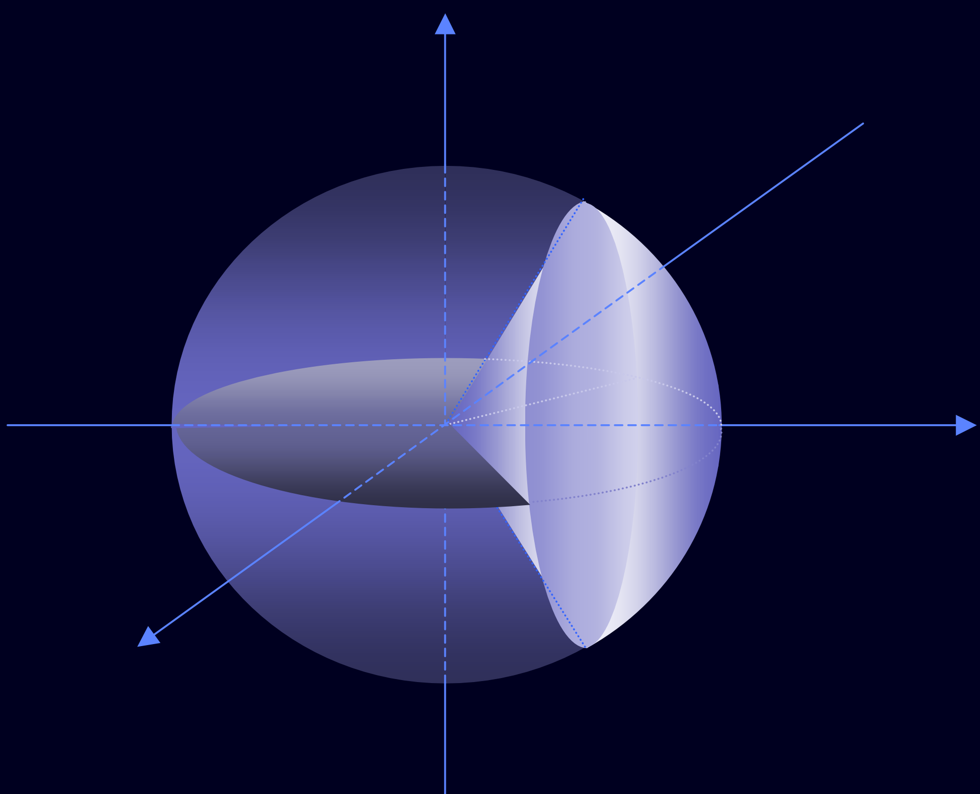
삼각함수

Trigonometry



김성렬

Seong R. Kim

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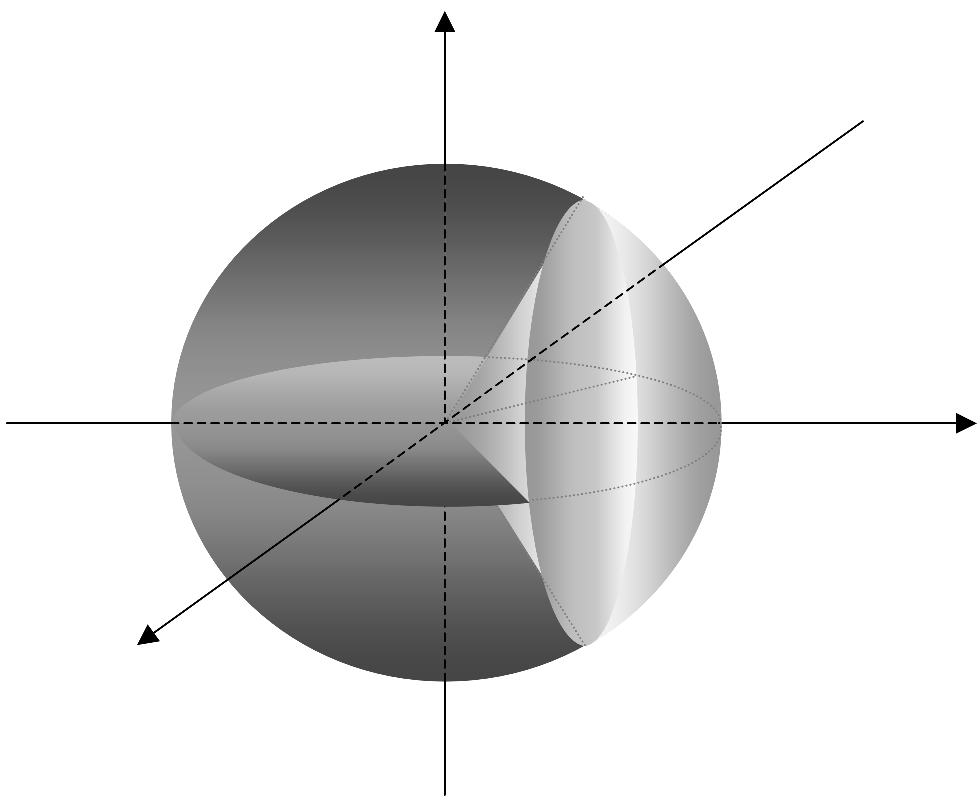

Seong R. KIM

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Dear students,

Students need the best teacher, so you need examples, because examples are the best teacher. This is thus, a book of examples. All the examples here are fully worked, and explain **how** the basic but essential tools in math are made, together with **what** they are, **how** they work, and **how** to work with them. Such tools include numbers, formulas, identities, equations, laws, etc.

