog.

Using thus, the log notation, we can put the exponent 3 in terms of the base 2 and the number 8, and we call 8 the antilogarithm.

And we know $8 = 2^3$. So we can use 2^3 as the antilogarithm, too.

That is to say that using the log form, we can express the exponent 3 in terms of not only 2 and 8 but also 2 and 2^3, which is a power.

So in the log notation, the antilog, that is, the antilogarithm can be either an ordinary number as 49 or a power as 7^2. What then is a log, that is, a logarithm?

A log is another name for an exponent, and thus, is an exponent.
As indicated above however, it's not just an exponent, and can be expressed in terms of two numbers. One is called the base, and the other is called the antilog.

How then can we put an exponent in terms of the two numbers, the base and the antilog?

Normally, specifying a logarithm, we use the log notation. And using the log notation, we use a special sign called a log sign, and as the sign, we use '**log**'. Sometimes though, instead of such a sign, we use '**ln**', too, which is called a natural log sign.
So together with a log sign, we can specify an exponent in terms of a base and an antilog, which is the number we get raising the base to the exponent.

And describing a log specifically, we say a log of a number to a base.

And if we say a log of a number to a base, the number is an antilog.

So for instance, saying the log of 49 to 7, we mean 7 is the base, and 49 is the antilog.

More specifically thus, we can say the log of 49 to base 7.

And of course, the **log of 49 to 7** is an ***exponent***, which is 2, since $49 = 7^2$.

So we can say that **2** is the **log of 49 to 7**. And in that case, 2 is an exponent.

• Thus, a ***log of a number to a base*** is an ***exponent***,
to which the base is raised to equal the number. And the number is called the antilog.

So using log notation, we can express an exponent in a structured manner, and show the exponent using a base and the number we get raising the base to the exponent.

How then do we use the **log** sign?   In other words, what does the log form look like?

Suppose we get $\boldsymbol{A}$ raising a base $\boldsymbol{b}$ to an exponent $\boldsymbol{x}$. That is, $\boldsymbol{A = b^x}$.

Then, putting the exponent $\boldsymbol{x}$ in the log form, we set $\boldsymbol{x = \log_b A}$, which can be read as $\boldsymbol{x}$ is equal to the logarithm of $\boldsymbol{A}$ to base $\boldsymbol{b}$.

Normally though, we just say that $\boldsymbol{x}$ is equal to the **log** of $\boldsymbol{A}$ to $\boldsymbol{b}$.

And we can say it this way, too: "$\boldsymbol{x}$ is equal to the **log** sub $\boldsymbol{b}$ of $\boldsymbol{A}$."

Also, $\boldsymbol{x}$ is called the value of $\boldsymbol{\log_b A}$, too, which is a log.
So $\boldsymbol{\log_b A}$ is called a log, and the value of it is $\boldsymbol{x}$, which can be thus, called a log-value.
Thus, a log-value is an exponent, and is the value of the log of a number to a base.
For instance:

$\mathbf{8 = 2^3 \Leftrightarrow 3 = \log_2 8}$, which can be read as the log of 8 to base 2. And the log value is 3.

$\mathbf{\sqrt{2} = 2^{\frac{1}{2}} = 2^{0.5} \Leftrightarrow \frac{1}{2} = \log_2 \sqrt{2}}$, can be read as the log of $\mathbf{\sqrt{2}}$ to base 2. And the value is $\mathbf{\frac{1}{2}}$.

$\mathbf{3 = 27^{\frac{1}{3}} \Leftrightarrow \frac{1}{3} = \log_{27} 3}$, can be read as the log of 3 to 27. And the value is $\mathbf{\frac{1}{3}}$.

$\mathbf{\frac{1}{3} = 81^{-\frac{1}{4}} \Leftrightarrow -\frac{1}{4} = \log_{81} \frac{1}{3}}$, which can be read as the log sub 81 of $\frac{1}{3}$. And the value is $\mathbf{-\frac{1}{4}}$.

$\mathbf{9.7385 \approx 5^{\sqrt{2}} \Leftrightarrow \sqrt{2} \approx \log_5 9.7385}$, which is saying that

the value of the log sub 5 of 9.7385 is approximately $\mathbf{\sqrt{2}}$.

$\mathbf{1385.45573 \approx 10^{\pi} \Leftrightarrow \pi \approx \log_{10} 1385.45573}$, which is saying that

the value of the log sub 10 of 1385.45573 is approximately π,
which is the circular ratio, which is approximately 3.141592.

So using log notation, that is, the log form, we can express a particular exponent in a structured manner, where we can show a particular base and the number we get raising the particular base to the particular exponent. Why particular, though?

Many powers can have the same value.

For instance, we can have $\mathbf{64 = 2^6 = 4^3 = 8^2 \approx 10^{1.806}}$, and also, $\mathbf{64 = 4096^{0.5} = 0.5^{-6} = \ldots}$

So we can get $6 = \log_2 64$, $3 = \log_4 64$, $2 = \log_8 64$, $1.806 \approx \log_{10} 64$, $0.5 = \log_{4096} 64$, …

Thus, we can put the idea above the way below, too:

If the value of a power or an antilog remains the same, but the base changes, then the exponent changes, too.

So for instance, assuming $\boldsymbol{A = b^x}$, that is, $\boldsymbol{x = \log_b A}$, and keeping $\boldsymbol{A}$ constant, but changing the value of $\boldsymbol{b}$, we have to change the value of $\boldsymbol{x}$, too.

In short, if $\boldsymbol{A = b^x}$ or $\boldsymbol{x = \log_b A}$, and $\boldsymbol{A}$ remains the same, but $\boldsymbol{b}$ changes, $\boldsymbol{x}$ changes, too.

Thus, given a particular base and an antilog, we can track down a particular exponent corresponding to the particular base.

On the other hand, keeping the base constant in a log, if we vary the antilog, what happens to the exponent?
That is, if $\boldsymbol{x = \log_b A}$, and we make $\boldsymbol{A}$ change keeping $\boldsymbol{b}$ constant, what happens to $\boldsymbol{x}$?

The exponent $\boldsymbol{x}$ changes, too, of course. So what?

We can set up a correlation between an antilog and an exponent.
That is to say that we can make a function, where the value of an exponent changes in accordance with the change of the value of an antilog.
Therefore, by means of such a function, we can keep track of an exponent (a variable exponent), which varies as the antilog changes with respect to a particular base fixed.

Such a function is called a ***logarithmic function***, often just called a ***log function***, which is a function of antilog. So each input of a log function is an antilog, and each output of a log function is an exponent.

On the other hand, we can set up another function where the reverse can be achieved.
Keeping the base constant in a power, if we vary the exponent, what happens to the value of the power?
That is, if $A = b^x$, and we make x change keeping b constant, what happens to A?

The value of A changes, too, of course.

So we can set up a correlation between an exponent and a power.
That is to say that we can make a function, where the value of a power changes in accordance with the change of the value of an exponent.

Therefore, by means of such a function, we can keep track of (the value of) a power, which varies as the exponent changes with respect to a particular base fixed.

Such a function is called an ***exponential function***, which is a function of exponent.
So each input of an exponential function is an exponent, and each output of an exponential function is the value of a power, which is in fact, an antilog.

Thus, a ***log function*** is taken as the ***inverse*** of an ***exponential function***, and vice versa.

(You can find more details on functions themselves, in the book **BASIC FUNCTIONS**.)

Now, what is the difference between the value of a power and the value of an antilog?

Basically, both are the same.
If we get a number raising a base to an exponent, what can we call the number?
We can call the number two different ways as follows.

> Putting the number in term of the base and the exponent, we call the number a power.
>
> Using log notation, we can put an exponent in terms of a base and the number we get raising the base to the exponent. Then, we can call the number an antilog.

So if we get a number raising a base to an exponent, we can call the number a power or an antilog. So an antilog can be not only a number as 8 but a power as 2^3, too.

Thus, for instance, we can have $\mathbf{3 = \log_2 8}$, and can put it this way, too: $\mathbf{3 = \log_2 2^3}$.

And putting the idea above more specifically, we can put it the way as follows.

Getting a number raising a base to an exponent, and putting the number in power notation, that is, the power form, we call the number a power.

Getting a number raising a base to an exponent, and putting the exponent in log notation, that is, the log form, we call the number an antilog.

That is to say that if $\boldsymbol{A = b^x}$, we get $\boldsymbol{x = \log_b A}$, and also, if $\boldsymbol{x = \log_b A}$, we get $\boldsymbol{A = b^x}$, too.

In other words, we have $\boldsymbol{A = b^x \Rightarrow x = \log_b A}$, and also, $\boldsymbol{x = \log_b A \Rightarrow A = b^x}$, too.

In short: $\boldsymbol{A = b^x \Leftrightarrow x = \log_b A}$, which is read as: $\boldsymbol{A = b^x}$ if and only if $\boldsymbol{x = \log_b A}$.

So for instance, we can have

$9 = 3^2 \Rightarrow 2 = \log_3 9$, and also, $2 = \log_3 9 \Rightarrow 9 = 3^2$.
In short, $9 = 3^2 \Leftrightarrow 2 = \log_3 9$.

$0.001 = 0.1^3 \Rightarrow 3 = \log_{0.1} 0.001$, and also, $3 = \log_{0.1} 0.001 \Rightarrow 0.001 = 0.1^3$.
In short, $0.001 = 0.1^3 \Leftrightarrow 3 = \log_{0.1} 0.001$.
What then, is a log value?

It's the value of a log, and is in fact, an exponent.

So for instance, 3 is the value of $\log_2 8$.
Also, since $\sqrt{2} = 2^{\frac{1}{2}}$, $\frac{1}{2}$ is an exponent, and is the value of $\log_2 \sqrt{2}$.

And we have $3 = \log_2 8 = \log_{0.1} 0.001 = \ldots$
So many logs looking different can have the same value, and thus, can be the same.

7. Definitions for Logarithms

To begin with, a logarithm is called a log for short, and is another name for an exponent. So in short, a log is an exponent. What then is a log-value?

Taking a log of a number, and then, taking the value of the log, we get a log-value.
For instance, $\log_2 8$ is a log, and the value of it is 3, which is a log-value.
So we set $3 = \log_2 8$, where 2 is called the base, and 8 is called the antilog.

So more specifically, a log-value is the value of the log of a number to a particular base. Taking thus, the value of the log of a number to a base, we get the log-value to the base.

So a log-value depends on the base to which we take a log of the number. For instance, the log of 81 to base 3 is 4, but the log of 81 to base 9 is 2, so in this case, 4 is the log-value to base 3, and 2 is the log-value to base 9. So what is a log-value?

It's an exponent that we use expressing a number in a power of a particular base.
So a log value is the value of a log, and is a number meaning an exponent.

> And a log value is real, but the antilog in a log is positive only.
> In other words, a log-value belongs to a set of all real numbers, and the antilog in a log belongs to a set of all positive numbers.
> That is, all real numbers need to be allowed to be a log-value, and all positive numbers need to be allowed to be the antilog in a log.
> And the base in a log has to be positive but unequal to 1.

What then is the definition for logarithms?

Suppose now, $\boldsymbol{R}$ is a set of all real numbers, and $\boldsymbol{A}$, $\boldsymbol{b}$, and $\boldsymbol{x} \in \boldsymbol{R}$.
Suppose also, $\boldsymbol{A}$ and $\boldsymbol{b}$ both $> \mathbf{0}$, but $\boldsymbol{b} \neq \mathbf{1}$.

Then, $\boldsymbol{A} = \boldsymbol{b}^{\boldsymbol{x}} \Leftrightarrow \boldsymbol{x} = \mathbf{log}_{\boldsymbol{b}}\, \boldsymbol{A}$.

The statements above can be called the definition for logarithms.

We can have two kinds in definitions. One is a full definition, and the other is a short definition. The entire set of statements above can be called the full definition for logs, and we can take as the short definition the last statement where **"$\boldsymbol{A} = \boldsymbol{b}^{\boldsymbol{x}} \Leftrightarrow \boldsymbol{x} = \mathbf{log}_{\boldsymbol{b}}\, \boldsymbol{A}$"**.

Note that '$\Leftrightarrow$' means 'if and only if'.

And we often use the short version, because it is convenient for a quick reference.
Keep in mind though, the full definition, too.
That's because the two suppositions in the full version are intrinsic to logarithms, and in particular, the first of the two is often a vital condition we need to satisfy solving many problems on logarithms.

So when solving problems with logs, we want to check to see if the solutions meet the condition. And in fact, we can readily solve many of log problems by means of the condition, too. What then is the condition?

The full definition has the condition, and is shown below.

Suppose that $\boldsymbol{R}$ is a set of all real numbers, and $\boldsymbol{A}$, $\boldsymbol{b}$, and $\boldsymbol{x} \in \boldsymbol{R}$.
Suppose also, $\boldsymbol{A}$ and $\boldsymbol{b}$ both $> \mathbf{0}$, but $\boldsymbol{b} \neq \mathbf{1}$.

Then, $\boldsymbol{A} = \boldsymbol{b}^{\boldsymbol{x}} \Leftrightarrow \boldsymbol{x} = \mathbf{log}_{\boldsymbol{b}}\, \boldsymbol{A}$.

In the full definition above, the statement that $\underline{\boldsymbol{A} > \mathbf{0}\text{, and } \boldsymbol{b} > \mathbf{0} \text{ but } \boldsymbol{b} \neq \mathbf{1}}$ is the condition.

And in the statement above,

"$A > 0$." is saying that *the antilog in a log* is *positive*.

In other words, the antilog in a log belongs to a set of all positive numbers.

So numbers positive only can be assigned to the antilog A in $\log_b A$, and thus, the antilog ***cannot*** be ***negative*** or **0**.

Next, in the statement above,

"$b > 0$ but $b \neq 1$" is saying that the base in a log is positive, but cannot be 1.

So if we can get $\log_b A$, that is, if we can take the log of A to b (or the log sub b of A), it has to be the case where A and $b > 0$, but $b \neq 1$. And the reverse is true, too.

So if A and $b > 0$, but $b \neq 1$, we can get $\log_b A$, that is, we can take the log sub b of A.

Thus, the statement "A and $b > 0$ but $b \neq 1$." is a favorite with many examiners including probably your teacher, too. So keep in mind that *in a log*, the antilog and the base have to be positive, but the base cannot be 1.
Why in the definition though, do we have this: $b > 0$ but $b \neq 1$?

Suppose first, $A = b^x$, and $\underline{b < 0}$.
Then, we ***cannot*** use all real numbers as the exponent x. Why not?

If $b = -2$ and $x = 0.5$, we get $A = b^x = (-2)^{0.5} = \sqrt{-2}$, which is however, not real.
So if $b < 0$, A can be in trouble, and thus, it can be the case where we cannot get $\log_b A$.

So we want the base b to be positive only. Then of course, A can be positive only, too.

If however, $b > 0$, we can always get $A = b^x$ no matter what real number x may be, so in turn, we can always get $x = \log_b A$. Thus, if $b > 0$, we can use all real numbers as x in b^x.

What then about the case where the base is 0 or 1?

If the base $\boldsymbol{b}$ is 0, we cannot use any number negative as the exponent $\boldsymbol{x}$, because it causes 0 to be a denominator. For instance, $0^{-2} = \frac{1}{0^2} = \frac{1}{0}$, which is not allowed.

And in fact, if $\boldsymbol{x} < \mathbf{0}$, $\mathbf{0}^x = \frac{1}{0^x} = \frac{1}{0}$, which is not allowed.

So if $\boldsymbol{b} = \mathbf{0}$, and $\boldsymbol{x} < \mathbf{0}$, we cannot get $\boldsymbol{A} = \boldsymbol{b}^x$, and in turn, we cannot get $\boldsymbol{x} = \mathbf{log}_{\boldsymbol{b}}\, \boldsymbol{A}$.

What's even worse is that we cannot take the log sub 0 of any number other than 1.
Why not?

Suppose for instance, $\boldsymbol{A} = \mathbf{2}$, and $\boldsymbol{b} = \mathbf{0}$. Then by definition, we get $\boldsymbol{x} = \mathbf{log}_{\boldsymbol{b}}\, \boldsymbol{A} = \mathbf{log}_{\mathbf{0}}\, \mathbf{2}$, which can be however, no number, because there is no $\boldsymbol{x}$ for which $\mathbf{0}^x = \mathbf{2}$.
Thus, 0 cannot be the base.

And next, if the base $\boldsymbol{b}$ is 1, we just get $\boldsymbol{A} = \boldsymbol{b}^x = \mathbf{1}^x = \mathbf{1}$, no matter what real number the exponent $\boldsymbol{x}$ may be.
So taking the log of 1 to 1, that is, the log sub 1 of 1, we do not get a particular number as the value of the log. That is, $\log_1 1$ can be any number. How?

We can get $\mathbf{1} = \mathbf{1}^x$ for all $\boldsymbol{x}$, and thus, for all $\boldsymbol{x}$, we can get $\boldsymbol{x} = \mathbf{log}_{\mathbf{1}}\, \mathbf{1}$. So no point doing it.
And what's even worse is that we cannot take the log sub 1 of any number other than 1.
Suppose for instance, $\boldsymbol{A} = \mathbf{2}$, and $\boldsymbol{b} = \mathbf{1}$. Then by definition, we get $\boldsymbol{x} = \mathbf{log}_{\boldsymbol{b}}\, \boldsymbol{A} = \mathbf{log}_{\mathbf{1}}\, \mathbf{2}$, which can be however, no number, because there is no $\boldsymbol{x}$ for which $\mathbf{1}^x = \mathbf{2}$.

So in sum, we can say that the *base* has to be *positive* but *unequal to* 1.
Thus, assuming $\boldsymbol{b}$ is the base in a log, and keeping the definition happy, we need to have this: $\mathbf{0} < \boldsymbol{b} < \mathbf{1}$, or $\mathbf{1} < \boldsymbol{b}$.

We have already covered the details on the problems with bases negative in the sections for **Problem Bases**. In the next section though, we will cover more details on the bases used in logarithms. That's because even if the base is positive, and is not 1, the base can still matter if it is used in a logarithm. And it does very much so.

Now, getting back to the full definition for logs, we have this:

Assuming that $\boldsymbol{R}$ is a set of all real numbers, and $\boldsymbol{A}$, $\boldsymbol{b}$, and $\boldsymbol{x} \in \boldsymbol{R}$, and also that $\boldsymbol{A}$ and $\boldsymbol{b}$ both $> \mathbf{0}$, but $\boldsymbol{b} \neq \mathbf{1}$.

Then, we get this: $\boldsymbol{A = b^x \Leftrightarrow x = \log_b A}$.

So keep in mind that the base in a log has to be positive but unequal to 1.

Thus, a log-value is real, and the antilog is positive.
In other words, a log-value belongs to a set of all real numbers, and the antilog belongs to a set of all positive numbers.
And usually, we work with the short version below:

$\boldsymbol{A = b^x \Leftrightarrow x = \log_b A}$ (where $\boldsymbol{A}$ and $\boldsymbol{b}$ both $> \mathbf{0}$, but $\boldsymbol{b} \neq \mathbf{1}$).

What do we mean by though, defining a log?

Defining a log, we make a log.
Of course, taking a log of a number to a base, we make a log, too.
And making a log, we define a log, too.

So for instance, defining a log where the base is 2 and the antilog is 8, we make a log, which is $\mathbf{\log_2 8}$, which can be therefore, called a log definition.

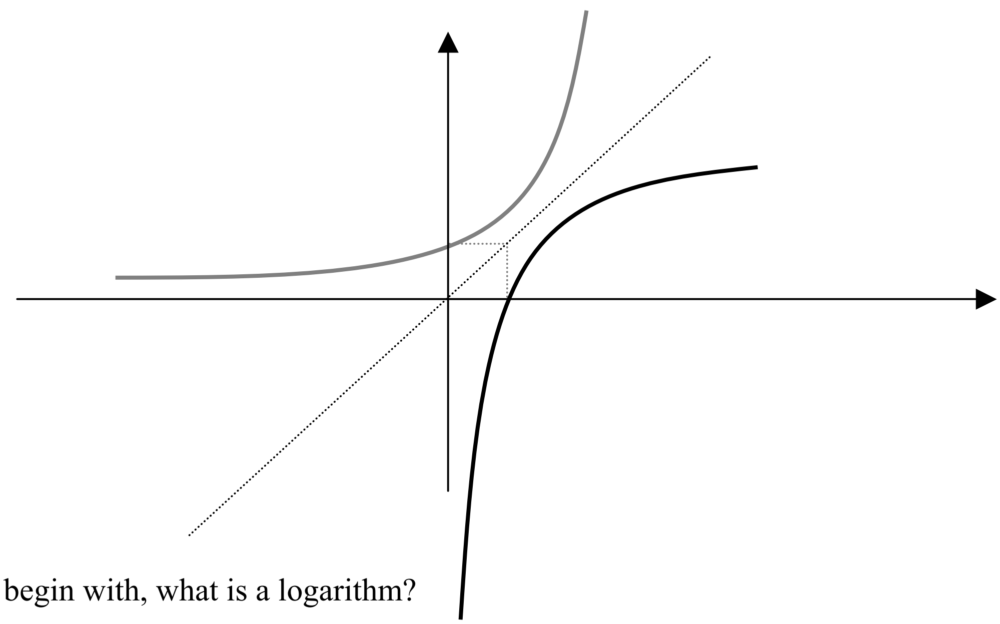
So a log definition is the definition of a log, and is a log, but is not the definition for logs.

The definition for logs explains or shows what logs are, and can be this:

$$\boldsymbol{A = b^x \Leftrightarrow x = \log_b A}$$

And a log definition shows or specifies a particular log, and for instance, can be $\log_3 9$.

8. Bases Matter in Logarithms



To begin with, what is a logarithm?

It is called a log for short, and is no other than an exponent, which is a number we use to make a power as 2^3, where 2 is called the base, and 3 is called the exponent.

And taking the log of a power to the base used in the power, we get the exponent used in the power.
So for instance, taking the log of 2^3 to the base 2, we get $\log_2 2^3$, which is the exponent used in the power 2^3. So $\log_2 2^3$ is the exponent 3. Thus, we get $3 = \log_2 2^3$.

And we know $2^3 = 8$, so we get $\log_2 2^3 = \log_2 8 = 3$.

Thus, we can put all the ideas above at once the way as follows.

$\mathbf{8 = 2^3 \Leftrightarrow 3 = \log_2 8}$ or $\mathbf{3 = \log_2 2^3}$.

In fact, the definition for logs is $\boldsymbol{A = b^x \Leftrightarrow x = \log_b A}$, where $\boldsymbol{A}$ and $\boldsymbol{b > 0}$, but $\boldsymbol{b \neq 1}$.

And since we have $\boldsymbol{A = b^x}$, we can get this, too: $\boldsymbol{x = \log_b b^x}$.

So using the *definition for logs*, we can *switch between a power and its exponent*.

We don't just switch between the two, though. How then do we do so?

We do so based on the base. What base though?

It is the base we use taking a log of a number, and we use it making a power, too.
We can make many different powers that have the same value. Those powers then, have different bases. For instance,

$16 = 16^1 = 4^2 = 2^4 = (\sqrt{2})^8 = (\sqrt[3]{2})^{12} = \ldots$

What then do we get taking a log of 16?

It depends on the base to which we take a log of 16.

Taking the log of 16 to base 4, we get 2, which is the exponent in the power 4^2.
Taking the log of 16 to base 2, we get 4, which is the exponent in the power 2^4.
Taking the log of 16 to base $\sqrt{2}$, we get 8, which is the exponent in the power $(\sqrt{2})^8$.
And so forth

Taking thus, a log of the same number to a different base, we get a different log, that is, a different exponent.

So the base matters. And we can get

$16^1 \Leftrightarrow 1 = \log_{16} 16$ $\quad 4^2 \Leftrightarrow 2 = \log_4 16$ $\quad 2^4 \Leftrightarrow 4 = \log_2 16$

$(\sqrt{2})^8 \Leftrightarrow 8 = \log_{\sqrt{2}} 16$ $\quad (\sqrt[3]{2})^{12} \Leftrightarrow 12 = \log_{\sqrt[3]{2}} 16$ $\quad \ldots$

Thus, the base can be the pivot of switching between the two, a power and its exponent.
Also, we can say that using the *definition for logs*, we can switch between an antilog and its log, too. And the switching is made based on the base.

And we have another case where the base matters, too. So let's see now, what the case is.

Assuming first, $\boldsymbol{b}$ is the base of a power, and the exponent is $\boldsymbol{x}$, we can put the power the way as follows: $\boldsymbol{b^x}$.
Then, if changing the value of the base $\boldsymbol{b}$ keeping constant the value of the power $\boldsymbol{b^x}$, we have to change the value of the exponent $\boldsymbol{x}$, too.

In short, changing $\boldsymbol{b}$ holding $\boldsymbol{b^x}$ constant, we have to change $\boldsymbol{x}$, too.
What then about a logarithm?

For instance, taking a log of a number $\boldsymbol{A}$ to a base $\boldsymbol{b}$, we get $\mathbf{log}_{\boldsymbol{b}}\boldsymbol{A}$.
And we know that a log is an exponent. So assuming $\boldsymbol{x}$ is the exponent to which the base $\boldsymbol{b}$ is raised to equal the number $\boldsymbol{A}$, we get $\boldsymbol{x} = \mathbf{log}_{\boldsymbol{b}}\boldsymbol{A}$. Then, we call $\boldsymbol{A}$ the antilog.

And we have this, too: $\boldsymbol{A} = \boldsymbol{b^x}$, since $\boldsymbol{A}$ is the number we get raising the base $\boldsymbol{b}$ to the exponent $\boldsymbol{x}$.
Also, we can get it by the definition, which is $\boldsymbol{A} = \boldsymbol{b^x} \Leftrightarrow \boldsymbol{x} = \mathbf{log}_{\boldsymbol{b}}\boldsymbol{A}$, where $\boldsymbol{A}$ and $\boldsymbol{b} > \mathbf{0}$, yet $\boldsymbol{b} \neq \mathbf{1}$.

Now, keeping $\boldsymbol{A}$ constant, if we change the base $\boldsymbol{b}$, the exponent $\boldsymbol{x}$ changes, too.

So keeping constant the antilog in a log, and using a different number as the base, we get a different value of the log.
Thus, when we take the log of a number to a base, the value of the log depends on the value of the base. In short, a log depends on the base we choose.

So the base matters in a log.

Depending on the extent of the base, a log works two different ways, which are opposite of each other. So the base in a log can be in one of two different cases.

In one, the <u>base is between 0 and 1</u>, and in the other, the <u>base is greater than 1</u>.

And the same is true for powers, too, of course.

> Thus, doing problems on powers and logarithms, we want to consider the two cases as follows: "$\mathbf{0} <$ the base $< \mathbf{1}$", and "the base $> \mathbf{1}$"

So let's now take a look at how powers and logs work in each case.

Suppose first, that in $\boldsymbol{A} = \boldsymbol{b^x}$, the exponent $\boldsymbol{x} \geq \mathbf{0}$, and that the base $\boldsymbol{b}$ is between 0 and 1.

In short, $\boldsymbol{x} \geq \mathbf{0}$, and $\mathbf{0} < \boldsymbol{b} < \mathbf{1}$ in $\boldsymbol{A} = \boldsymbol{b^x}$. Then, we get $\mathbf{0} < \boldsymbol{A} \leq \mathbf{1}$. How?

Assuming first, $\boldsymbol{c > 1}$, we can get $\frac{1}{c} > \mathbf{0}$, as well as $\frac{1}{c} < \mathbf{1}$.

So we get $\mathbf{0} < \frac{1}{c} < \mathbf{1}$. How though, do we get $\frac{1}{c} > \mathbf{0}$, too?

Since $\boldsymbol{c} > \mathbf{1}$, as $\boldsymbol{c}$ increases, $\frac{1}{c}$ decreases, but stays positive, and also, cannot be 0 no matter how large $\boldsymbol{c}$ may be. So we get $\frac{1}{c} > \mathbf{0}$, together with $\frac{1}{c} < \mathbf{1}$. Thus, $\mathbf{0} < \frac{1}{c} < \mathbf{1}$.

So assuming next, $\boldsymbol{b} = \frac{1}{c}$, we get $\mathbf{0} < \boldsymbol{b} < \mathbf{1}$.

Now, we have $\boldsymbol{A} = \boldsymbol{b}^x$, where $\boldsymbol{b} = \frac{1}{c}$. So we get $\boldsymbol{A} = (\frac{1}{c})^x = \dfrac{\mathbf{1}}{\boldsymbol{c}^x}$, and thus, we get $\boldsymbol{A} = \dfrac{\mathbf{1}}{\boldsymbol{c}^x}$.

Meanwhile, $\boldsymbol{x} \geq \mathbf{0} \Rightarrow \boldsymbol{c}^x \geq \mathbf{1}$, because $\boldsymbol{c} > \mathbf{1}$, and if $\boldsymbol{x} = \mathbf{0}$, we get $\boldsymbol{c}^x = \boldsymbol{c}^0 = \mathbf{1}$.

So we can get not only $\boldsymbol{A} = \dfrac{\mathbf{1}}{\boldsymbol{c}^x} \leq \mathbf{1}$ but $\boldsymbol{A} = \dfrac{\mathbf{1}}{\boldsymbol{c}^x} > \mathbf{0}$, too.

That's because since $\boldsymbol{c}^x \geq \mathbf{1}$, as $\boldsymbol{c}^x$ increases, $\dfrac{\mathbf{1}}{\boldsymbol{c}^x}$ decreases, but stays positive, and cannot be 0 no matter how large $\boldsymbol{c}^x$ may be. So we get $\mathbf{0} < \dfrac{\mathbf{1}}{\boldsymbol{c}^x} \leq \mathbf{1}$.

And we know: $\boldsymbol{A} = \dfrac{\mathbf{1}}{\boldsymbol{c}^x}$. So we get $\mathbf{0} < \boldsymbol{A} \leq \mathbf{1}$.

Now, we have set $\boldsymbol{A} = \boldsymbol{b}^x$, where $\boldsymbol{b} = \frac{1}{c}$ for $\boldsymbol{c} > \mathbf{1}$.

So we now have $\boldsymbol{A} = \dfrac{\mathbf{1}}{\boldsymbol{c}^x} = \boldsymbol{b}^x$, where $\mathbf{0} < \boldsymbol{b} < \mathbf{1}$.

Therefore, if $\boldsymbol{x} \geq \mathbf{0}$ when $\mathbf{0} < \boldsymbol{b} < \mathbf{1}$, we get $\mathbf{0} < \boldsymbol{A} \leq \mathbf{1}$.

Let's see now, more specifically, how the value of $\boldsymbol{A} = \boldsymbol{b}^x$ changes as that of $\boldsymbol{x}$ changes. We are now considering the case where $\mathbf{0} < \boldsymbol{b} < \mathbf{1}$.

So suppose for instance, $\boldsymbol{b} = \mathbf{0.5} = \frac{1}{2}$. Then, if $\boldsymbol{x} = \mathbf{0}$, we get $\boldsymbol{A} = (\frac{1}{2})^0 = \mathbf{1}$.

If $\boldsymbol{x} = \frac{1}{2}$, we get $\boldsymbol{A} = (\frac{1}{2})^{\frac{1}{2}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$, which is between 0 and 1, since $\sqrt{\mathbf{2}} < \mathbf{2}$.

If $\boldsymbol{x} = \mathbf{1}$, we get $A = (\frac{1}{2})^1 = \frac{1}{2}$.

If $\boldsymbol{p} = \mathbf{3}$, we get $A = (\frac{1}{2})^3 = \frac{1}{8}$. **… … …**

If $\boldsymbol{x} = \mathbf{1000}$, we get $A = (\frac{1}{2})^{1000} = \frac{1}{2^{1000}}$, which is very close to 0.

So we can see that if the value of the exponent $\boldsymbol{x}$ begins with 0, and gets larger, the value of the power $\boldsymbol{b^x}$ begins with 1, but gets smaller, and approaches 0, but will never be 0.

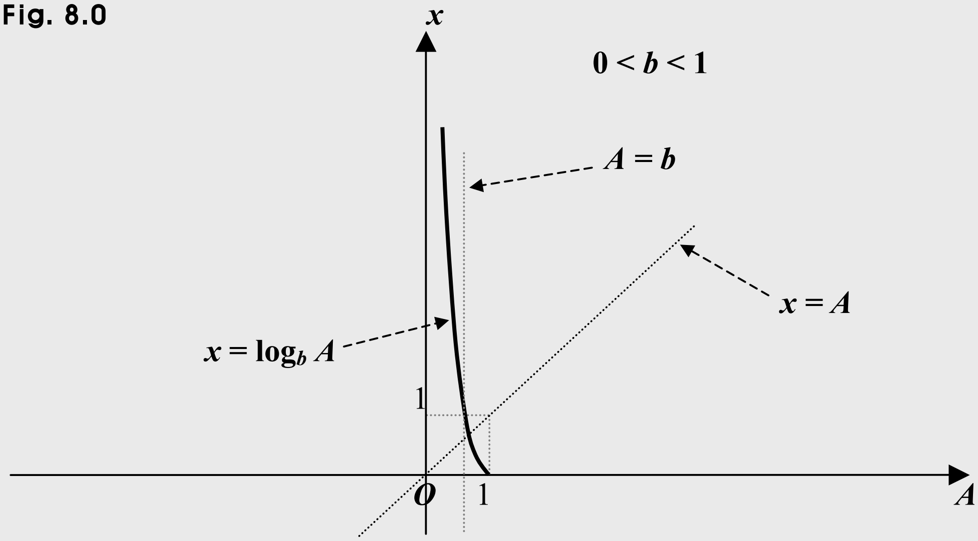
Let's now, put the ideas above in log's perspective.
To begin with, the short definition for logs is as follows: $\boldsymbol{A = b^x \Leftrightarrow x = \log_b A}$.

So $\boldsymbol{x}$ is the value of $\boldsymbol{\log_b A}$, as well as the exponent in the power $\boldsymbol{b^x}$.

Referring thus, to the values of $\boldsymbol{x}$ and $\boldsymbol{A}$ listed in the case where $\boldsymbol{b = 0.5}$, we can say that when the base $\boldsymbol{b}$ is between 0 and 1, if the value of the antilog $\boldsymbol{A}$ begins with 1, decreases, and approaches 0, the value of $\boldsymbol{x}$, that is, the value of $\boldsymbol{\log_b A}$ begins with 0, and keeps increasing.
And in particular, the change of $\boldsymbol{x}$ is far greater than the change of $\boldsymbol{A}$.

Let's now, put the ideas above in a graph.

Fig. 8.0



Suppose next, the exponent $\boldsymbol{x \le 0}$, and $\mathbf{0 < \boldsymbol{b} < 1}$ in $\boldsymbol{A = b^x}$.
Then, we get $\boldsymbol{A \ge 1}$.   How?

Suppose again, that $\boldsymbol{b} = \frac{1}{c}$ where $\boldsymbol{c > 1}$.

Then, we get $\mathbf{0} < \frac{1}{c} < \mathbf{1}$, and thus, we get $\mathbf{0 < \boldsymbol{b} < 1}$.

Now, we have $\boldsymbol{A = b^x}$, where $\boldsymbol{b} = \frac{1}{c}$. So $\boldsymbol{A} = (\frac{1}{c})^x = \boldsymbol{c}^{-x}$, and thus, we get $\boldsymbol{A = c^{-x}}$.

Meanwhile, $\boldsymbol{x \le 0 \Rightarrow -x \ge 0 \Rightarrow c^{-x} \ge 1}$, because $\boldsymbol{c > 1}$, and if $\boldsymbol{-x = 0}$, we ge: $\boldsymbol{c^{-x} = c^0 = 1}$.

And we have $\boldsymbol{A = c^{-x} = b^x}$.
So if $\boldsymbol{x \le 0}$ when $\mathbf{0 < \boldsymbol{b} < 1}$, we get $\boldsymbol{A \ge 1}$.

Let's see now, more specifically, how the value of $\boldsymbol{A = b^x}$ changes as that of $\boldsymbol{x}$ changes.

Suppose again, that $\boldsymbol{b} = \mathbf{0.5} = \frac{1}{2}$. Then, if $\boldsymbol{x = 0}$, we get $\boldsymbol{A} = (\frac{1}{2})^0 = \mathbf{1}$.

If $\boldsymbol{x} = -\frac{1}{4}$, we get $\boldsymbol{A} = (\frac{1}{2})^{-\frac{1}{4}} = \mathbf{2}^{\frac{1}{4}} = \sqrt[4]{\mathbf{2}} > \mathbf{1}$. How do we get $\sqrt[4]{\mathbf{2}} > \mathbf{1}$?