Doing math, we work with ideas and run ideas, because every thing in math is an idea. What idea though?

It's a concept. So we need to understand it if we want to use it. A number is an idea, for instance, and the same is true for a line or circle, too. And putting ideas together, we can build another, which becomes the base or an element of another, and each is connected. And that's the way your math grows.

So you get to build a circuit, and sometimes, need to fill the gap or repair the circuit so that you can get the sense of it. Your calculation thus, runs properly, and gets to the solution.

Getting the concept doing good examples, and gaining the fluency doing right examples, you can build the circuit strong.

This book is however, nothing but a bunch of examples until you get it powered. How then to get it powered, and make it run and work for you?

Read it closely at your pace, and then, do each example yourself. It is quite important to note that you do it in **your** own writing. Just watching someone doing it, you just only feel that you can do it, too, but can hardly do it when you need it. You do it, you get it. And you’ve got to do it over with different examples. It’s a cliché, of course, but is always true that knowing is one thing and doing is another.

I’ve been helping students get conceptual understanding and fluency so that they can grow, take care of, and run their own math. The area covers the basics in algebra and geometry, and includes equations for unknowns and those for curves, functions and their graphs, and other math tools, which are the basic elements in calculus, which’s been the core of my interest from my early age in high school.

Of my students, some are quite poor in math, and thus, are afraid of or even hate math, some require special education because of exceptional intelligence, some are smart enough, some are naïve and diligent, and many are clever but lazy. All the students are badly after though, one thing in common.

A Strong Foundation in Algebra

It is of course, the prime objective of my work, and I'm always happy and eager to help them achieve it. I can only help though. Your math grows as much as you grow and take care of it yourself. So does mine. What math then do students most often do or use in high schools or colleges?

Algebra and geometry.   What algebra then?

Elementary algebra, of course. Doing the algebra, you should be able to not only work with, but manipulate, that is, change, alter, and convert math expressions the way you want, so for instance, you can break apart and factor polynomials in many kinds. Also, you can handle properly ratios and rates, trigonometry, factors or divisors, constants and variables, powers and logarithms or exponents, equations and functions, along with their graphs, equalities and inequalities, identities and formulas, laws and rules, etc.

And if you want to handle them properly, you want to know their natures and how they mingle with each other.

So studying those stated above, you want to know **what** they are, **how** they work, and **how** to work with them or **what** to do with them. What then about the geometry?

Basically, the geometry is about shapes, and has much to do with positions and angles in algebra. Why algebra?

You often need to compute lengths, areas, or volumes in many situations. The shapes begin with points, lines, triangles, and circles, and move on to rectangles, squares, parallelograms or rhombuses, trapezoids, tetragons, other polygons as pentagons, polyhedrons as hexahedrons, etc.

Doing the geometry, too, we need to do the algebra stated above such as trigonometry. So it is analytic geometry, often called coordinate geometry, too. And doing it, we can specify positions using coordinates. So in that geometry, basically, we work with graphs. Putting a math idea in a graph, we can not only effectively think about it, but actually see it, too, and therefore, can not just easily, but efficiently work with it, too. What idea then is it?

The idea begins with a point, a line, a parabola, a circle, an ellipse, and a hyperbola, called a conic section or basic curve, and then, moves on to other curves, planes, surfaces, volumes, and other objects in various dimensional spaces, together with vectors. And using an angle, we can specify and work with an amount of turning or change in direction.

So learning, using, or applying those ideas or tools in math, we get to solve problems. And this book can help. It can help you build and run your own math engine in your brain, and thus, can help achieve your solid math skill.

It is however, not a magic book giving you a math skill of high caliber overnight. And it's just a book of examples, and can have many mistakes. I'm a human, too. There's no mojo making a math genius, and math is science, and is full of facts and ideas. And it is after all, not me, not your teacher, and not any book but you who actually make you put together some of those facts and ideas, and understand it.

Putting facts and ideas together, understanding it, and taking care of what you have learned doing examples, you grow your math. And this book is ready to help and can help you help yourself.

Solving a problem, we put it many different ways. For instance, while expanding or reducing it, or modifying or converting it, we keep searching for the solution, keep approaching the solution, and eventually, can get there.