Using a calculator, of course, we can readily see that $\sqrt[4]{\mathbf{2}} > \mathbf{1.189}$.

Let's check algebraically though, to see if $\sqrt[4]{\mathbf{2}} > \mathbf{1}$.

To begin with, we can readily see that if $\boldsymbol{m > 1}$, we get $\boldsymbol{m^4 > 1}$, since $1^4 = 1$, and $2^4 = 16$.

If however, we are given ($\boldsymbol{m^4 > 1}$) only, then can we just say that $\boldsymbol{m > 1}$, too?

No, we can't, because $\boldsymbol{m}$ can be negative, too. For instance, $\mathbf{(-2)^4 = 16 > 1}$ but $\mathbf{-2 < 1}$.

If we have though, $\boldsymbol{m > 0}$ and $\boldsymbol{m^4 > 1}$, we can say that we get $\boldsymbol{m > 1}$.

Now, we know $\sqrt[4]{\mathbf{2}} > \mathbf{0}$, and we have $(\sqrt[4]{\mathbf{2}})^4 = \mathbf{2} > \mathbf{1}$, so we get $\sqrt[4]{\mathbf{2}} > \mathbf{1}$.

Now, getting back to the point where we left off, we have $\boldsymbol{b} = \mathbf{0.5} = \frac{1}{2}$, and have taken two steps as follows.

If $\boldsymbol{x} = \mathbf{0}$, we get $\boldsymbol{A} = (\tfrac{1}{2})^{\mathbf{0}} = \mathbf{1}$. And if $\boldsymbol{x} = -\tfrac{1}{4}$, we get $\boldsymbol{A} = \sqrt[4]{\mathbf{2}}$.

Let's now, take several more steps.

If $\boldsymbol{x} = -\tfrac{1}{2}$, we get $\boldsymbol{A} = (\tfrac{1}{2})^{-\frac{1}{2}} = \mathbf{2}^{\frac{1}{2}} = \sqrt{\mathbf{2}}$.

If $\boldsymbol{x} = \mathbf{-1}$, we get $\boldsymbol{A} = (\tfrac{1}{2})^{-1} = \mathbf{2}$.

If $\boldsymbol{x} = \mathbf{-3}$, we get $\boldsymbol{A} = (\tfrac{1}{2})^{-3} = \mathbf{2}^{3} = \mathbf{8}$.

…

If $\boldsymbol{x} = \mathbf{-10}$, we get $\boldsymbol{A} = (\tfrac{1}{2})^{-10} = \mathbf{2}^{10} = \mathbf{1024}$.

Thus, if the value of the exponent $\boldsymbol{x}$ starts at 0, and gets smaller, the value of the power $\boldsymbol{b}^{x}$ begins with 1, and gets larger and larger.

Let's now, put the ideas above in log's perspective.

First, beginning with the short definition for logs, we have $\boldsymbol{A} = \boldsymbol{b}^{x} \Leftrightarrow \boldsymbol{x} = \log_{b} \boldsymbol{A}$.

So $\boldsymbol{x}$ is the value of $\log_{b} \boldsymbol{A}$, as well as the exponent in the power $\boldsymbol{b}^{x}$.

Therefore, referring to the values of $\boldsymbol{x}$ and $\boldsymbol{A}$ in the case where $\boldsymbol{b} = \mathbf{0.5}$, and $\boldsymbol{x} < \mathbf{0}$,

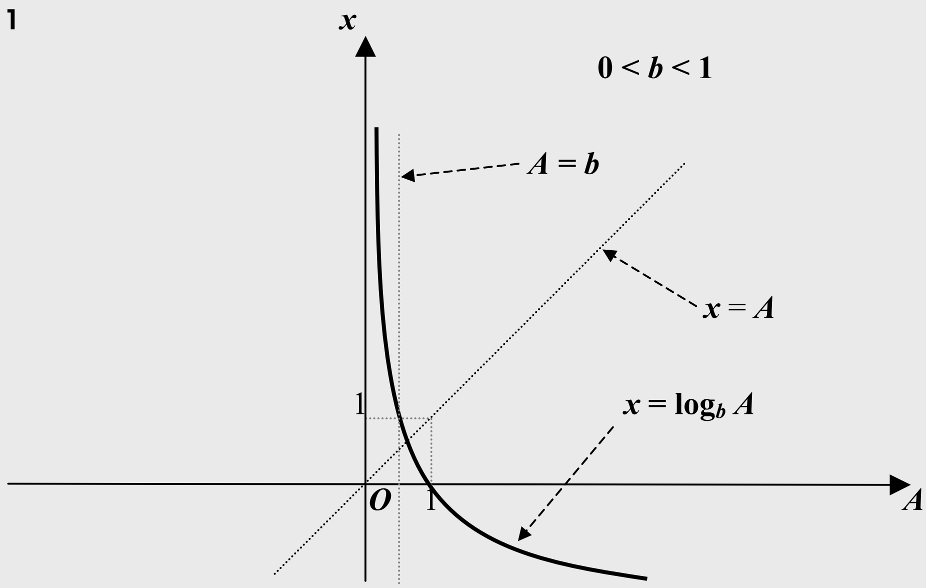
we can say that when the base $\boldsymbol{b}$ is between 0 and 1,
if the value of the antilog $\boldsymbol{A}$ begins with 1, and increases,

the value of $\boldsymbol{x}$, i.e., the value of $\log_{b} \boldsymbol{A}$ begins with 0, and keeps decreasing.

This time however, the change of $\boldsymbol{x}$ is much smaller than the change of $\boldsymbol{A}$.

Let's now, put in a graph the ideas above, together with the curve in the **Fig. 8.0**.

Fig. 8.1



Therefore, we can see that if $\mathbf{0 < Y \leq X}$, we get $\mathbf{\log_b Y \geq \log_b X}$ where $\mathbf{0 < b < 1}$.

That is to say that if the base is between 0 and 1, the larger the antilog is, the smaller the log gets.
Let's now, put together the two cases where $\boldsymbol{x \geq 0}$ and $\boldsymbol{x \leq 0}$, and then, reconsider both.

This time though, we will put the steps in reverse order.

We have $\boldsymbol{b = \frac{1}{2}}$, and $\boldsymbol{A = b^x \Leftrightarrow x = \log_b A}$. Then,

if $\boldsymbol{x = -10}$, we get $\boldsymbol{A = (\frac{1}{2})^{-10} = 2^{10} = 1024}$.

… …

If $\boldsymbol{x = -3}$, we get $\boldsymbol{A = (\frac{1}{2})^{-3} = 2^3 = 8}$.

If $\boldsymbol{x = -1}$, we get $\boldsymbol{A = (\frac{1}{2})^{-1} = 2}$.

If $\boldsymbol{x = -\frac{1}{2}}$, we get $\boldsymbol{A = (\frac{1}{2})^{-\frac{1}{2}} = 2^{\frac{1}{2}} = \sqrt{2}}$.

If $\boldsymbol{x} = -\frac{1}{4}$, we get $\boldsymbol{A} = \sqrt[4]{\mathbf{2}}$.

If $\boldsymbol{x} = \mathbf{0}$, we get $\boldsymbol{A} = (\frac{1}{2})^0 = \mathbf{1}$.

If $\boldsymbol{x} = \frac{1}{2}$, we get $\boldsymbol{A} = (\frac{1}{2})^{\frac{1}{2}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$.

If $\boldsymbol{x} = \mathbf{1}$, we get $\boldsymbol{A} = (\frac{1}{2})^1 = \frac{1}{2}$.

If $\boldsymbol{x} = \mathbf{3}$, we get $\boldsymbol{A} = (\frac{1}{2})^3 = \frac{1}{8}$.

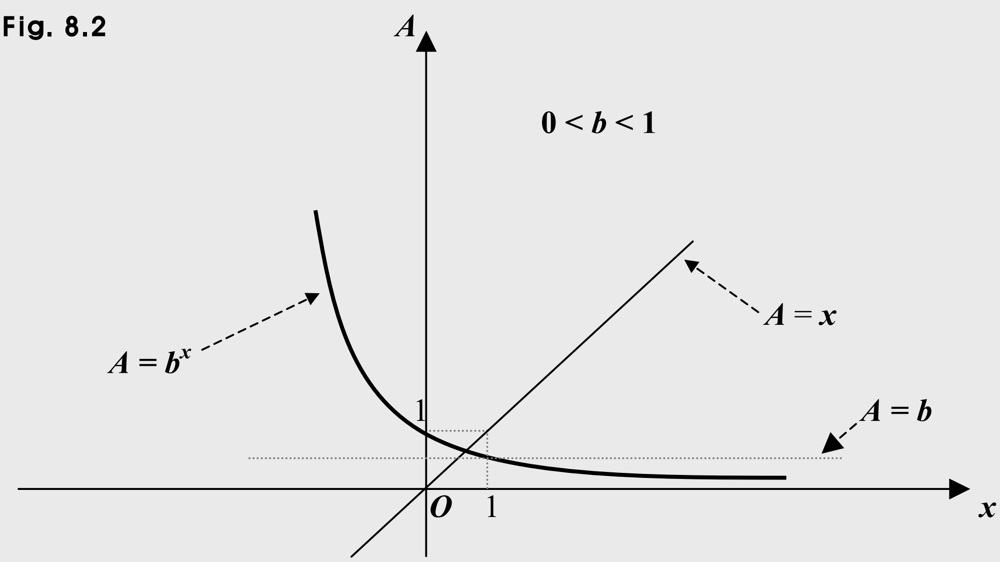
… … …

If $\boldsymbol{x} = \mathbf{1000}$, we get $\boldsymbol{A} = (\frac{1}{2})^{1000} = \frac{1}{2^{1000}}$, which is very close to 0.

So we can see that as the exponent $\boldsymbol{x}$ gets larger, the antilog $\boldsymbol{A}$, that is, the value of the power $\boldsymbol{b}^{\boldsymbol{x}}$ gets smaller, and approaches 0, but will never be 0.

Let's now, put in a graph the ideas above.

Fig. 8.2



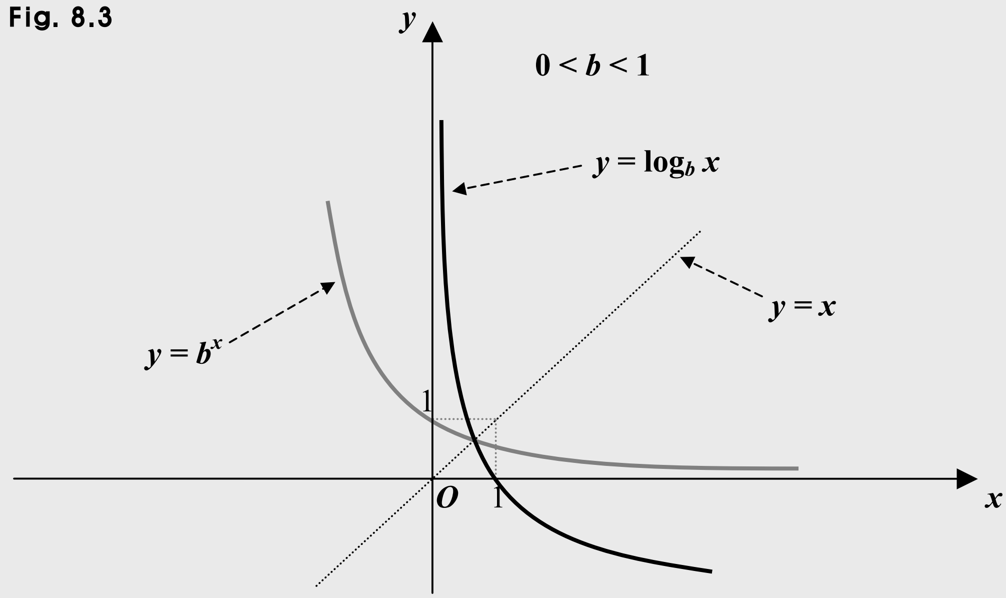
In the graph above, we can see that if $Y \le X$, then we get $b^Y \ge b^X$ where $\mathbf{0} < b < \mathbf{1}$.

And putting together the curve in the graph above and the one in the **Fig. 8.1**, we can put the two curves in one graph as shown below.

Note that the two curves share the same variables since we put the two in one graph.

Each value of x in $y = b^x$ is an exponent, but in $y = \log_b x$, each value of x is an antilog.

Fig. 8.3



We can notice that there is symmetry between the two curves. The axis of symmetry is a line $y = x$.   Why?

You may want to refer to **Basic Functions** or **Function Transformations**.

Suppose next, the exponent $x \ge \mathbf{0}$, and the base $b > \mathbf{1}$ in $A = b^x$.
Then, we get $A \ge \mathbf{1}$.   How?

Assuming $b > \mathbf{1}$, we get $x \ge \mathbf{0} \Rightarrow b^x \ge \mathbf{1}$, because $b > \mathbf{1}$ and if $x = \mathbf{0}$, we get $b^x = b^0 = \mathbf{1}$.
And we have $A = b^x$.
Therefore, if $x \ge \mathbf{0}$ when $b > \mathbf{1}$, we get $A \ge \mathbf{1}$. Let's see now, how the value of A changes as the value of x changes.

Suppose for instance, $\boldsymbol{b = 2}$. Then,

if $\boldsymbol{x = 0}$, we get $\boldsymbol{A = 2^0 = 1}$.

If $\boldsymbol{x = \frac{1}{2}}$, we get $\boldsymbol{A = \sqrt{2}}$.

If $\boldsymbol{x = 1}$, we get $\boldsymbol{A = 2}$.

If $\boldsymbol{x = 3}$, we get $\boldsymbol{A = 2^3 = 8}$.

…

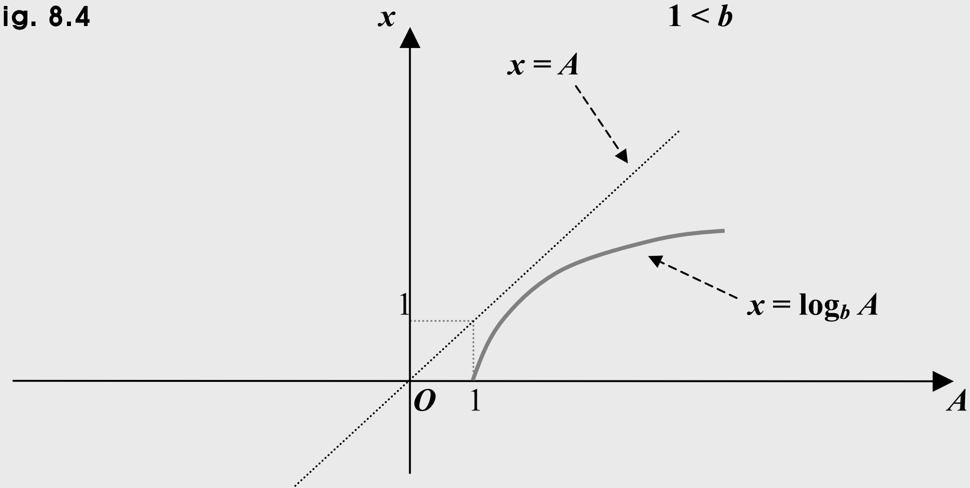
If $\boldsymbol{x = 10}$, we get $\boldsymbol{A = 2^{10} = 1024}$.

So we can see that if the value of the exponent $\boldsymbol{x}$ begins with 0, and gets larger, the value of the antilog $\boldsymbol{A}$ begins with 1, and gets larger.

And we can put it this way, too: if the value of the antilog $\boldsymbol{A}$ begins with 1, and gets larger, the value of $\boldsymbol{x}$ begins with 0, and gets larger, that is, the value of $\boldsymbol{\log_b A}$ gets larger.

Let's now, put the ideas in a graph.

Fig. 8.4



Suppose next, the exponent $\boldsymbol{x \le 0}$, and the base $\boldsymbol{b > 1}$ in $\boldsymbol{A = b^x}$.
Then, we get $\boldsymbol{0 < A \le 1}$. Why?

We have $\boldsymbol{A = b^x}$. So $\boldsymbol{A = (b^{-1})^{-x} = (\frac{1}{b})^{-x}}$, and thus, we get $\boldsymbol{A = (\frac{1}{b})^{-x}}$.

And we have $\boldsymbol{b} > \mathbf{1}$. So we get $\frac{1}{b} < \mathbf{1}$.

Suppose now, that $\frac{1}{b} = \boldsymbol{c}$. Then, we get $\boldsymbol{c} < \mathbf{1}$, since we have $\frac{1}{b} < \mathbf{1}$.

And we get $\mathbf{0} < \boldsymbol{c}$, too, because as $\boldsymbol{b}$ increases, $\boldsymbol{c} = \frac{1}{b}$ decreases, but stays positive, but cannot be 0 no matter how large $\boldsymbol{b}$ may be. So we get $\mathbf{0} < \boldsymbol{c} < \mathbf{1}$.

Let's next, put $\boldsymbol{A}$ this way: $\boldsymbol{A} = \boldsymbol{c}^{-x}$, since we have $\boldsymbol{c} = \frac{1}{b}$, and $\boldsymbol{A} = (\frac{1}{b})^{-x}$.

Next, $\boldsymbol{x} \le \mathbf{0} \Rightarrow -\boldsymbol{x} \ge \mathbf{0} \Rightarrow \mathbf{0} < \boldsymbol{c}^{-x} \le \mathbf{1}$, because $\mathbf{0} < \boldsymbol{c} < \mathbf{1}$, and if $-\boldsymbol{x} = \mathbf{0}$, we get $\boldsymbol{c}^{-x} = \boldsymbol{c}^{0} = \mathbf{1}$.

So we now have $\boldsymbol{A} = \boldsymbol{c}^{-x} = (\frac{1}{b})^{-x} = \boldsymbol{b}^{x}$.

Therefore, if $\boldsymbol{x} \le \mathbf{0}$ when $\boldsymbol{b} > \mathbf{1}$, we get $\mathbf{0} < \boldsymbol{A} \le \mathbf{1}$.

Let's see now, in $\boldsymbol{A} = \boldsymbol{b}^{x}$, how the value of $\boldsymbol{A}$ changes as that of $\boldsymbol{x}$ changes.
Suppose again, that $\boldsymbol{b} = \mathbf{2}$. Then, if $\boldsymbol{x} = \mathbf{0}$, we get $\boldsymbol{A} = \mathbf{2}^{0} = \mathbf{1}$.