It is often the case a problem itself is the solution. We can put a problem in many different ways, and eventually, can end up with the solution. How come then is the solution no other than the problem?

For instance, the solution to 3232 $\div$ 101 is 32. And we can put it this way:

3232 $\div$ 101 $= \frac{3232}{101} = \frac{32 \text{ x } 101}{101} = \frac{32}{1} = 32 \;\Rightarrow$ 3232 $\div$ 101 = 32.

And we can get this, too: 32 = 3232 $\div$ 101.   How?

$32 = \frac{32}{1} = \frac{32 \text{ x } 101}{101} = \frac{3232}{101} =$ 3232/101 = 3232 $\div$ 101.

Too easy?

For another instance, the solution to $\boldsymbol{ax^2} + \boldsymbol{bx} + \boldsymbol{c} = \mathbf{0}$ is

$\boldsymbol{x} = \frac{-\boldsymbol{b} \pm \sqrt{\boldsymbol{b}^2 - 4\boldsymbol{ac}}}{2\boldsymbol{a}}$, which is called the quadratic formula.

How come then is the solution no other than the problem?

We can put it this way:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow 2ax = -b \pm \sqrt{b^2 - 4ac}$$

$$\Rightarrow 2ax + b = \pm\sqrt{b^2 - 4ac} \Rightarrow (2ax + b)^2 = b^2 - 4ac$$

$$\Rightarrow 4a^2x^2 + 4abx + b^2 = b^2 - 4ac \Rightarrow 4a^2x^2 + 4abx = -4ac$$

$$\Rightarrow ax^2 + bx = -c \Rightarrow ax^2 + bx + c = 0.$$

And we can get this, too:

$$ax^2 + bx + c = 0 \quad \Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{How?}$$

$$ax^2 + bx + c = a(x^2 + \frac{b}{a}x) + c = a(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}) + c$$

$$= a(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}) - \frac{b^2}{4a} + c = a(x + \frac{b}{2a})^2 - \frac{b^2 - 4ac}{4a} = 0$$

$$\Rightarrow a(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a} \Rightarrow (x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\Rightarrow x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \Rightarrow x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

So when solving a problem, we expand or reduce it, modify or convert it, and keep searching for and approaching the solution. Then eventually, we can get there. What then do we do when doing the set of processes above?

We do algebra. You need a strong foundation in algebra. So if a problem is well defined, that is, if it makes sense, we should be able to get it solved the way as follows.

A problem $\Rightarrow$ **…** $\Rightarrow$ **…** $\Rightarrow$ **the solution**,

and thus, **the problem** $\Rightarrow$ **the solution**.

So solving a problem, we put it many different ways so that we can get to the solution. And that's the way math runs.

Many more examples are available for free, and you can get them downloaded at **www.runmath.com**. So visit the website, get them downloaded, and do the examples. The website is getting updated weekly or monthly.

May your math run very well.

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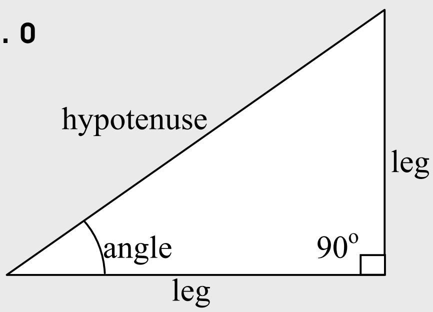
Contents

0.0. The Basics

To begin with, what is trigonometry, and what is it about?

It's about ***ratio***s. And we have three basic ratios in trigonometry. Each is between two sides in a triangle where an angle is 90^o, called a right angle. So such a triangle is called a right triangle.

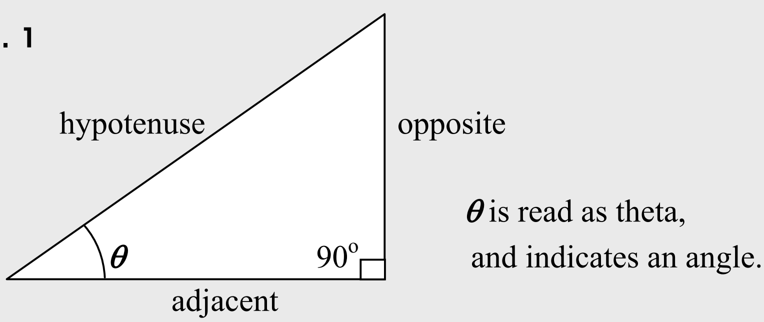
Fig. 0



In a right triangle, an angle is 90^o, called a right angle, and is between the two sides called legs. And showing a right angle, we use a small rectangle

And doing trigonometry, we name the sides in a right triangle the way as follows.

Fig. 1



As in Fig. 1, the side slanted and connecting the two legs is called the ***hypotenuse***,

the side adjacent to the angle θ is called the ***adjacent*** for short,
so the angle θ is between the hypotenuse and the adjacent,

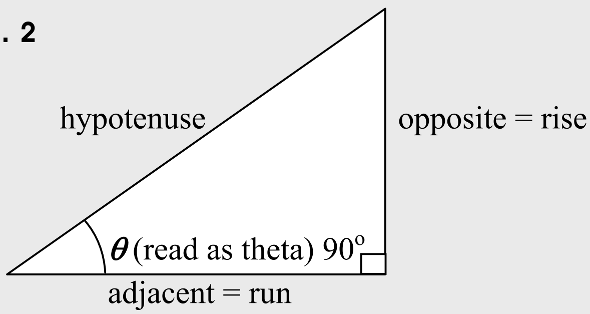
and the side opposite to the angle is just called the ***opposite***.

Thus, a right triangle is made of the hypotenuse, the adjacent, and the opposite.
What then are the three basic ratios?

They are the sine, the cosine, and the tangent. What are they about?

To get straight to the point, the ***sine*** is about the side called the ***opposite***, and the ***cosine*** is about the side called the ***adjacent.*** What about the tangent?

Fig. 2



The ***tangent*** is the ***slope*** of the hypotenuse, is often called ***rise over run***, and is in fact, ***the sine over the cosine***, and thus, is the opposite over the adjacent.

$$\text{the tangent} = \frac{\textbf{\textit{opposite}}}{\textbf{\textit{adjacent}}} = \frac{\text{the sine}}{\text{the cosine}} = \frac{\textbf{\textit{rise}}}{\textbf{\textit{run}}}$$

And if the hypotenuse is 1, the sine is the opposite, and the cosine is the adjacent.
Opposite to what and adjacent to what?

Opposite to an angle, adjacent to the angle,
and the angle is between the adjacent and the hypotenuse.

So the angle matters, since the ratios depend on the angle.

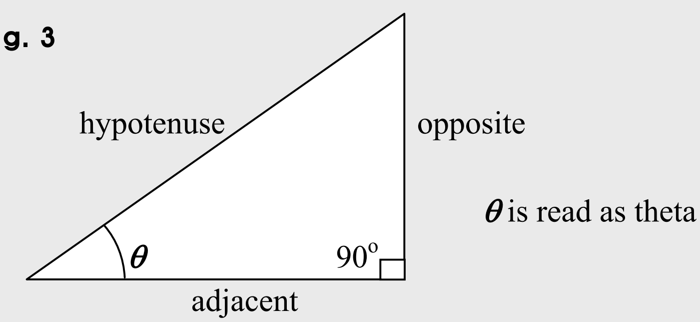
In short,
the ***sine*** is about the ***opposite***,
the ***cosine*** is about the ***adjacent***,
the ***tangent*** is the ***slope***,
and the adjacent and the hypotenuse make the angle.

In fact, if the hypotenuse is 1, the ***sine*** is just the ***opposite***, and the ***cosine*** is simply the ***adjacent***. And anyway, the ***tangent*** is the ***slope***. What slope?

The slope of the hypotenuse. Let's now cover a bit more of the detail.
So first, trigonometry is about ratios. What ratios then?

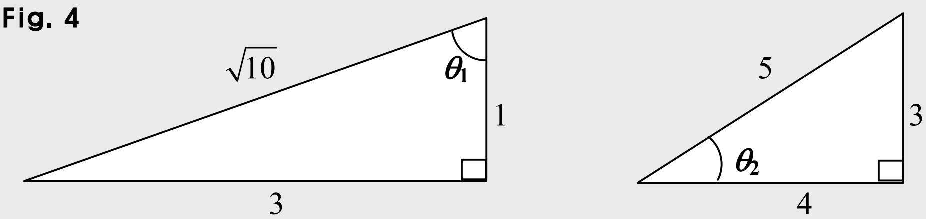
Trigonometric ratios. Each ratio is between two sides in a right triangle.

Fig. 3



So it's a *ratio* of one side to the other. In a right triangle, we can make a ratio using two from the three sides, and make three ratios. Each is called the ***sine***, the ***cosine***, or the ***tangent***. In fact, more ratios exist, but are derived ones. So basically, trigonometry is about trigonometric ratios, often called ***trig ratios***. That's not it, though. One more to it.

Fig. 4



In a right triangle, two sides make an angle, three angles are made, and one of the three matters, since trig ratios depend on that angle. Which angle is it, and how does it matter?

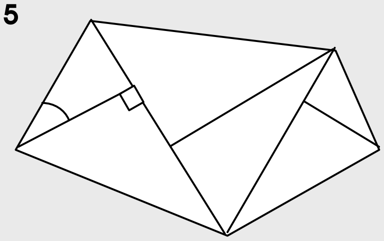
Finding a trig ratio, we need to know ***which angle***. The angle is between the two sides that we use finding the cosine. So the hypotenuse makes that angle with the adjacent. Wrong angle, wrong ratio. (By the way, in math, a right angle means 90°.) If the angle changes, all ratios change. So the angle matters, and in this book, the angle is called the ***governing*** angle, since the angle governs the ratios.