If $\boldsymbol{x} = -\frac{1}{4}$, we get $\boldsymbol{A} = \mathbf{2}^{-\frac{1}{4}} = \sqrt[4]{\frac{1}{2}} = \sqrt[4]{\frac{2^3}{2^4}} = \frac{\sqrt[4]{2^3}}{\sqrt[4]{2^4}} = \frac{\sqrt[4]{2^3}}{2}$.

If $\boldsymbol{x} = -\frac{1}{2}$, we get $\boldsymbol{A} = \mathbf{2}^{-\frac{1}{2}} = \sqrt{\frac{1}{2}} = \sqrt{\frac{2}{2^2}} = \frac{\sqrt{2}}{\sqrt{2^2}} = \frac{\sqrt{2}}{2}$.

If $\boldsymbol{x} = \mathbf{-1}$, we get $\boldsymbol{A} = \mathbf{2}^{-1} = \frac{1}{2}$.

If $\boldsymbol{p} = \mathbf{-3}$, we get $\boldsymbol{A} = \mathbf{2}^{-3} = \frac{1}{2^3} = \frac{1}{8}$.

… … …

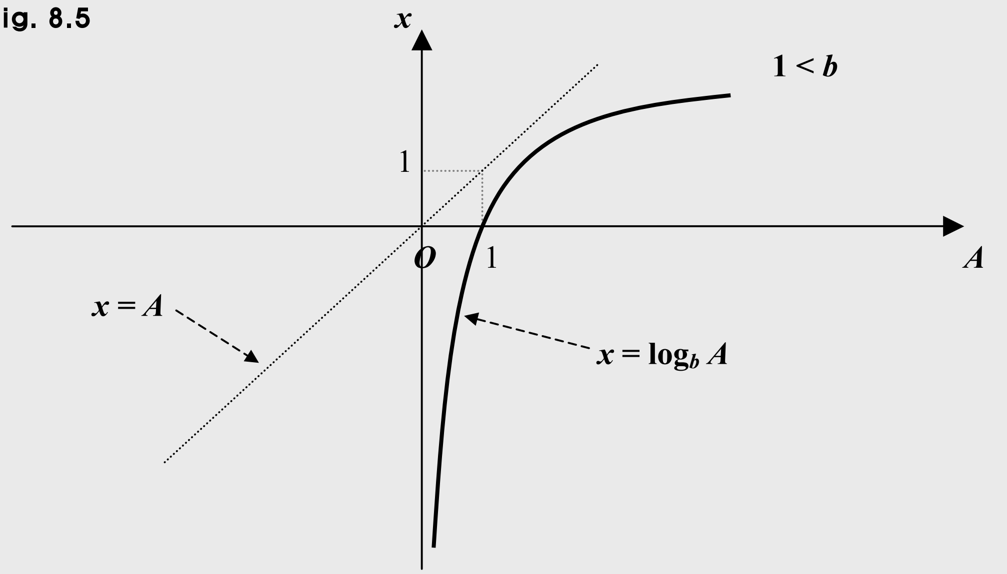
If $\boldsymbol{x} = \mathbf{-1000}$, we get $\boldsymbol{A} = \mathbf{2}^{-1000} = \frac{1}{2^{1000}}$, which is pretty close to 0.

So if the exponent $\boldsymbol{x}$ starts at 0, and gets smaller, the antilog $\boldsymbol{A}$ begins with 1, gets smaller, and approaches 0, but will never be 0.
Then again, considering $\boldsymbol{A} = \boldsymbol{b}^{x} \Leftrightarrow \boldsymbol{x} = \log_b \boldsymbol{A}$, we can put the idea the way below, too.

If the antilog $\boldsymbol{A}$ starts at 1, gets smaller, and approaches 0, the value of $\boldsymbol{x} = \log_b \boldsymbol{A}$ starts at 0, and gets smaller.

Let's now put in a graph the ideas above, together with the curve in Fig. 8.4.

Fig. 8.5



Therefore, we can see that if $\mathbf{0 < \mathit{Y} \le \mathit{X}}$, we get $\mathbf{\log_b \mathit{Y} \le \log_b \mathit{X}}$ where $\mathbf{1 < \mathit{b}}$.

Let's now, put together the two cases where $\boldsymbol{x \ge 0}$ and $\boldsymbol{x \le 0}$, and then, reconsider both. This time though, we will take the steps backward.

We have $\boldsymbol{b = 2}$, and $\boldsymbol{A = b^x \Leftrightarrow x = \log_b A}$. Then, if $\boldsymbol{x = 10}$, we get $\boldsymbol{A = 2^{10} = 1024}$. …

If $\boldsymbol{x = 3}$, we get $\boldsymbol{A = 2^3 = 8}$. If $\boldsymbol{x = 1}$, we get $\boldsymbol{A = 2}$.

If $\boldsymbol{x = \frac{1}{2}}$, we get $\boldsymbol{A = \sqrt{2}}$. If $\boldsymbol{x = 0}$, we get $\boldsymbol{A = 2^0 = 1}$.

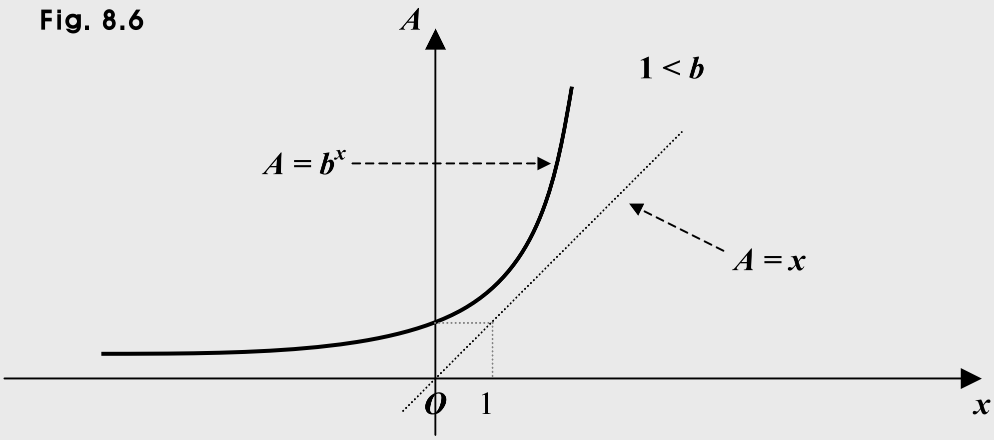
If $\boldsymbol{x = -\frac{1}{4}}$, we get $\boldsymbol{A = 2^{-\frac{1}{4}} = \frac{1}{2^{\frac{1}{4}}} = \frac{2^{\frac{3}{4}}}{2^{\frac{4}{4}}} = \frac{\sqrt[4]{2^3}}{2}}$. If $\boldsymbol{x = -\frac{1}{2}}$, we get $\boldsymbol{A = 2^{-\frac{1}{2}} = \frac{1}{2^{\frac{1}{2}}} = \frac{2^{\frac{1}{2}}}{2^{\frac{2}{2}}} = \frac{\sqrt{2}}{2}}$.

If $\boldsymbol{x = -1}$, we get $\boldsymbol{A = 2^{-1} = \frac{1}{2}}$. If $\boldsymbol{x = -3}$, we get $\boldsymbol{A = 2^{-3} = \frac{1}{2^3} = \frac{1}{8}}$. … … …

If $\boldsymbol{x = -1000}$, we get $\boldsymbol{A = 2^{-1000} = \frac{1}{2^{1000}}}$, which is quite close to 0.

So we can see that if the exponent $\boldsymbol{x}$ begins with 10, gets smaller, and proceeds to negative infinity, the antilog $\boldsymbol{A}$ starts at 1024, gets smaller, and approaches 0, but will never be 0. Let's now, put the ideas in a graph.

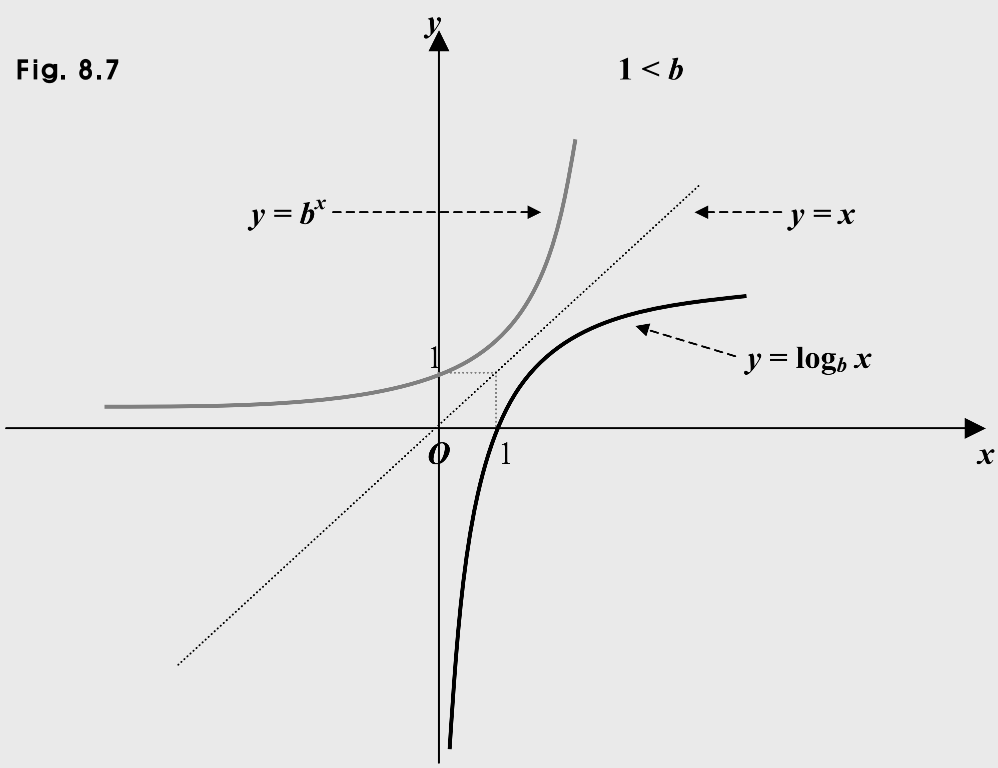
Fig. 8.6



In the graph above, we can see that if $Y \leq X$, we get $b^Y \leq b^X$ where $1 < b$.

And putting together the curve in the graph above and the one in Fig. 8.5, we get a graph as shown below. Note that the two curves share the same variables since we put the two in one graph. Each value of y in $y = b^x$ is an antilog, but in $y = \log_b x$, each value of y is a log value, that is, an exponent.

Fig. 8.7



We can notice that there is symmetry between the two curves. The axis of symmetry is a line $y = x$. How?

That's because the two equations are inverses of each other. If not sure of inverses, you may want to refer to **Basic functions** or **Function Transformations**.

In short,

if we have $\mathbf{0 < b < 1}$ and $X \geq Y$, we get $\log_b X \leq \log_b Y$.

Besides, if we have $\mathbf{0 < b < 1}$ and $X \geq Y$, we get $b^X \leq b^Y$.

If we have $\mathbf{b > 1}$ and $X \geq Y$, we get $\log_b X \geq \log_b Y$.

Besides, if we have $\mathbf{b > 1}$ and $X \geq Y$, we get $b^X \geq b^Y$.

And there are some categories in logarithms.
If we use 10 as the base in a log, the log is called a *common* log.
In such a case though, we usually do not specify the base 10.
So $\log 7 = \log_{10} 7$, for instance.

Some people call a common log, 'Briggs log' or 'Briggsian log', too.
It is said that Henry Briggs did much work regarding logarithms to base 10. Henry was a British astronomer and mathematician specialized in geometry. So he must have been very much interested in logarithms, since astronomers usually handle numbers with excessively large magnitudes. Historically though, a British mathematician, John Napier is said to have first found logarithms in early 17th century.

Besides, many scientists often use a special base, called Euler's number denoted by e. Such a logarithm is called a *natural logarithm*, briefly called a natural log.

In such a case, we usually use as the sign, **Ln** or **ln** instead of **log**.
Using a natural log, we do not usually specify the base e as in the case of a common log. For instance, $\text{Ln } 7 = \log_e 7$. What then is the Euler's number e?

It is an irrational number, and is conceptually $(1 + 0)^{\infty}$ where 0 is called an infinitesimal, which is as good as zero but not zero itself, and ∞ is called infinity.

Actually, $\boldsymbol{e} = \sum_{n=1}^{\infty} \frac{\mathbf{1}}{\boldsymbol{n}!} = \mathbf{1} + \frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{1}}{\mathbf{6}} + \frac{\mathbf{1}}{\mathbf{24}} + \ldots$, which converges to 2.718181828459045…, which is an irrational number.   Where do we use such a number, though?

We'll get to see a lot of ***e***s working in calculus.

9. Identities on Logarithms

In this section, we are going to take a look at some math tools that can make life easier. The tools are for algebra, which is on logarithms. So using the tools, we can handle better and work faster with expressions when doing log algebra.

The tools are called identities, which are on logarithms, and thus, called *log identities*. And we are going to see how the tools can get made, and how they work.

Talking about logs, we are talking about exponents. And in fact, a log is an exponent. We can expect therefore, that the rules or principles on exponential identities apply to log identities, too. So let's begin with some exponential identities as follows.

Suppose that both a and $b > 0$. Then, we get:

0. $a^m a^n = a^{m+n}$

1. $a^m \div a^n = a^m/a^n = \dfrac{a^m}{a^n} = a^{m-n} = 1/a^{n-m} = \dfrac{1}{a^{n-m}}$

2. $(a^m)^n = a^{mn}$

3. $(ab)^n = a^n b^n$

4. $(b/a)^n = \left(\dfrac{b}{a}\right)^n = \dfrac{b^n}{a^n} = b^n/a^n$

Let's now consider a log identity for each of the identities above.

• Beginning with the exponential identity **2** above, we have $(a^m)^n = a^{mn}$.

What then, can we see from the log's perspective?

Assuming first, A and $b > 0$, but $b \neq 1$, we can get $x = \log_b A$.

Then, we can get $A = b^x$ by the definition for logs.

That's because the definition is $A = b^x \Leftrightarrow x = \log_b A$, which means:

We get $A = b^x \Rightarrow x = \log_b A$, and also, can get $x = \log_b A \Rightarrow A = b^x$, too.

Next, assuming y is real, and applying the exponential identity above, we can get this:

$A^y = (b^x)^y = b^{xy} = b^{yx} \Rightarrow A^y = b^{yx}$.

Next, by the definition for logs again, we can get $yx = \log_b A^y$.

And we know $x = \log_b A$. So we can get $yx = y \log_b A$, too.   What then can we get?

We have $yx = \log_b A^y$.   So we get $yx = y \log_b A = \log_b A^y$.

That is to say that we can get $y \log_b A = \log_b A^y$, which is usually taken as a log identity.

The identity is saying that we get $y \log_b A \Rightarrow \log_b A^y$, and also, get $\log_b A^y \Rightarrow y \log_b A$, too.

For instance: we can have $\log_3 8^2 = 2 \log_3 8$.

And we know $8 = 2^3$. So we can get $2 \log_3 8 = 2 \log_3 2^3 = 2{\cdot}3 \log_3 2 = 6 \log_3 2$.

In other words, we can have $\log_3 8^2 = \log_3 (2^3)^2 = \log_3 2^6 = 6 \log_3 2$.

And for another instance, setting $\boldsymbol{A = b}$ in the identity above, we get $\boldsymbol{y \log_b b = \log_b b^y}$.

So we get $\boldsymbol{y = \log_b b^y}$, because $\boldsymbol{\log_b b = 1}$. How though, can we have $\boldsymbol{\log_b b = 1}$?

We know $\boldsymbol{b = b^1}$. So we get $\boldsymbol{1 = \log_b b}$ by the definition for logs.

Thus, we now have a log identity where $\boldsymbol{n \log_b M = \log_b M^n}$.

So if the antilog is a power, we can put in front of the log the exponent of the power.

• Next, moving on to the exponential identity $\mathbf{0}$, which is $\boldsymbol{a^m a^n = a^{m+n}}$, we can expect that assuming $\boldsymbol{M}$, $\boldsymbol{N}$, and $\boldsymbol{b > 0}$, but $\boldsymbol{b \neq 1}$, we can get $\boldsymbol{\log_b M + \log_b N = \log_b MN}$.

That's because we add together exponents when taking a product of powers with the same base. For instance, we can get $\mathbf{2^3 2^4 = 2^{2+3} = 2^7}$.

Let's see now though, how we can get $\boldsymbol{\log_b M + \log_b N = \log_b MN}$.

To begin with, assuming $\boldsymbol{A = \log_b M}$, and $\boldsymbol{B = \log_b N}$, and using the definition for logs, we can get

$\boldsymbol{A = \log_b M \Rightarrow M = b^A}$, and $\boldsymbol{B = \log_b N \Rightarrow N = b^B}$.

So we can get $\boldsymbol{MN = b^A b^B = b^{(A+B)}}$, which is $\boldsymbol{MN = b^{(A+B)}}$, which is a power.

Thus next, using the definition for logs again, we get $\boldsymbol{A + B = \log_b MN}$.

And we know $\boldsymbol{A = \log_b M}$, and $\boldsymbol{B = \log_b N}$. So we get $\boldsymbol{A + B = \log_b M + \log_b N}$, too.

So we get $\boldsymbol{\log_b M + \log_b N = \log_b MN}$, which is in fact, another log identity often used.

Thus, taking the sum of logs with the same base, we can take the product of the antilogs, and then, take the log of the product to the same base. And vice versa.

So we can put a particular log into a sum of logs, where the product of the antilogs is the antilog in the particular log.   For instance,

> $\mathbf{\log_3 9 + \log_3 27 = \log_3 9{\cdot}27 = \log_3 243}$. In fact, we have $\mathbf{\log_3 9} = \mathbf{\log_3 3^2 = 2 \log_3 3 = 2}$ since $\mathbf{\log_b b = 1}$, $\mathbf{3 = \log_3 27}$ since $\mathbf{3^3 = 27}$, and $\mathbf{\log_3 243 = \log_3 3^5 = 5 \log_3 3 = 5}$.

> $\mathbf{\log_3 19683 = \log_3 9{\cdot}27{\cdot}81 = \log_3 9 + \log_3 27 + \log_3 81}$. And in fact, we have $\mathbf{\log_3 9 = 2}$, $\mathbf{\log_3 27 = \log_3 3^3 = 3}$, $\mathbf{\log_3 81 = \log_3 3^4 = 4}$, and $\mathbf{\log_3 19683 = 9}$, since $\mathbf{19683 = 3^9}$.