Thus, trigonometry has a lot to do with angles. And we can say that the choice of the adjacent matters, too, so we need to make sure which side the adjacent is, because the governing angle depends on the choice of the adjacent, since the adjacent makes the governing angle with the hypotenuse. So right adjacent, right ratios.

(Please note that governing angle is not an official designation. I just call it that way for convenience, so it's effective in this book only.) What then is the use of those ratios?

Using trig ratios, we can find the length, force or weight, speed or velocity, etc. in structural settings involving angles or triangles. How to use the ratios then?

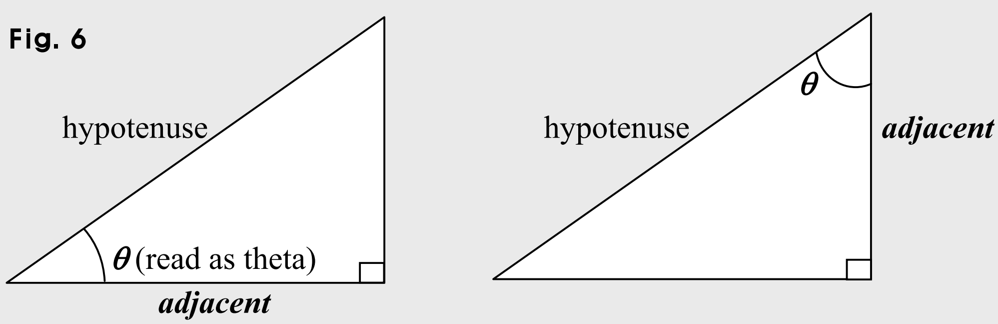
Fig. 5



In a structure made of right triangles as shown in Fig. 5 above, multiplying known components by proper trig ratios, we can get unknown components. The proper ratios depend on how each unknown component is related to its governing angle. We'll cover such relations and how the ratios work later.

Let's now briefly, cover the basics of three ratios. Their names are the sine, the cosine, and the tangent. So to begin with, what is the cosine, and what is it about?

Fig. 6



It's about the ***adjacent***. Adjacent to what?

Adjacent to the angle. What angle then?

As shown in Fig. 7 below, θ is the angle, and is called the governing angle in this book. So the governing angle is between the hypotenuse and the adjacent. That angle matters. Specifying a trig ratio, we use the governing angle, along with the name. So in the case of Fig. 7, we can say ***the cosine of θ***, often just say cosine θ, and θ is read as theta.

For instance, if $\theta = 30^{\circ}$, we can say the cosine of 30°, or just cosine 30°. Usually though, it's written as **cos 30°**, read as cosine thirty degrees.
By the way, we can express angles in two different ways. One is in degrees, and the other is in radians, covered in a later section. Note that the length of the side adjacent to the governing angle is often just called ***the adjacent***.

Fig. 7

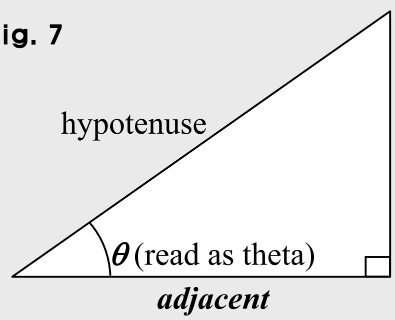
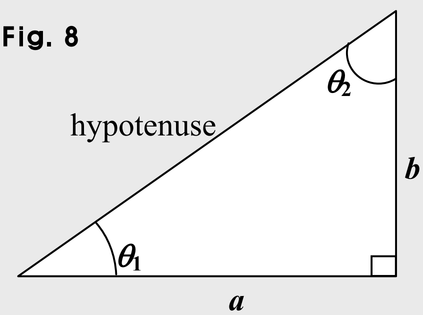


Fig. 8



In Fig. 8, if θ_2 is the governing angle, b is the side adjacent to the governing angle, and if θ_1 is the governing angle, a is the side adjacent to the governing angle. So we need to make sure where the governing angle is, or which side the adjacent is.

> Now, the ***cosine*** is the ***ratio*** between the ***two sides forming the governing angle***, one of the two is the ***adjacent***, and the other is the ***hypotenuse***.

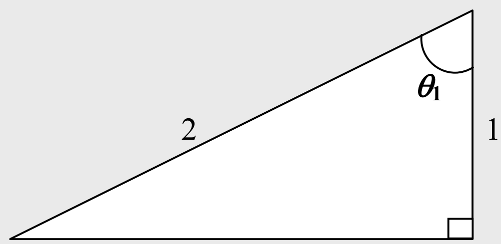
We can put the cosine in several different ways as follows.

The cosine is ***the adjacent*** *compared* to the hypotenuse. In other words, the cosine is *the ratio of* ***the adjacent*** to the hypotenuse, that is, ***the adjacent*** *over* the hypotenuse.
The same story in different words, of course.

For instance, if in Fig. 9, the cosine is $\frac{1}{2}$, the adjacent is half the hypotenuse.

Fig. 9

$\cos\theta_1 = 1/2$

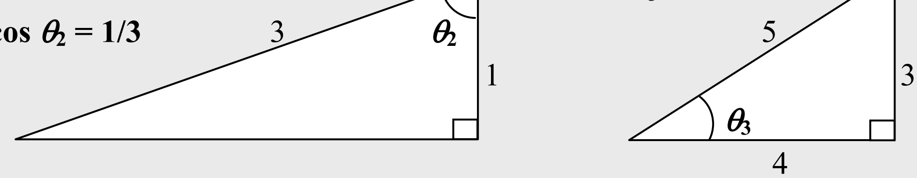


If the cosine is $\frac{1}{3}$ as shown in Fig. 10, the hypotenuse is 3 times the adjacent.

Fig. 10

$\cos\theta_2 = 1/3$

$\cos\theta_3 = 4/5$



As in Fig. 10 above, if the adjacent is 4, and the hypotenuse is 5, the cosine is $\frac{4}{5}$, so we can say that the ***cosine*** is the ***adjacent over*** the ***hypotenuse***.

$$\cos\theta = \frac{\textbf{\textit{adjacent}}}{\textbf{\textit{hypotenuse}}}$$

How then can we use the cosine?

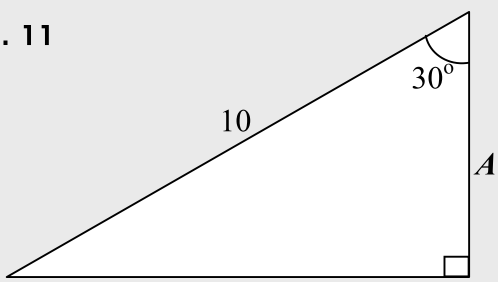
Suppose in a right triangle, we know an angle between the hypotenuse and a leg, and the leg is the adjacent. Then, if multiplying the hypotenuse by the cosine of the angle, what do we get?

We get the adjacent.

$$\cos\theta = \frac{\textbf{\textit{adjacent}}}{\textbf{\textit{hypotenuse}}}, \text{ so } \textbf{\textit{hypotenuse}}\cdot\cos\theta = \textbf{\textit{hypotenuse}}\cdot\frac{\textbf{\textit{adjacent}}}{\textbf{\textit{hypotenuse}}} = \textbf{\textit{adjacent}}$$

Thus, doing a problem where we know the hypotenuse, but the adjacent is unknown, we can use the cosine to get the adjacent. And using the cosine in this case, we multiply the hypotenuse by the cosine to get the adjacent.
For instance, suppose in a right triangle, the hypotenuse is 10, the governing angle is 30°, and the adjacent is ***A***, which is unknown.

Fig. 11



Multiplying the hypotenuse 10 by the cosine 30°, we get $10\cdot\cos 30^{\circ} = 10\cdot\frac{\boldsymbol{A}}{10} = \boldsymbol{A}$.

And we have $\cos 30^{\circ} = 0.5 = \frac{1}{2}$. So we get $10\cdot\cos 30^{\circ} = 10\cdot\frac{1}{2} = 5$.

Thus, we get ***A* = 5**. In sum, $\boldsymbol{A} = 10\cdot\cos 30^{\circ} = 10\cdot\frac{1}{2} = 5$, where 10 is the hypotenuse, and ***A*** is the adjacent. So we can use the cosine to find the adjacent.

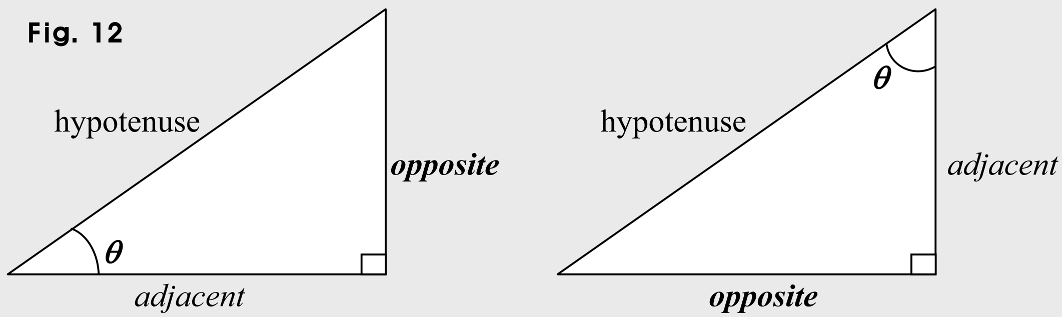
So what is the ***cosine*** about?