Now, getting back to the exponential identity **0** above, we can say that taking a product of powers with the same base, we take the sum of the exponents keeping base intact.

For instance, $\mathbf{3^2 3^3 = 3^{2+3} = 3^5}$, and $\mathbf{3^2 3^3 3^4 = 3^{2+3+4} = 3^9}$.

> Note that $\mathbf{\log_b PQ = \log_b (PQ)}$, and $\mathbf{\log_c PQR = \log_c (PQR)}$, but that $\mathbf{\log_k x^2 + 2y \neq \log_k (x^2 + 2y)}$.

> • Next, moving on to the exponential identity **1**, we have $\boldsymbol{a^m/a^n = \dfrac{a^m}{a^n} = a^{m-n}}$.

What then can we see from the log's perspective?

Assuming $\boldsymbol{M}$, $\boldsymbol{N}$, and $\boldsymbol{b > 0}$, but $\boldsymbol{b \neq 1}$, we can expect that $\boldsymbol{\log_b M - \log_b N = \log_b \dfrac{M}{N}}$.

It seems to be the case, since we take the difference between exponents doing a division with powers with the same base. For instance, we can get $\dfrac{2^4}{2^3} = \mathbf{2^{4-3} = 2^1}$.

So let's see now, if it really is the case where we can get $\boldsymbol{\log_b M - \log_b N = \log_b \dfrac{M}{N}}$.

To begin with, assuming $\boldsymbol{A = \log_b M}$, and $\boldsymbol{B = \log_b N}$, and using the definition for logs, we can get

$\boldsymbol{A = \log_b M \Rightarrow M = b^A}$, and $\boldsymbol{B = \log_b N \Rightarrow N = b^B}$.

So we can get $\boldsymbol{\dfrac{M}{N} = \dfrac{b^A}{b^B} = b^{(A-B)}}$. Thus, we get $\boldsymbol{\dfrac{M}{N} = b^{(A-B)}}$, which is a power.

Thus next, using the definition for logs again, we get $\boldsymbol{A - B = \log_b \dfrac{M}{N}}$

And we know $\boldsymbol{A = \log_b M}$, and $\boldsymbol{B = \log_b N}$. So we get $\boldsymbol{A - B = \log_b M - \log_b N}$, too.

Thus, we get $\boldsymbol{\log_b M - \log_b N = \log_b \dfrac{M}{N}}$, which is in fact, another log identity often used.

So <u>subtracting a log from another to the same base</u>, we can take a <u>fraction made of the antilogs</u>, and then, take the log of the fraction to the same base. And vice versa.

Taking thus, a log of a fraction to a base, we can subtract the log of the denominator from the log of the numerator to the same base.

So for instance,

$\boldsymbol{\log_2 8 - \log_2 16 = \log_2 \tfrac{8}{16} = \log_2 \tfrac{1}{2}}$. $\boldsymbol{\log_{0.2} \tfrac{0.04}{125} = \log_{0.2} 0.04 - \log_{0.2} 125 = 2 - (-3) = 5}$.

$\boldsymbol{125 = 0.2^{-3}}$, and $\boldsymbol{\tfrac{0.04}{125} = 0.000032 = 0.2^5}$.

• Suppose next, $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{C}$, and $\boldsymbol{D > 0}$, but $\boldsymbol{a}$ and $\boldsymbol{b \neq 1}$, and we have $\boldsymbol{\log_a C = \log_b D}$.

Then, we get $\boldsymbol{\log_a C = \log_b D = \log_{ab} CD}$.

Let's now, confirm that it really is the case.
First, setting $\boldsymbol{\log_a C = \log_b D = k}$, we get $\boldsymbol{C = a^k}$ and $\boldsymbol{D = b^k}$ by the definition for logs.
Next, looking at the exponential identity **3**, we have $\boldsymbol{(ab)^n = a^n b^n}$.
Thus, we get $\boldsymbol{CD = a^k b^k = (ab)^k}$. So we get $\boldsymbol{k = \log_{ab} CD}$ by the definition for logs.

And we know $\log_a C = \log_b D = k$. So we can get $\log_a C = \log_b D = \log_{ab} CD$.
And of course, we can take it as a log identity.

In exponential algebra, taking a product of antilogs with different bases and the same exponents, we take the product of the bases, and then, use the product as the new base keeping the exponent intact. For instance, $2^3 3^3 = (2 \cdot 3)^3 = 6^3$.

So in log algebra, we can have $3 = \log_2 2^3 = \log_3 3^3 \Rightarrow \log_2 8 = \log_3 27 = \log_6 8 \cdot 27 = 3$.

• Suppose next, a, b, C, and $D > 0$, but a and $b \neq 1$, and we have $\log_a C = \log_b D$.

Then, we can get $\log_a C = \log_b D = \log_{\frac{a}{b}} \frac{C}{D} = \log_{\frac{b}{a}} \frac{D}{C}$. How?

In exponential algebra, we have $(\frac{b}{a})^n = \frac{b^n}{a^n}$ for $a \neq 0$, and $(\frac{a}{b})^n = \frac{a^n}{b^n}$ for $b \neq 0$.

So let's see now, if the same is true for the case of logs, too.

First, setting $\log_a C = \log_b D = k$, we get $C = a^k$ and $D = b^k$ by the definition for logs.

So we get $\dfrac{C}{D} = \dfrac{a^k}{b^k} = \left(\dfrac{a}{b}\right)^k$, and also, $\dfrac{D}{C} = \dfrac{b^k}{a^k} = \left(\dfrac{b}{a}\right)^k$.

Thus, we get $k = \log_{\frac{a}{b}} \frac{C}{D}$ by the definition for logs, and $k = \log_{\frac{b}{a}} \frac{D}{C}$ by the definition, too.

And we have $\log_a C = \log_b D = k$. So we can get $\log_a C = \log_b D = \log_{\frac{a}{b}} \frac{C}{D} = \log_{\frac{b}{a}} \frac{D}{C}$.

In exponential algebra, doing a division with two antilogs, we divide one base by the other if the exponents are the same, and then, use the quotient as the new base keeping the exponent intact, of course. For instance $\frac{4^3}{2^3} = (\frac{4}{2})^3 = 2^3$.
In log algebra, we can have $3 = \log_4 4^3 = \log_2 2^3$. And we know $4^3 = 64$, and $2^3 = 8$.

So we get $\log_4 64 = \log_2 8 = \log_{\frac{4}{2}} \frac{64}{8} = \log_2 8 = \log_{\frac{2}{4}} \frac{8}{64} = \log_{\frac{1}{2}} \frac{1}{8} = \log_{\frac{1}{2}} (\frac{1}{2})^3 = 3$.

So we can see that the same rules or principles on exponents do apply to logarithms, too, which sounds quite natural, since in nature, logs are no different than exponents.

Now, let's take a look at some more log identities, which are more intrinsic to the idea of logarithms.

So let's begin with assuming that $\boldsymbol{A}$, $\boldsymbol{b}$, and $\boldsymbol{c > 0}$, but $\boldsymbol{A}$, $\boldsymbol{b}$, and $\boldsymbol{c \neq 1}$. Then, we get

5. $\log_b b = 1$, and $\log_b 1 = 0$.

6. $\log_b A = \dfrac{\log_c A}{\log_c b}$.

7. $\log_b A = \dfrac{1}{\log_A b}$.

Now, we want to see how the identities above can hold, don't we?
Then again, we may want to first recall the short definition for logs as follows.
Assuming $\boldsymbol{x}$ and $\boldsymbol{y > 0}$, but $\boldsymbol{x \neq 1}$, we get $\log_x y = z \Leftrightarrow y = x^z$.

5. $\log_b b = 1$, and $\log_b 1 = 0$.

To begin with, we get $\log_b b = 1 \Leftrightarrow b^1 = b$ by the definition for logs.

And next, we get $\log_b 1 = 0 \Leftrightarrow b^0 = 1$ by the definition, too.

6. $\log_b A = \dfrac{\log_c A}{\log_c B}$.

Doing log algebra, we often need to put a log in a fractional form.
In addition, we frequently need to modify logs so that they have the same base.
And we can quickly do so by means of the log identity above.

To begin with, suppose that $\boldsymbol{m} = \log_b \boldsymbol{A}$.

Then, we can see that $\boldsymbol{b^m = A}$ by the definition for logs.
Next, assuming that $\boldsymbol{c > 0}$ but $\boldsymbol{\neq 1}$, let's take a log sub $\boldsymbol{c}$ of each side in the equality above.

That is, we take $\mathbf{log}_c$ of each of the two sides of the equality $\boldsymbol{b^m = A}$.
How can we do that though?

As far as we do the same to both sides of an equality, the equality remains the same.
That is, it still holds. For instance, $\boldsymbol{A = B \Rightarrow A + 1 = B + 1, \frac{A}{2} = \frac{B}{2}, 3A = 3B}$, etc.

By the same token, if we take the same log of each side of an equation or equality, the equal sign remains valid. What then do we mean by taking the same log of each side?

Taking a log of each side to the same base, we take the same log of each side.
Note that though, we don't want to use a negative number or 1 as a base of any log.

Suppose for instance, $\boldsymbol{b > 0}$ but $\boldsymbol{\neq 1}$, and $\boldsymbol{U = V}$.
Then, taking a $\mathbf{log}_b$ of each side of the equality $\boldsymbol{U = V}$, we get $\boldsymbol{\log_b U = \log_b V}$.

We know for instance, 100 = 100. So taking $\mathbf{log}_{10}$ of each of the two sides, we get

$$\mathbf{\log_{10} 100 = \log_{10} 100 \Rightarrow \log_{10} 10^2 = \log_{10} 10^2 \Rightarrow 2 \log_{10} 10 = 2 \log_{10} 10 \Rightarrow 2 = 2.}$$

So we can say that '$\mathbf{log}_b$' is a *log operator*. And of course, the base $\boldsymbol{b}$ can be any number that can be a base of a log. So the base $\boldsymbol{b}$ has to be $> \mathbf{0}$, but $\neq \mathbf{1}$.

If for instance, we take a log of $\boldsymbol{N}$ to a base $\boldsymbol{b}$, we can say that $\mathbf{log}_b$ is applied to $\boldsymbol{N}$, so $\mathbf{log}_b$ is the operator, and $\boldsymbol{N}$ is the operand. What then do we get?

Taking a log of $\boldsymbol{N}$ to a base $\boldsymbol{b}$, we get: $\boldsymbol{\log_b N}$, which is an exponent to which the base $\boldsymbol{b}$ is raised to equal the number $\boldsymbol{N}$. For instance, taking **log** of 9 to base 3, we apply $\mathbf{log}_3$ to 9, and we get $\mathbf{log_3 9}$, which is 2, because $9 = 3^2$. And we can put it the way below, too.

Taking **log sub 3** of 9, we apply $\mathbf{log}_3$ to 9, and we get $\mathbf{log_3 9}$, which is 2.

Thus, $\mathbf{log_3 9}$ can be read as **log sub 3 of 9**, as well as **log of 9 to base 3**, or **log of 9 to 3**.

Now, let's get back to the point where we left off.
We are taking $\mathbf{log}_c$ of each side of the equality where $\boldsymbol{b^m = A}$.

Then, we get $\mathbf{log}_c\, \boldsymbol{b^m} = \mathbf{log}_c\, \boldsymbol{A}$. And next, we know $\mathbf{log}_c\, \boldsymbol{b^m} = \boldsymbol{m}\, \mathbf{log}_c\, \boldsymbol{b}$.

So we get $\boldsymbol{m}\, \mathbf{log}_c\, \boldsymbol{b} = \mathbf{log}_c\, \boldsymbol{A}$, since $\boldsymbol{b^m = A}$. Thus, we get $\boldsymbol{m} = \dfrac{\mathbf{log}_c\, \boldsymbol{A}}{\mathbf{log}_c\, \boldsymbol{B}}$.

And we have $\boldsymbol{m} = \mathbf{log}_b\, \boldsymbol{A}$, too. Therefore, we can see that $\mathbf{log}_b\, \boldsymbol{A} = \dfrac{\mathbf{log}_c\, \boldsymbol{A}}{\mathbf{log}_c\, \boldsymbol{b}}$.

For instance, in exponential algebra, we can have $\mathbf{27} = \frac{81}{3} = \frac{9^2}{9^{\frac{1}{2}}} = 9^{2-\frac{1}{2}} = 9^{\frac{3}{2}}$.

Thus, in log algebra, we can get $\mathbf{log}_9\, \mathbf{27} = \mathbf{log}_9\, 9^{\frac{3}{2}} = \frac{3}{2}$, and also, $\mathbf{log}_9\, \mathbf{27} = \frac{\log_3 27}{\log_3 9} = \frac{3}{2}$.

In short, suppose first, $\boldsymbol{m} = \mathbf{log}_b\, \boldsymbol{A}$. Then, we get $\boldsymbol{b^m = A}$ by the definition for logs.

Suppose next, that $\boldsymbol{c} > \mathbf{0}$ and $\neq \mathbf{1}$, and that we take $\mathbf{log}_c$ of each of the two sides of the equality where $\boldsymbol{b^m = A}$. Then, we get $\mathbf{log}_c\, \boldsymbol{b^m} = \mathbf{log}_c\, \boldsymbol{A}$.

We know $\mathbf{log}_c\, \boldsymbol{b^m} = \boldsymbol{m}\, \mathbf{log}_c\, \boldsymbol{b}$. So we get $\boldsymbol{m}\, \mathbf{log}_c\, \boldsymbol{b} = \mathbf{log}_c\, \boldsymbol{A}$. Then, we get $\boldsymbol{m} = \dfrac{\mathbf{log}_c\, \boldsymbol{A}}{\mathbf{log}_c\, \boldsymbol{b}}$.

And we have $\boldsymbol{m} = \mathbf{log}_b\, \boldsymbol{A}$, too. Therefore, we can get $\mathbf{log}_b\, \boldsymbol{A} = \dfrac{\mathbf{log}_c\, \boldsymbol{A}}{\mathbf{log}_c\, \boldsymbol{b}}$.

Note that the base $\boldsymbol{c}$ above can be any number if the number can be a base in a log.
And we know that a base in a log has to be positive but unequal to 1.
Then, can we use the base $\boldsymbol{b}$ in the $\mathbf{log}_b\, \boldsymbol{A}$ as the base $\boldsymbol{c}$ in the $\mathbf{log}_c\, \boldsymbol{b}$, too?

Yes, we can. In $\dfrac{\mathbf{log}_c\, \boldsymbol{A}}{\mathbf{log}_c\, \boldsymbol{b}}$, replacing $\boldsymbol{c}$ with $\boldsymbol{b}$, we get $\mathbf{log}_b\, \boldsymbol{A} = \dfrac{\mathbf{log}_b\, \boldsymbol{A}}{\mathbf{log}_b\, \boldsymbol{b}} = \dfrac{\mathbf{log}_b\, \boldsymbol{A}}{\mathbf{1}} = \mathbf{log}_b\, \boldsymbol{A}$.

Well, the replacement with $\boldsymbol{b}$ doesn't give us much of significance, since it doesn't make any difference. Nevertheless, it still works. That is, the equal sign remains valid.

Then, can we use the antilog $\boldsymbol{A}$ in the $\mathbf{log}_b\, \boldsymbol{A}$ as the base $\boldsymbol{c}$ in the $\mathbf{log}_c\, \boldsymbol{b}$, too?

Yes, we can ***unless A*** is equal to 1. What if ***A*** is not positive?

A is already an antilog in a log, so it is positive. We'll see how it can be the case in the next log identity, which is the identity numbered **7**, and will be explained shortly.

Changing the base in a log keeping intact the value of the log, we usually put the log in such a fractional form as described above.
Suppose for instance, $\boldsymbol{A}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d} > \mathbf{0}$, but $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d} \neq \mathbf{1}$. Then, we can get

$\log_b A = \dfrac{\log_c A}{\log_c b} = \dfrac{\log_d A}{\log_d b} = \dfrac{\log_3 A}{\log_3 b} = \dfrac{\log_{0.3} A}{\log_{0.3} b} = \dfrac{\log_\pi A}{\log_\pi b} = \ldots$ where π is the circular ratio, which is 3.141592…, which is an irrational number.

By the way, in a natural log, where we use as the log sign, **ln** (or **Ln**) instead of **log**, we use as the base, the Euler's number, denoted by ***e***. The number ***e*** is an irrational number, and is approximately 2.72. To be more accurate, it is approximately 2.718181828459045.

So we can set: $\log_b A = \dfrac{\log_e A}{\log_e b}$, too, which is normally put in $\dfrac{\ln A}{\ln b}$.

Note:

Suppose $\boldsymbol{A}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{S}$, and $\boldsymbol{t} > \mathbf{0}$, but $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{t} \neq \mathbf{1}$.

Then, $\log A$ is a common log of $\boldsymbol{A}$, and is the same as $\log_{10} A$. So we get $\log 10 = 1$.

And we have a log identity where $\log_b A = \dfrac{\log_c A}{\log_c b}$.

So we get $\log_b A = \dfrac{\log_{10} A}{\log_{10} b} = \dfrac{\log A}{\log b}$, and thus, we get $\log_b A = \dfrac{\log A}{\log b}$.

By the same token, we get $\dfrac{\log S}{\log t} = \log_t S$.

7. $\log_b A = \dfrac{1}{\log_A b}$ where A and b **both** are > 0 but $\neq 1$.

Basically, this log identity is no different from the one explained above.

Suppose now, that $m = \log_b A$. Then, we get $b^m = A$ by the definition for logs.

Next, we know that $A > 0$ but $\neq 1$.

So taking $\log_A$ of each side of $b^m = A$, we get $\log_A b^m = \log_A A = 1 \Rightarrow \log_A b^m = 1$.

And we have $\log_A b^m = m \log_A b$, too.

So we can see that $\log_A b^m = 1 \Rightarrow m \log_A b = 1 \Rightarrow m = \dfrac{1}{\log_A b}$.

And we have $m = \log_b A$. Therefore, we can see that $\log_b A = \dfrac{1}{\log_A b}$.

In short, $\log_b A = \dfrac{\log_A A}{\log_A b} = \dfrac{1}{\log_A b}$.