About the side adjacent to the governing angle.
The cosine is about ***the adjacent***.

Using the ***cosine***,
we multiply the hypotenuse by the cosine
to get the ***adjacent***.

Note that when the hypotenuse is 1, the cosine is actually, the adjacent itself, because the cosine is the adjacent over the hypotenuse. We'll see that it is convenient when we work with trigonometric functions. For the cosine function $\boldsymbol{y = f(x) = \cos x}$, where $\boldsymbol{x}$ is an angle, the graph shows how the adjacent changes as the angle $\boldsymbol{x}$ changes.
Let's now, move on to the sine, and talk about what the sine is about.

Fig. 12



The ***sine*** is about the side ***opposite***. What opposite?

Opposite to the angle. What angle then?

The governing angle. The side ***opposite to*** an angle is the side ***facing*** the angle.
As in Fig. 13 below, $\boldsymbol{\theta}$ is the governing angle, since the side labeled ***opposite*** is *facing* the angle $\boldsymbol{\theta}$.
Specifying a trig ratio, we need to use the governing angle, also. So in Fig. 11, we can say ***the sine of*** $\boldsymbol{\theta}$, or just say ***sine*** $\boldsymbol{\theta}$, written as **sin** $\boldsymbol{\theta}$, read as sine theta.
For instance, if $\boldsymbol{\theta} = 60^{\circ}$, we say the sine of 60°, or say sine 60°, written as **sin** 60°.

Fig. 13

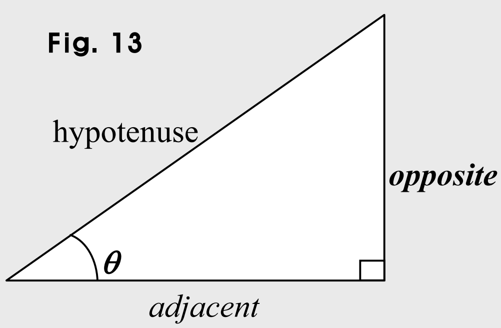
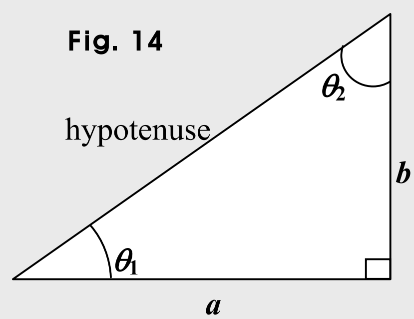


Fig. 14



In Fig. 14 above, if $\boldsymbol{\theta_2}$ is the governing angle, then $\boldsymbol{a}$ is the side ***opposite to*** the angle $\boldsymbol{\theta_2}$, and if $\boldsymbol{\theta_1}$ is the governing angle, $\boldsymbol{b}$ is the side ***opposite to*** the angle $\boldsymbol{\theta_1}$. So we need to make sure which angle the governing angle is.

The length of the side opposite to the angle is often just called ***the opposite***.

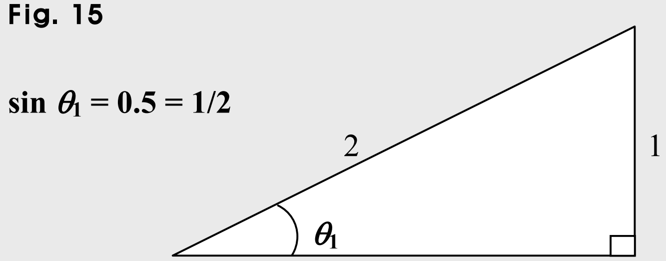
> Now, the ***sine*** is the ***ratio*** between two sides,
> one is the ***opposite***, and the other is the ***hypotenuse***.

We can put the same idea of the sine in several different ways as follows.

The sine is ***the opposite*** *compared* to the hypotenuse. In other words, the sine is *the ratio of **the opposite*** to the hypotenuse, that is, ***the opposite*** *over* the hypotenuse.

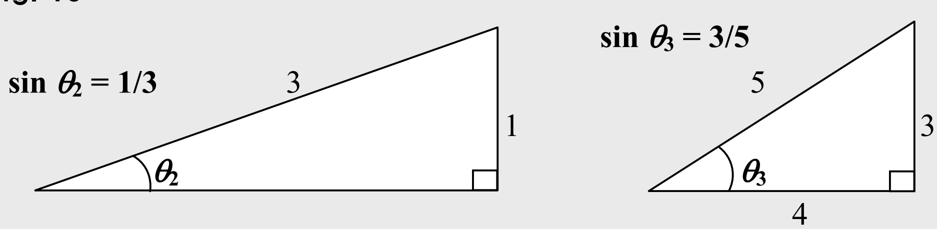
For instance, as shown in Fig. 15, if the sine is 0.5, the opposite is half the hypotenuse.