Note:

Suppose $A > 0$.
Then, $\ln A$ is a natual log of A, and is the same as $\log_e A$ where e is the Euler's number.
We can have $\ln e$, which is 1, because $\ln e = \log_e e = 1$.
However, we do not have such a log as $\ln_b A$, which does not make sense.
So we have $\log_b A = \dfrac{\log_e A}{\log_e b} = \dfrac{\ln A}{\ln b}$, which is however, not $\ln_b A$, which doesn't exist.

Now, putting threads together, we have these:

$$y \log_b A = \log_b A^y.$$

$$\log_b M + \log_b N = \log_b MN. \qquad \log_b M - \log_b N = \log_b \tfrac{M}{N}.$$

$$\log_a C = \log_b D = \log_{ab} CD. \qquad \log_a C = \log_b D = \log_{\frac{a}{b}} \tfrac{C}{D} = \log_{\frac{b}{a}} \tfrac{D}{C}.$$

$\log_b b = 1$, and $\log_b 1 = 0$.

$$\log_b A = \frac{\log_c A}{\log_c b}. \qquad \log_b A = \frac{1}{\log_A b}.$$

A. Common Logs

To begin with, what kind of number can we take a log of?

If a number is positive, we can take a log of it.
Next, a log has its base, which is a number. What number then can we use as the base?

Taking a log, we can use as the base any number positive and unequal to 1.
Quite usually though, when taking logs, we use 10 as the base. And such a log is called a ***common log***. That's because it is commonly used, that is, it is used quite often.

> So taking a log of a number to base 10, or taking a log sub 10 of a number, we take a *common log* of the number. And taking a common log, we use 10 as the base.

When working with numbers very large or small, we often find common logs are convenient and effective. Taking a common log of a number, we can readily see how big or small the numbers is. How can we see it though?

The value of a common log is specified by two parts.
One is called a ***characteristic***, and the other is called a ***mantissa***.

And taking the value of the common log of a number, and finding the characteristic, we can quickly see the highest place value of the number. What is the highest place value?

It is the place value of the highest digit. So for instance, the highest place value of 5023 is $1000 = 10^3$, and the highest place value of 0.04082 is $0.01 = 10^{-2}$. That is, in 5023, 5 is in the 1000's place, and in 0.04082, 4 is in the one hundredth's place.

And looking at the characteristic, we can readily see the highest place value.
How then can we get or identify the characteristic of a common log?

Math begins with a definition. So let's now, begin with the definition for common logs.

Normally, taking a common log of a number, we omit the base 10. Thus, seeing a log of a number to no base, we can just take it as the common log of the number.

So the definition for common logs can be as follows.

$$\boldsymbol{y = \log x \Leftrightarrow x = 10^y}, \text{ where } \boldsymbol{x > 0}$$

That is to say that if $\boldsymbol{x > 0}$, $\boldsymbol{y = \log x \Rightarrow x = 10^y}$, and also, $\boldsymbol{x = 10^y \Rightarrow y = \log x}$, too.
So $\boldsymbol{\log x}$ is the same as $\boldsymbol{\log_{10} x}$, and is taken as a common log.

For instance, $\mathbf{2 = \log 100 \Rightarrow 100 = 10^2}$, and also, $\mathbf{100 = 10^2 \Rightarrow 2 = \log 100}$, too.
So $\mathbf{\log_{10} 100 = \log 100}$, which is taken as a common log.

How then can we get or identify the characteristic of a common log?

Suppose $\boldsymbol{N > 0}$.
Then, we can put the value of the common log of $\boldsymbol{N}$ in terms of the characteristic and mantissa the way as follows. $\boldsymbol{\log_{10} N = c + m}$, where $\boldsymbol{c}$ is an *integer*, and $\boldsymbol{0 \le m < 1}$.

And in the equality above, $\boldsymbol{c}$ is called a *characteristic*, and $\boldsymbol{m}$ is called a *mantissa*.

Normally though, taking a common log of N, we don't specify the base, so we put it this way: $\log N$.

And set it this way: $\log N = c + m$, where c is an *integer*, and $0 \le m < 1$.

For instance, if $\log A = 3.1405$, the characteristic is 3, and the mantissa is 0.1405.

So what's good about a common log?

First, the integer c called characteristic can quickly tell us the highest place value of N. And the highest place value of N is simply 10^c.

So for instance, taking a common log of 3851, we can set it the way below:

$\log 3851 = 3 + m$ where $0 \le m < 1$.

So the characteristic $c = 3$ can tell us immediately the highest place value is $10^c = 10^3$. And the same is true, also, for any number where the highest place value is 10^3.

Thus for instance, taking a common log of 9759.024, we can set it the way below:

$\log 9759.024 = 3 + n$ where $0 \le n < 1$.

And by the same token, we can set $\log 8076.1205 = 3 + p$ where $0 \le p < 1$.

So if the highest place value of every antilog is the same, only the mantissa is different.

And next, if for instance, we take common logs of 305, 3050, and 30500, we get the same mantissas. That is to say that we can set

$\log 305 = 2 + m$, $\log 3050 = 3 + m$, and $\log 30500 = 4 + m$, where $0 \le m < 1$.

Taking care of the characteristic though, we need to pay attention to the sign of it.
In other words, the sign of the characteristic matters.

And the sign depends on the *magnitude* of the antilog.

For instance, assuming $\log N = c + m$, where c is an integer, and $0 \le m < 1$, we can get

If N is ≥ 1, we get $c \ge 0$.
If N is greater than 0 but less than 1, that is, $0 < N < 1$, we get c negative.

In sum, $N \ge 1 \Rightarrow c \ge 0$, and $0 < N < 1 \Rightarrow c < 0$.

Next, in the case where $\log N = c + m$, $(c + m)$ is a log value, which is the value of $\log N$.

When getting the value of the $\log N$ though, how do we get the characteristic c?

If the antilog N has n digits above the decimal point, we get $c = n - 1$.

In other words, N has $c + 1$ digits above the point, since $n = c + 1$.

For instance, if $N = 3809.07042$, we get $c = 3$, since N has 4 digits above the point.

And it doesn't matter how many digits N has below the point.
What matters is *the number of digits above the point*.
So for instance, the characteristic of $\log 102.31$ is 2, and so is that of $\log 927.993527$.

On the other hand, if N has no nonzero digit above the decimal point as 0.01, and its first nonzero digit appears in the n^{th} place below the point, then we get $c = -n$.

For instance, if the antilog $N = 0.004208$, we get $c = -3$, since the first nonzero digit appears in the third place below the point.

Anyhow, the highest place value of the antilog N is 10^c, where c is the characteristic.

In the examples above,

$N = 3809.07042 \Rightarrow c = 3 \Rightarrow 10^c = 10^3$, so the highest place value of N is 1000.

$N = 0.004208 \Rightarrow c = -3 \Rightarrow 10^c = 10^{-3} = 0.001$, so the highest place value of N is 0.001.

So putting threads together, what does the characteristic have to do with?

It has to do with the magnitude of the highest digit in the number we take a common log of. And in fact, it can tell us such a magnitude, which is $\mathbf{10}^{c}$, where c is the characteristic. That is, it says the highest place value of the antilog in a common log.

• Now, what about the ***mantissa***?

Suppose two positive numbers have the same sequence of digits, but the positions of the decimal point are different.

For instance, if $x = \mathbf{1202.387}$ and $y = \mathbf{0.01202387}$, then x and y have the same sequence of digits, but the positions of the decimal point are different.

Then, if we take common logs of x and y above, the characteristics of the logs will be different, but the mantissas will be the same.

That is, taking the difference between $\log x$ and $\log y$, we get an integer.

The highest place value of x above is 10^3, so the characteristic c of $\log x$ is 3.
The highest place value of y above is 10^{-2}, and therefore, c of $\log y$ is -2.

So the characteristics are different, because the highest place values are different. However, the mantissas will be the same, because the sequences of digits are the same.

Thus, we can set $\log x = 3 + m$, and $\log y = -2 + m$, where $0 \le m < 1$.

Therefore, we get $\log x - \log y = 5$, which is an integer, of course.

So what does the mantissa have to do with?

It has to do with the sequence of digits in the number we take a common log of. And in fact, if two numbers have the same sequence of digits as 1023 and 1.023, and we take common logs of the two numbers, their mantissas are the same.

Let's next, have a look at cases where $0 < N < 1$, that is, the antilog is between 0 and 1.
So suppose $\log N = c + m$, where $0 < N < 1$, c is an integer, and $0 \le m < 1$.
Then, the characteristic c is negative.

That is, if the antilog in a common log is between 0 and 1, the characteristic c is negative. In such a case, we use special notation for the characteristic c of $\log N$.

In the special notation, we place the minus sign (–) on top of the characteristic.
So for instance, we normally set $\log y = \bar{2}.abcd$ instead of $\log y = -2.abcd$.

Thus, unlike cases of ordinary numbers, we put a minus sign not in front but on top of the characteristic.

Why is the characteristic c negative though, if $0 < N < 1$, and also, why do we bother putting the minus sign above c?

The answer to the first part of the question begins with a statement that the highest place value of a number between 0 and 1 is less than or equal to 10^{-1}, which is one tenth.

So of the common log of such a number, the characteristic is ≤ -1, since the highest place value of the number is 10^c, where c is the characteristic of the common log.

Suppose for instance, that the antilog $N = 0.3$. Then, we can get $N = 0.3 = 0.1 \cdot 3 = 10^{-1} \cdot 3$.

So we can get $\log N = \log 10^{-1} \cdot 3 = \log 10^{-1} + \log 3 = -1 + \log 3$.

Thus, we get $\log 0.3 = -1 + \log 3$.

And we know $\log 1 = 0$ and $\log 10 = 1$.

So we get $1 \leq 3 < 10 \Rightarrow \log 1 \leq \log 3 < \log 10 \Rightarrow 0 \leq \log 3 < 1$.

Thus, we get $\log 0.3 = -1 + s$, where $0 \leq s < 1$.

So the characteristic is -1, which is negative.

Suppose for another instance, that $N = 0.999$. Then, $N = 0.999 = 0.1 \cdot 9.99 = 10^{-1} \cdot 9.99$.

So $\log N = \log 0.999 = \log 10^{-1} + \log 9.99 = -1 + \log 9.99 \Rightarrow \log 0.999 = -1 + \log 9.99$.

And since $\log 1 = 0$ and $\log 10 = 1$, we get $1 \leq 9.99 < 10 \Rightarrow 0 \leq \log 9.99 < 1$.

Therefore, $\log 0.999 = -1 + t$, where $0 \leq t < 1$. What if $N = 0.0299$?

Then, we can get $N = \mathbf{0.0299 = 0.01 \cdot 2.99 = 10^{-2} \cdot 2.99}$.

So we can get $\log N = \log 0.0299 = \log 10^{-2} + \log 2.99 = -2 + \log 2.99$.

And since $\log 1 = 0$ and $\log 10 = 1$, we get $1 \le 2.99 < 10 \Rightarrow 0 \le \log 2.99 < 1$.

Therefore, $\log 0.0299 = -2 + u$ where $0 \le u < 1$.

Now, going back to the second part of the question, we have this:

If $0 < N < 1$, why do we bother putting the minus sign above c?

The answer begins with a statement that the value of a common log is not just a value but a string of digits showing two values separated by the decimal point, one value is called a characteristic, and the other is called a mantissa.

So we need to differentiate them first.

And next, if we put the negative sign in front, the mantissa is negative, too, which is against the definition where **0 ≤ the mantissa < 1**.

Now in a common log, $\log N$, if $0 < N < 1$, then we get $\log N < 0$.

So going back to the examples above, we have these:

$\log 0.3 = -1 + s$ where $0 \le s < 1$, so $\log 0.3 < 0$, since $-1 + s < 0$ since $0 \le s < 1$.

$\log 0.999 = -1 + t$ where $0 \le t < 1$, so $\log 0.999 < 0$, since $-1 + t < 0$ since $0 \le t < 1$.

$\log 0.0299 = -2 + u$ where $0 \le u < 1$, so $\log 0.0299 < 0$, since $-2 + u < 0$ since $0 \le u < 1$.

Suppose for another instance, we set $\log N = -1.2345$.

Then, the characteristic is not -1, and the mantissa is not 0.2345, either.

That's because we have **0 ≤ the mantissa < 1**, and $-1.2345 = -1 - 0.2345$.

In fact, the mantissa of $\log N$ is $1 - 0.2345 = 0.7655$, and the characteristic is -2.

That's because $\log N = -1.2345 = -1 - 1 + 1 - 0.2345 = -2 + 0.7655 = \bar{2}.7655$.

Suppose for another instance, $\log N = \bar{1}.3582$.

Then, we get $\log N = -1 + 0.3582$, so it shows the characteristic and mantissa properly.

So if N in $\log N$ is between 0 and 1, then,

we need to note that the characteristic is negative, but the ***mantissa*** is ***positive***, which is the nature of a common log, and the nature doesn't change no matter what positive value the antilog N may have. In other words, we want to keep the mantissa positive.

Thus, putting the minus sign in front of the characteristic, we cannot keep the mantissa positive. So such an unusual placement of the minus sign is the means for that purpose.

Suppose for another instance, $\log N = -1.1812$, and $\log M = \bar{1}.1812$. Then, we get thses:

$\log N = -1.1812 \Rightarrow \log N = -1 + (-0.1812) = -1 - 0.1812$.

$\log M = \bar{1}.1812 \Rightarrow \log M = -1 + 0.1812$.

Note:

Using a calculator, and taking the common log of a number between 0 and 1, we just get a negative number in the display panel.

For instance, using a calculator, we get $\log 0.02 \approx -1.69897$ instead of $\bar{2}.30103$.

That's because $-1.69897 = -2 + 0.30103$, where -2 is the characteristic, and 0.30103 is the mantissa.

Now, reconsidering those x and y taken for example above, we have

$x = 1202.387$, and $y = 0.01202387$.

And taking common logs of x and y, we get $\log x = 3.abcd$, and $\log y = \bar{2}.abcd$.

More specifically,

$\log x = \log 1000(1.202387) = \log 10^3 + \log 1.202387 = 3 + \log 1.202387$

$\log y = \log 0.01(1.202387) = \log 10^{-2} + \log 1.202387 = -2 + \log 1.202387$

So the mantissas of both logs are the same, so for instance, both are **0.*abcd***.
Thus, the mantissa of $\log y$ is not negative. Neither is that of $\log x$, of course.

So $\log y$ is not $-2 + (-0.abcd)$ but $-2 + 0.abcd$.

Now, if $N \geq 1$, why does the characteristic c of $\log N$ have to be ≥ 0?

That's because the highest place value of a number ≥ 1 has to be $\geq 10^0$.

So of the common log of such a number, the characteristic is ≥ 0 since the highest place value of a number is 10^c where c is the characteristic of the common log of the number.
Still not quite sure?

Let's take a look at some examples that can help us see the reason better.
Suppose for instance, that the antilog $N = 1$. Then, $N = 1 = 10^0$.

So $\log N = \log 10^0 = 0 \log_{10} 10 = 0 \cdot 1 = 0$.

We have $\log N = c + m$ where c is an integer, and $0 \leq m < 1$. Thus, $c = 0$ and $m = 0$.

So anyway, it is true that $c \geq 0$, since $c = 0$.

Suppose for another instance, that $N = 3$. Then, $N = 3 = 1 \cdot 3 = 10^0(3)$.

Thus, $\log N = \log 3 = \log 10^0 + \log 3 = 0 + \log 3$. So $\log N = 0 + \log 3$.

And since $\log 1 = 0$ and $\log 10 = 1$, we get $1 \leq 3 < 10 \Rightarrow 0 \leq \log 3 < 1$.

Thus, $\log 3 = 0 + v$ where $0 \leq v < 1$.

And we have $\log N = c + m$ where c is an integer, and $0 \leq m < 1$.

Therefore, $c = 0$ and $m = v$. So anyway, it is true that $c \geq 0$, since $c = 0$.

Suppose for one more instance, that $\boldsymbol{N} = \mathbf{358}$. Then, $\boldsymbol{N} = \mathbf{358 = 100(3.58) = 10^2(3.58)}$.

Thus, $\mathbf{log}\ \boldsymbol{N} = \mathbf{log\ 358 = log\ 10^2 + log\ 3.58 = 2 + log\ 3.58}$.

And since $\mathbf{log\ 1 = 0}$ and $\mathbf{log\ 10 = 1}$, we get: $\mathbf{1 \leq 3.58 < 10 \Rightarrow 0 \leq log\ 3.58 < 1}$.

Thus, $\mathbf{log\ 358 = 2} + \boldsymbol{w}$ where $\mathbf{0} \leq \boldsymbol{w} < \mathbf{1}$.

And we have $\mathbf{log}\ \boldsymbol{N} = \boldsymbol{c} + \boldsymbol{m}$ where $\boldsymbol{c}$ is an integer, and $\mathbf{0} \leq \boldsymbol{m} < \mathbf{1}$.

Therefore, $\boldsymbol{c} = \mathbf{2}$ and $\boldsymbol{m} = \boldsymbol{w}$. So anyway, it is true that $\boldsymbol{c} \geq \mathbf{0}$, since $\boldsymbol{c} = \mathbf{2}$.

Besides a common log, we have another frequent log, where the base is normally not specified either.

Such a log is called a natural log, which is quite popular with many scientists.
In a natural log, the base is $\boldsymbol{e}$, called Euler's number, and we can see it quite often taking courses in calculus.

Conceptually, the number $\boldsymbol{e}$ can be described as $(1 + \boldsymbol{0})^{\infty}$ where $\boldsymbol{0}$ is called infinitesimal, and ∞ is called infinity. It is in fact, an irrational number, and can be roughly estimated to be 2.72. To be more accurate, it is approximately 2.718181828459045.

And taking a natural log of a number, we can take the log of the number to base $\boldsymbol{e}$.
That is, we can take log sub $\boldsymbol{e}$ of the number.
So for instance, taking the natural log of 3, we get $\mathbf{log}_{\boldsymbol{e}}\,\mathbf{3}$.

However, we often use a different sign, which is **ln**. And we use it without the base $\boldsymbol{e}$.

So for instance, taking the natural log of 3, we get $\mathbf{ln\ 3}$.

Thus, we have $\mathbf{log}_{\boldsymbol{e}}\ \mathbf{3 = ln\ 3}$, which can be read as natural log of 3 or just LN of 3.

In addition, we often use a capital **L**, too, instead of the lower case **l**.

So we have $\mathbf{Ln}\ \boldsymbol{N} = \mathbf{ln}\ \boldsymbol{N} = \mathbf{log}_{\boldsymbol{e}}\ \boldsymbol{N}$.

Anyway, taking a common log of $\boldsymbol{A}$, we can set $\mathbf{log}\ \boldsymbol{A} = \boldsymbol{c} + \boldsymbol{m}$, where $\boldsymbol{c}$ is an integer, and $\mathbf{0} \leq \boldsymbol{m} < \mathbf{1}$. And we call $\boldsymbol{c}$ the characteristic, and call $\boldsymbol{m}$ the mantissa.

How can we set it that way, though?

We can see it doing the examples in common logs.

B. Number Systems and Logs

Saying just a decimal number, we mean a number indicating a fraction as 0.23.

Saying a decimal number referring to a number system though, we mean a number that is made of powers of 10.

For instance, we can call 256 a decimal number if considering a number system, because we can put it this way: $256 = 2{\cdot}10^2 + 5{\cdot}10^1 + 6{\cdot}10^0$, which is a sum of powers of 10.

Such numbers are said to be in the decimal number system, called the decimal system, for short, and those numbers can be briefly called decimals, too.

And we define a number system using a base. What then is the base?

Each digit in a number has to have one single-digit number only. And the base is the number of single-digit integers we can use in each digit in every number in the number system having the base. So for instance, 10 is the base in the decimal number system.

That's because each digit in such a number has one of ten numbers from 0 to 9.
And we can call a decimal number a number to base 10, too.

And if the base is 2, we use in each digit, one of two numbers, which are 0 and 1. And such a number is called a binary number. Also, it can be called a number to base 2.

And from the decimal number's perspective, a binary number is a sum of powers of 2.

For instance, if we put a binary 101 into the decimal system, we get $1{\cdot}2^2 + 0{\cdot}2^1 + 1{\cdot}2^0$, which is $2^2 + 0 + 1 = 5$. So the decimal equivalent of a binary 101 is 5.

And we know taking the common log of a number, we can put the value of the log in terms of the characteristic and mantissa.

So if for instance, a positive integer $\boldsymbol{A}$ has $\boldsymbol{d}$ digits, what is the characteristic $\boldsymbol{c}$ of $\mathbf{log}\ \boldsymbol{A}$?

It is $\boldsymbol{d} - \mathbf{1}$. So we can set $\mathbf{log}\ \boldsymbol{A} = \boldsymbol{c} + \boldsymbol{m}$, where $\boldsymbol{c} = \boldsymbol{d} - \mathbf{1}$, and $\mathbf{0} \leq \boldsymbol{m} < \mathbf{1}$.
Then, we get $\boldsymbol{d} - \mathbf{1} \leq \mathbf{log}\ A < \boldsymbol{d}$. So we get $\mathbf{10}^{\boldsymbol{d}-\mathbf{1}} \leq \boldsymbol{A} < \mathbf{10}^{\boldsymbol{d}}$.

Thus, the highest place value in a $\boldsymbol{d}$-digit integer is $\mathbf{10}^{\boldsymbol{d}-\mathbf{1}}$, which is $\mathbf{10}^{\boldsymbol{c}}$, since $\boldsymbol{c} = \boldsymbol{d} - \mathbf{1}$.
How can we get this, though: $\mathbf{log}\ \boldsymbol{A} = \boldsymbol{c} + \boldsymbol{m}$, where $\boldsymbol{c} = \boldsymbol{d} - \mathbf{1}$, and $\mathbf{0} \leq \boldsymbol{m} < \mathbf{1}$?

Let's see now, how we can get this: $\mathbf{log}\ \boldsymbol{A} = \boldsymbol{c} + \boldsymbol{m}$, where $\boldsymbol{c}$ is an integer, and $\mathbf{0} \leq \boldsymbol{m} < \mathbf{1}$.

If we take a log of a number, the number is positive. Normally, if we take a common log of a number, the number is not only positive but ***decimal***, too. In fact, taking a log of a number, we normally use a decimal number weather the log is a common log or not.

Now, taking a common log of a number, we can get the characteristic, which can tell us the highest place value in the number. Such a characteristic can tell us the highest place value in the number, because both bases used in the log and the number are 10. The base in a common log is 10, and a decimal number is a number to base 10.

What if however, we want to find the highest place value in a number not decimal?
That is, the number is in a number system to base other than 10.

Then, it depends on the number system the number belongs to.
What do we mean by the number system though?

To begin with, we use a base in a number system, as well as in a log and a power.

If a number is decimal, the number is in the decimal number system, where the ***base*** is ***ten***. A decimal number is a number to base ten, and can be briefly called a decimal, too.

In a decimal number, each digit holds one of ten single-digit numbers, and they are 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.

A binary number is a number to base two, and can be briefly called a binary, too. In a binary, each digit holds one of two single-digit numbers, which are 0 and 1.

Next for instance, in a decimal integer where the base is **10**, the place values are $\mathbf{10^0}$, $\mathbf{10^1}$, $\mathbf{10^2}$, **…**, $\mathbf{10^{n-1}}$ if the integer has ***n*** digits.

By the same token, in a binary integer where the base is **2**, the place values are $\mathbf{2^0}$, $\mathbf{2^1}$, $\mathbf{2^2}$, **…**, $\mathbf{2^{n-1}}$ if the binary integer has ***n*** digits.

Suppose now, we want to find the highest place value in a number that is not decimal. Suppose for instance, we want to find the highest place value in a binary 101.

Then, the highest place value is not 10^2 but 2^2. Why?

That's because 101 is not $1{\cdot}10^2 + 0{\cdot}10^1 + 1{\cdot}10^0 = 100 + 1$ but $1{\cdot}2^2 + 0{\cdot}2^1 + 1{\cdot}2^0$, which is $4 + 1 = 5$, which is the decimal equivalent of a binary 101.

Now, what if we want to find the highest place value in a binary, which is the binary equivalent of a decimal 20435910?

Then, of course, converting the decimal to the binary equivalent, we can see the highest place value counting the digits in the binary equivalent.

Not all decimals can get readily converted, though.

Converting 20435910 to the binary equivalent, we might not have to spend too much time, but converting a decimal like 3^{3412}, we probably cannot avoid spending a long time if not using a calculator or computer.

We don't actually need to use such a gadget, though.
We just run math, instead. Running it, we can readily get the highest place value taking a log of the decimal 3^{3412} to base 2. How?

Suppose ***A*** is the decimal 3^{3412}, which is the decimal equivalent of the binary in which we want to find the highest place value.

Then, taking a log of A to base 2, we get $\log_2 A$, which can be put the way below:

$\log_2 A = u + v$, where u is an integer, and $0 \le v < 1$.

Then, the highest place value in the binary equivalent of the decimal 3^{3412} is 2^u.
More specifically, if the binary equivalent is an n-digit binary, u is $n - 1$.

That's because the highest place value an n-digit binary is 2^{n-1}.
For instance, in a 3-digit binary 101, the highest place value is: 2^{3-1}, which is: 2^2.

So if A is the decimal equivalent of an n-digit binary, taking a log of A to base 2, we get $\log_2 A = u + v$, where $u = n - 1$, and $0 \le v < 1$. How though, is it the case?

Suppose first, A is a k-digit integer positive and decimal, and d_m is each digit in A.
Then, expanding A in terms of digits, we can put it the way as follows.

$$A = d_0 10^0 + d_1 10^1 + d_2 10^2 + \ldots + d_{k-1} 10^{k-1}.$$

Normally, putting a number in writing, we just show its digits, so we put A in this form:
$d_{k-1}d_{k-2}d_{k-3} \ldots d_2d_1d_0$, of which the expansion is $d_0 10^0 + d_1 10^1 + d_2 10^2 + \ldots + d_{k-1} 10^{k-1}$.

For instance, $35028 = 8 \cdot 10^0 + 2 \cdot 10^1 + 0 \cdot 10^2 + 5 \cdot 10^3 + 3 \cdot 10^4$, which is a 5-digit decimal integer.

Suppose next, that we convert the decimal integer A into an n-digit binary integer B, and that b_m is each digit in B.
That is, B is equivalent to A, and can only look different, so both are the same in value.

Then, we can put B in $b_{n-1}b_{n-2}b_{n-3} \ldots b_2b_1b_0$, and call B, the binary equivalent of A.

So we can set $A = b_0 2^0 + b_1 2^1 + b_2 2^2 + \ldots + b_{n-1} 2^{n-1}$, too.

That is because A is the same as B in value. Thus, we can set

$$A = d_0 10^0 + d_1 10^1 + d_2 10^2 + \ldots + d_{k-1} 10^{k-1} = b_0 2^0 + b_1 2^1 + b_2 2^2 + \ldots + b_{n-1} 2^{n-1} = B.$$

So as a decimal integer, the highest place value in A is 10^{k-1}.
And as a binary integer, the highest place value in A is 2^{n-1}.

Now, let's quickly go over the ideas above.

Suppose A is a k-digit integer decimal and positive, and d_m is each digit in A.
Then, expanding A, we get

$$A = d_{k-1}d_{k-2}d_{k-3} \dots d_2d_1d_0 = d_0 10^0 + d_1 10^1 + d_2 10^2 + \dots + d_{k-1}10^{k-1}.$$

Suppose next, B is an n-digit binary integer, which is the binary equivalent of A, and b_m is each digit in B. Then, expanding B, we get

$$B = b_{n-1}b_{n-2}b_{n-3} \dots b_2b_1b_0 = b_0 2^0 + b_1 2^1 + b_2 2^2 + \dots + b_{n-1}2^{n-1}.$$

Now, A is equivalent to B, and can only look different. So we can set

$$\begin{aligned}
&A \\
&= d_{k-1}d_{k-2}d_{k-3} \dots d_2d_1d_0 \\
&= d_0 10^0 + d_1 10^1 + d_2 10^2 + \dots + d_{k-1}10^{k-1} \\
&= b_0 2^0 + b_1 2^1 + b_2 2^2 + \dots + b_{n-1}2^{n-1} \\
&= b_{n-1}b_{n-2}b_{n-3} \dots b_2b_1b_0 \\
&= B.
\end{aligned}$$

In sum, we have

$$A = d_0 10^0 + d_1 10^1 + d_2 10^2 + \dots + d_{k-1}10^{k-1} = b_0 2^0 + b_1 2^1 + b_2 2^2 + \dots + b_{n-1}2^{n-1} = B.$$

Then first, taking a common log of A, we can get
$\log_{10} A = \log A = c + m$, where c is an integer, and $0 \le m < 1$. (We will see why shortly.)

Then anyway, c is the characteristic, and m is the mantissa. More specifically, c is the characteristic for decimal numbers, and m is the mantissa for decimal numbers.

So as a decimal integer, the highest place value in A is 10^c. (We will see why shortly.)

Thus anyway, the characteristic $\boldsymbol{c = k - 1}$.

That is because if $\boldsymbol{A}$ is a $\boldsymbol{k}$-digit decimal integer, the highest place value in $\boldsymbol{A}$ is $\mathbf{10}^{\boldsymbol{k}\text{-}\mathbf{1}}$.

Next, taking a log of $\boldsymbol{A}$ to base 2, we can get

$\log_2 A = u + v$, where $\boldsymbol{u}$ is an integer, and $\mathbf{0} \le v < \mathbf{1}$.

Then, $\boldsymbol{u}$ is the characteristic for binary numbers, and $\boldsymbol{v}$ is the mantissa for binaries.

So the highest place value in the binary equivalent of $\boldsymbol{A}$ is $\mathbf{2}^{\boldsymbol{u}}$. (We will see why shortly.)

Thus anyway, the characteristic $\boldsymbol{u = n - 1}$.

That is because the highest place value in the $\boldsymbol{n}$-digit binary equivalent of $\boldsymbol{A}$ is $\mathbf{2}^{\boldsymbol{n}\text{-}\mathbf{1}}$.

Now, we are going to see the reason that we can set

$\log_2 A = u + v$, where $\boldsymbol{u}$ is an integer, and $\mathbf{0} \le v < \mathbf{1}$.

That is, we will get to see the answers to all the whys above. So to begin with, we know

$$A = d_0 10^0 + d_1 10^1 + d_2 10^2 + \dots + d_{k-1} 10^{k-1} = b_0 2^0 + b_1 2^1 + b_2 2^2 + \dots + b_{n-1} 2^{n-1} = B.$$

So we can set $A = b_0 2^0 + b_1 2^1 + b_2 2^2 + \dots + b_{n-1} 2^{n-1}$. Then first, we can get

$$A = b_0 2^0 + b_1 2^1 + b_2 2^2 + \dots + b_{n-1} 2^{n-1} = 2^{n-1} \cdot \frac{b_0 2^0 + b_1 2^1 + b_2 2^2 + \dots + b_{n-1} 2^{n-1}}{2^{n-1}}.$$

So taking a $\log_2$ of $\boldsymbol{A}$, we get

$$\begin{aligned}
&\log_2 A = \log_2 (2^{n-1} \cdot \frac{b_0 2^0 + b_1 2^1 + b_2 2^2 + \dots + b_{n-1} 2^{n-1}}{2^{n-1}}) \\
&= \log_2 2^{n-1} + \log_2 \frac{b_0 2^0 + b_1 2^1 + b_2 2^2 + \dots + b_{n-1} 2^{n-1}}{2^{n-1}} \\
&= (n-1) + \log_2 \frac{b_0 2^0 + b_1 2^1 + b_2 2^2 + \dots + b_{n-1} 2^{n-1}}{2^{n-1}}.
\end{aligned}$$

Meanwhile, we have

$$\frac{b_0 2^0 + b_1 2^1 + b_2 2^2 + ... + b_{n-1} 2^{n-1}}{2^{n-1}} = b_0 2^{0-n+1} + b_1 2^{1-n+1} + b_2 2^{2-n+1} + ... + b_{n-2} 2^{n-2-n+1} + b_{n-1}$$

$$= b_0 2^{1-n} + b_1 2^{2-n} + b_2 2^{3-n} + ... + b_{n-2} 2^{-1} + b_{n-1} = \frac{b_0}{2^{n-1}} + \frac{b_1}{2^{n-2}} + \frac{b_2}{2^{n-3}} + ... + \frac{b_{n-2}}{2} + b_{n-1}.$$

Now, for $m = 0, 1, 2, \ldots, n-1$, every b_m is 1 at most, since 1 and 0 are all the digits that can be used in a binary number, and for the same reason, every b_m is 0 at least.

Thus, $\frac{b_0}{2^{n-1}} + \frac{b_1}{2^{n-2}} + \frac{b_2}{2^{n-3}} + ... + \frac{b_{n-2}}{2} + b_{n-1}$ is at most $\frac{1}{2^{n-1}} + \frac{1}{2^{n-2}} + \frac{1}{2^{n-3}} + ... + \frac{1}{2} + 1$, and is at least $\frac{0}{2^{n-1}} + \frac{0}{2^{n-2}} + \frac{0}{2^{n-3}} + ... + \frac{0}{2} + 0 = 0$.

Next, we have $1 \le \frac{1}{2^{n-1}} + \frac{1}{2^{n-2}} + \frac{1}{2^{n-3}} + ... + \frac{1}{2} + 1 < 2$.

In other words, we have $1 \le \frac{1}{2^{n-1}} + ... + \frac{1}{2^3} + \frac{1}{2^2} + \frac{1}{2} + 1 < 2$. How, though?

We have $\frac{1}{2^{n-1}} + ... + \frac{1}{2^2} + \frac{1}{2} + 1 = \frac{1\{1-(\frac{1}{2})^n\}}{1-\frac{1}{2}} = \frac{1-\frac{1}{2^n}}{\frac{1}{2}} = 2(1-\frac{1}{2^n}) = 2 - \frac{1}{2^{n-1}}$. How?

Suppose $S = a + ar + ar^2 + ar^3 + \ldots + ar^{k-2} + ar^{k-1}$.

Then, $rS = ar + ar^2 + ar^3 + \ldots + ar^{k-1} + ar^k$.

Thus, we get $S - rS = S(1 - r)$

$= (a + ar + ar^2 + ar^3 + \ldots + ar^{k-2} + ar^{k-1}) - (ar + ar^2 + ar^3 + \ldots + ar^{k-1} + ar^k)$

$= a - ar^k = a(1 - r^k)$.

So we get $S(1 - r) = a(1 - r^k)$.

Thus, we get $\boldsymbol{S = a + ar + ar^2 + ar^3 + \ldots + ar^{k\text{-}2} + ar^{k\text{-}1} = \frac{a(1-r^k)}{1-r}}$.

Now, we have $\mathbf{\frac{1}{2^{n-1}} + \ldots + \frac{1}{2^3} + \frac{1}{2^2} + \frac{1}{2} + 1 = 1 + 1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2^2} + 1 \cdot \frac{1}{2^3} + \ldots + 1 \cdot \frac{1}{2^{n-1}}}$.

So taking 1 as $\boldsymbol{a}$ as 1 and taking $\frac{1}{2}$ as $\boldsymbol{r}$ in the sum $\boldsymbol{S}$ above, we get

$$\boldsymbol{S = \frac{a(1-r^k)}{1-r} = \frac{1\{1-(\frac{1}{2})^n\}}{1-\frac{1}{2}} = \frac{1-\frac{1}{2^n}}{\frac{1}{2}} = 2(1-\frac{1}{2^n}) = 2 - \frac{1}{2^{n-1}}}.$$

Next, since $\boldsymbol{n}$ is the number of digits in an integer, $\boldsymbol{n}$ is an integer $\geq \mathbf{1}$.

So if $\boldsymbol{n} = \mathbf{1}$, we get $\mathbf{2 - \frac{1}{2^{n-1}} = 2 - \frac{1}{2^{1-1}} = 2 - 1 = 1}$, and if $\boldsymbol{n}$ is large, we get $\mathbf{2 - \frac{1}{2^{n-1}} \approx 2}$.

Thus, we get $\mathbf{1 \leq 2 - \frac{1}{2^{n-1}} < 2}$, so we get $\mathbf{1 \leq \frac{1}{2^{n-1}} + \frac{1}{2^{n-2}} + \frac{1}{2^{n-3}} + \ldots + \frac{1}{2} + 1 < 2}$.

We know $\boldsymbol{\frac{b_0}{2^{n-1}} + \frac{b_1}{2^{n-2}} + \frac{b_2}{2^{n-3}} + \ldots + \frac{b_{n-2}}{2} + b_{n-1}}$ is at most $\mathbf{\frac{1}{2^{n-1}} + \frac{1}{2^{n-2}} + \frac{1}{2^{n-3}} + \ldots + \frac{1}{2} + 1}$.

So we get $\boldsymbol{1 \leq \frac{b_0}{2^{n-1}} + \frac{b_1}{2^{n-2}} + \frac{b_2}{2^{n-3}} + \ldots + \frac{b_{n-2}}{2} + b_{n-1} < 2}$.

We have $\boldsymbol{\frac{b_0 2^0 + b_1 2^1 + b_2 2^2 + \ldots + b_{n-1} 2^{n-1}}{2^{n-1}} = \frac{b_0}{2^{n-1}} + \frac{b_1}{2^{n-2}} + \frac{b_2}{2^{n-3}} + \ldots + \frac{b_{n-2}}{2} + b_{n-1}}$.

So we get $\boldsymbol{1 \leq \frac{b_0 2^0 + b_1 2^1 + b_2 2^2 + \ldots + b_{n-1} 2^{n-1}}{2^{n-1}} < 2}$.

Thus, we get $\boldsymbol{\log_2 1 \leq \log_2 \frac{b_0 2^0 + b_1 2^1 + b_2 2^2 + \ldots + b_{n-1} 2^{n-1}}{2^{n-1}} < \log_2 2}$.

So we get $\boldsymbol{0 \leq \log_2 \frac{b_0 2^0 + b_1 2^1 + b_2 2^2 + \ldots + b_{n-1} 2^{n-1}}{2^{n-1}} < 1}$.

Thus, we get: $\mathbf{\log_2 A = (n-1) + \log_2 \frac{b_0 2^0 + b_1 2^1 + b_2 2^2 + ... + b_{n-1} 2^{n-1}}{2^{n-1}}}$.

Now, setting $\boldsymbol{u = n - 1}$, and $\mathbf{v = \log_2 \frac{b_0 2^0 + b_1 2^1 + b_2 2^2 + ... + b_{n-1} 2^{n-1}}{2^{n-1}}}$, we get

$\mathbf{\log_2 A = u + v}$, where $\boldsymbol{u = n - 1}$, which is an integer, and $\mathbf{0 \le v < 1}$.

Thus, we get $\mathbf{u \le \log_2 A < u + 1 \Rightarrow 2^u \le A < 2^{u+1}}$.

Now, we know $\boldsymbol{A}$ is the decimal equivalent of the binary $\boldsymbol{B}$.
So the highest place value in the binary $\boldsymbol{B}$ is $\mathbf{2^u}$ where $\boldsymbol{u}$ is the characteristic for binary numbers. Of course, $\boldsymbol{u}$ is not a number to base two, that is, not a binary number.

We have $\boldsymbol{u = n - 1}$, where $\boldsymbol{n}$ is the number of digits in the binary $\boldsymbol{B}$.
Thus, the highest place value in the binary $\boldsymbol{B}$ is $\mathbf{2^{n-1}}$.

And the same is true for $\mathbf{\log_b A}$, where $\boldsymbol{A}$ is the decimal equivalent of a number to base $\boldsymbol{b}$.

That is, we get $\mathbf{\log_b A = s + t}$, where $\boldsymbol{s}$ is an integer, and $\mathbf{0 \le t < 1}$.

Thus, we get $\mathbf{s \le \log_b A < s + 1 \Rightarrow b^s \le A < b^{s+1}}$.

So the highest place value in the number to base $\boldsymbol{b}$ which is equivalent to $\boldsymbol{A}$ is $\underline{\boldsymbol{b}}^{\boldsymbol{s}}$.

Let's for instance, see if taking a $\mathbf{\log_8}$ of 98 works.

$\mathbf{\log_8 98 = \log_8 64 \cdot \frac{98}{64} = \log_8 64 + \log_8 \frac{98}{64} = 2 + \log_8 \frac{98}{64}}$.

We know $\mathbf{1 < \frac{98}{64} < 8 \Rightarrow \log_8 1 < \log_8 \frac{98}{64} < \log_8 8 \Rightarrow 0 < \log_8 \frac{98}{64} < 1}$.

So we can set $\mathbf{\log_8 98 = 2 + t}$ where $\mathbf{0 < t < 1}$, and thus, $\mathbf{s = 2}$.

In fact, $98 = 1{\cdot}8^2 + 4{\cdot}8^1 + 2{\cdot}8^0$, which is 142 to base eight.

That is, 98 is the decimal equivalent of an octal 142.

Now, let's see if it is the case where $\mathbf{8^s \le 98 < 8^{s+1}}$.

We know $\boldsymbol{s} = \mathbf{2}$, so $\mathbf{8}^{s} = \mathbf{8}^{2} = \mathbf{64}$, and $\mathbf{8}^{s+1} = \mathbf{8}^{3} = \mathbf{512}$.   So it is the case.

In general, we can set $\mathbf{log}_{b}\, \boldsymbol{A} = \boldsymbol{s} + \boldsymbol{t}$, where $\boldsymbol{s}$ is an integer, and $\mathbf{0} \leq \boldsymbol{t} < \mathbf{1}$.
Then, the integer $\boldsymbol{s}$ is called the *index*, and $\boldsymbol{t}$ is called the mantissa of $\mathbf{log}_{b}\, \boldsymbol{A}$.

More specifically,

the index $\boldsymbol{s}$ can be called the index for the numbers to base $\boldsymbol{b}$.
The mantissa $\boldsymbol{t}$ can be called the mantissa for the numbers to base $\boldsymbol{b}$.
So the highest place value in the number to base $\boldsymbol{b}$ is $\boldsymbol{b}^{s}$, where $\boldsymbol{s}$ is the index (called the characteristic, too) for the numbers to base $\boldsymbol{b}$.

Therefore, $\mathbf{log}_{b}\, \boldsymbol{A} = \boldsymbol{s} + \boldsymbol{t} \Leftrightarrow \boldsymbol{s} \leq \mathbf{log}_{b}\, \boldsymbol{A} < \boldsymbol{s} + \mathbf{1} \Leftrightarrow \boldsymbol{b}^{s} \leq \boldsymbol{A} < \boldsymbol{b}^{s+1}$, where $\boldsymbol{A}$ is positive, of course, and is the decimal equivalent of a number to base $\boldsymbol{b}$.

C. Approximations

To begin with, what is a log-value?

Taking the value of the log of a number to a base, we get the log-value to the base.
So a log-value depends on the base to which we take a log of the number.
For instance, the log of 16 to base 2 is 4, but the log of 16 to base 4 is 2, so in this case, 4 is the log-value to base 2, and 2 is the log-value to base 4. So what is a log-value?

It's an exponent that we use expressing a number in a power of a particular base.
Quite usually though, saying just a log, we mean a common log, where the base is 10.
So saying just a log-value, we usually mean the value of a common log.
How do we get such a value, though?

We can get it using a calculator, of course. What if no calculator or none of that sort?

If we want to get a log-value as the value of **log 2**, a log table can help.
Usually, such a log table is for values of common logs.

So if we want to get the value of the common log of a particular number, we can use a log table, which is a list of some antilogs and their log-values, that is, some numbers and the values of the common logs of those numbers. Thus, if the list includes the particular number we want to take the common log of, we can get the log-value at once.

Usually though, log-values listed in a log table are only mantissas, and antilogs are only between 1 and 10. (If not sure of mantissas, refer to the section, **Common Logs**.)

Thus, getting the value of a log of a particular number using a log table, we can get a problem where the particular number is not listed in the table.   What then can we do?

We can get an *approximation*. We can get a value close enough to the value of the log of the particular number. So let's see now, how we can get the approximation.

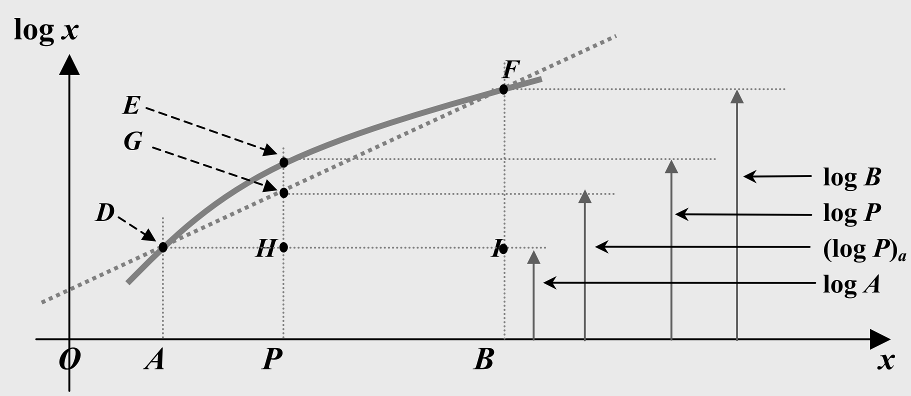
Suppose now, ***P*** is positive, the table has two antilogs ***A*** and ***B***, which are sufficiently close to ***P***, and ***A* < *P* < *B***, and we want to get the value of **log *P***, but ***P*** is not in the table.

Then, we can get an approximate value of **log *P***, and can get it the way as follows.

First, we can get the values of **log *A*** and **log *B***, since they are listed in the table.

Next, putting in a graph the two log-values, along with a part of a log-curve, we can get

Fig. 0



The distance from ***E*** to ***G*** should be sufficiently small, which means negligibly small. Then, we can get a good approximation by means of ratios that apply to similar triangles.

To begin with, we can see that two triangles ***DIF*** and ***DHG*** are similar to each other.

So next, assuming ***AB*** is the distance from ***A*** to ***B***, we can see this:
AB* : *AP* = *DI* : *DH* = *FI* : *GH.

That is to say that we get $\frac{AB}{AP} = \frac{DI}{DH} = \frac{FI}{GH}$. So we get $\frac{AB}{AP} = \frac{FI}{GH}$.

And assuming $(\log P)_a$ is an approximation of $\log P$, we can get $GH + \log A = (\log P)_a$.

So finding GH, we can an approximate value of $\log P$. How?

We can get it from the ratio equation above.

So first, we can get $\frac{AB}{AP} = \frac{FI}{GH} \Rightarrow GH \cdot \frac{AB}{AP} = FI \Rightarrow GH = FI \cdot \frac{AP}{AB}$.

And next, we can get $FI = \log B - \log A$, $AP = P - A$, and $AB = B - A$.

So we get $GH = (\log B - \log A) \cdot \frac{P - A}{B - A}$.

Now, conversely, what if a log-value is given, and we want to get the antilog, but the log-value is not in the table?

For instance, using a log table, we want to find x for which we get $0.3581 = \log x$, but the table does not have **0.3581**.

The exact solution is $x = 10^{0.3581}$, and using a calculator, we can get the value of x, that is, the value of the power $10^{0.3581}$. What if however, no calculator is allowed?

We can get an approximate value, called an approx value for short.
And we can get it using a log-table, too. How?

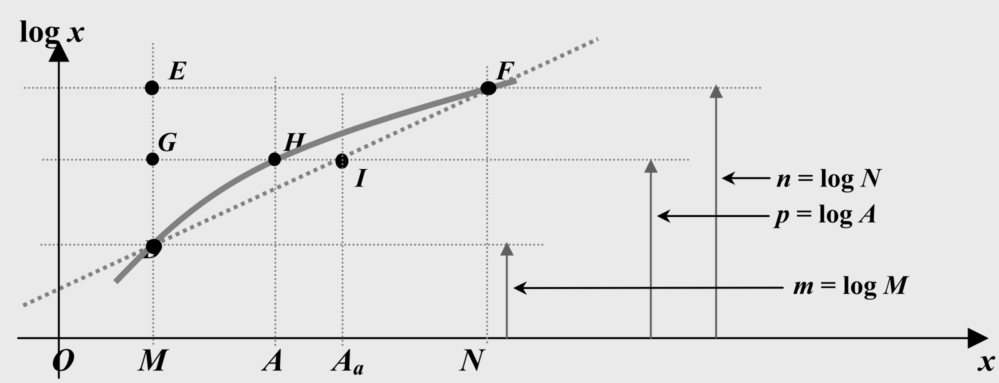
We can use basically the same idea as the one for an approximation of a log-value.
So suppose now, $p = \log A$, the table has two log-values m and n, which are sufficiently close to p, where $m < p < n$, and we want to get A, but A is not in the table.

Then, we can get an *approx value* of $\boldsymbol{A}$, and can get it the way as follows.

To begin with, we are given two antilogs, which are the values of $\mathbf{10}^{\boldsymbol{m}}$ and $\mathbf{10}^{\boldsymbol{n}}$, for they are listed in the table.

Next, setting $\boldsymbol{M} = \mathbf{10}^{\boldsymbol{m}}$, and $\boldsymbol{N} = \mathbf{10}^{\boldsymbol{n}}$, and putting in a graph the two antilogs $\boldsymbol{M}$ and $\boldsymbol{N}$, along with a part of a log-curve, we get

Fig. 1. The distance between $\boldsymbol{H}$ and $\boldsymbol{I}$ should be small enough.



Then again, we can get the approximation of $\boldsymbol{A}$ using ratios apply to similar triangles.

First, we can see that two triangles $\boldsymbol{DGI}$ and $\boldsymbol{DEF}$ are similar to each other.

So next, we can see that $\boldsymbol{DG} : \boldsymbol{DE} = \boldsymbol{GI} : \boldsymbol{EF}$. That is, we can get $\dfrac{\boldsymbol{DG}}{\boldsymbol{DE}} = \dfrac{\boldsymbol{GI}}{\boldsymbol{EF}}$.

And assuming $\boldsymbol{A_a}$ is an approximation of $\boldsymbol{A}$, we can get $\boldsymbol{M} + \boldsymbol{GI} = \boldsymbol{A_a}$.
So finding $\boldsymbol{GI}$ using the ratio equation above, we can an approximation of $\boldsymbol{A}$.

Thus, to begin with, we can get $\dfrac{\boldsymbol{DG}}{\boldsymbol{DE}} = \dfrac{\boldsymbol{GI}}{\boldsymbol{EF}} \Rightarrow \boldsymbol{GI} = \boldsymbol{EF} \cdot \dfrac{\boldsymbol{DG}}{\boldsymbol{DE}}$.

And next, we can get $\boldsymbol{EF} = \boldsymbol{N} - \boldsymbol{M}$, $\boldsymbol{DG} = \mathbf{log}\ \boldsymbol{A} - \mathbf{log}\ \boldsymbol{M}$, and $\boldsymbol{DE} = \mathbf{log}\ \boldsymbol{N} - \mathbf{log}\ \boldsymbol{M}$.

So we get $\boldsymbol{GI} = (\boldsymbol{N} - \boldsymbol{M}) \cdot \dfrac{\mathbf{log}\ \boldsymbol{A} - \mathbf{log}\ \boldsymbol{M}}{\mathbf{log}\ \boldsymbol{N} - \mathbf{log}\ \boldsymbol{M}}$.

Now, usually, log-values listed in a log table are mantissas, which are between 0 and 1, and antilogs are numbers between 1 and 10.

Then, what if the log-value we want to get an approximation of is bigger than 1 or the antilog we need to get an approximation of is larger than 10 or between 0 and 1?

For instance, the log-value can be between 23.4 and 23.5, and the antilog can be between 87 and 88, or between 0.123 and 0.124. What then can we do?

Shifting the entire digits in a number, and then, taking the log of the new number, we get a new log-value, but the new log-value still has the same mantissa. Not quite sure?

We know a log-value is composed of an integer and a fractional number.
The integer is called the characteristic, and the fractional number is between 0 and 1, can be 0, too, and is called the mantissa.
(If not sure of characteristics, and mantissas, refer to the section, **Common Logs**.)

So a log-value is made of a characteristic and a mantissa, and we can put a log of $\boldsymbol{A}$ in such a way as follows:

$\mathbf{log}\ \boldsymbol{A = c + m}$, where $\boldsymbol{c}$ is the characteristic, $\boldsymbol{m}$ is the mantissa, and $\mathbf{0} \leq \boldsymbol{m} < \mathbf{1}$. So what?

Suppose now, if we take a log of a number, the characteristic is 0, but the mantissa is not.
Suppose next, we shift the entire digits in the number.
That is, we move the decimal point several digits to the left or right.
In other words, $\boldsymbol{A}$ gets multiplied or divided by a power of 10.

Then, we get a new number, of course, but the new number still has the same sequence of digits. That is, the new number keeps the same sequence of number-alphabets.
What number-alphabets?

Number-alphabets are 1-digit numbers as 0, 3, and 8.

So if we take the log of the new number, we get a new characteristic, but the mantissa is the same. That is to say that, ***c*** changes, but ***m*** remains the same.

(Why the same mantissa? Refer to the section, **Common Logs**.)

Therefore, using such a shifting method, together with the approximation method, we can get the approximations of the numbers that are larger or smaller than the ones listed in a log table.

For instance, we can have **log 2.34 = 0.3692** in a log table.

And shifting the entire digits in 2.34 to the left by 3 decimal places, we get 2340.

That is, moving the decimal point 3 digits to the right, we get 2340.

In other words, multiplying 2.34 by 10^3, we get 2340.

Then, the highest place value in 2340 is 10^3, where the exponent 3 is the characteristic.

So we get **log 2340 = 3.3692**.

Shifting, for another instance, the entire digits in 23.4 to the right by 2 decimal places, we get 0.234.

That is, moving the decimal point 2 digits to the left, we get 0.234.

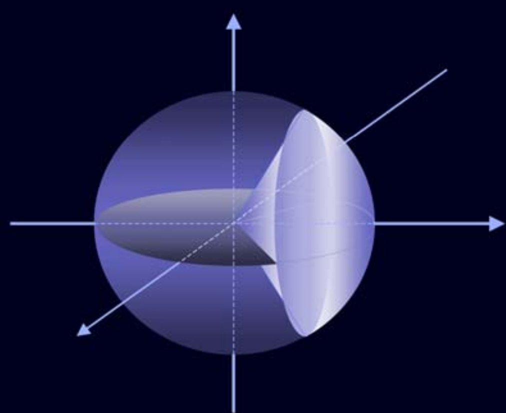
In other words, dividing 23.4 by 10^2, we get 0.234.

So the highest place value in 0.234 is 10^{-1}, where the exponent -1 is the characteristic.

Thus, we get **log 0.234 = $\bar{1}$.3692**, which means however, not –1.3692 but –1 + 0.3692.

So if using a calculator, we get **log 0.234 = -0.6308**, which is equivalent to **$\bar{1}$.3692**.

(Why the bar on top of 1 though? Refer to the section, **Common Logs**.)



거듭제곱과 로그

Powers and Logarithms

www.runmath.com