Fig. 15



If as shown in Fig. 16, the sine is $\frac{1}{3}$, the hypotenuse is 3 times the opposite.

Fig. 16



As shown in Fig. 16 above, if the opposite is 3 and the hypotenuse is 5, the sine is $\frac{3}{5}$, and thus, the sine is the ***opposite over*** the ***hypotenuse***.

$$\sin\theta = \frac{\textbf{\textit{opposite}}}{\textbf{\textit{hypotenuse}}}$$

How then can we use the sine?

Suppose in a right triangle, we know an angle opposite to a leg. Then, the leg is the opposite, so if multiplying the hypotenuse by the sine of the angle, what do we get?

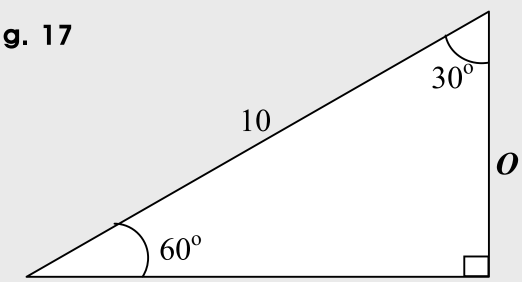
.

We get the opposite.

$$\sin\theta = \frac{\textbf{\textit{opposite}}}{\textbf{\textit{hypotenuse}}}\text{, so }\ \textbf{\textit{hypotenuse}}\cdot\sin\theta = \textbf{\textit{hypotenuse}}\cdot\frac{\textbf{\textit{opposite}}}{\textbf{\textit{hypotenuse}}} = \textbf{\textit{opposite}}$$

For instance, suppose in a right triangle, the hypotenuse is 10, the governing angle is 60°, and the opposite is ***O***, which is unknown.

Fig. 17



Multiplying the hypotenuse 10 by the sine, we get $10\cdot\sin 60^{\circ} = 10\cdot\frac{\boldsymbol{O}}{10} = \boldsymbol{O}$.

And we have sin 60° = 0.5 $=\frac{1}{2}$. So we get $10\cdot\sin 60^{\circ} = 10\cdot\frac{1}{2} = 5$.

Thus, we get ***O*** **= 5**. In sum, we get $\boldsymbol{O} = 10\cdot\sin 60^{\circ} = 10\cdot\frac{1}{2} = 5$.

Using the sine, we multiply by the sine, the known amount, which is the hypotenuse. What then do we get by the multiplication?

We get the opposite, which is the unknown. Therefore, using the sine, we multiply the hypotenuse by the sine to get the opposite.

So what is the ***sine*** about?

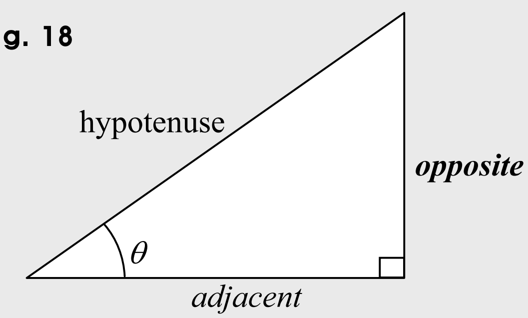
About the side opposite to the governing angle.
The sine is about ***the opposite***.

Using the ***sine***,
we multiply the hypotenuse by the sine
to get the ***opposite***.

Note that when the hypotenuse is 1, the sine is actually, the opposite itself, because the sine is the opposite over the hypotenuse. We'll see that it's convenient when talking about trig functions. For the sine function $\boldsymbol{y = f(x) = \sin x}$, where $\boldsymbol{x}$ is an angle, the graph shows how the opposite changes as the angle $\boldsymbol{x}$ changes.

Let's now move on to the tangent. What is the tangent?

Fig. 18



The ***tangent*** is a ***slope***. It is the slope of the hypotenuse.
The slop of the hypotenuse is the ratio of the opposite to the adjacent.
So the tangent is a slope, and is the ratio of the opposite to the adjacent.

The ***tangent*** is the ***ratio*** between two sides,
one is the ***opposite***, and the other is the ***adjacent***.

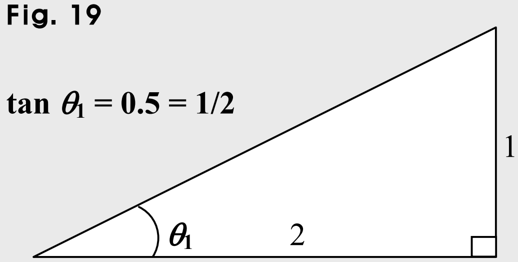
We can put the tangent in several different ways as follows.

The tangent is ***the opposite*** *compared* to ***the adjacent***. In other words, the tangent is *the ratio of* ***the opposite*** to ***the adjacent***, that is, ***the opposite*** *over* ***the adjacent***.

Exactly the same story in different words.

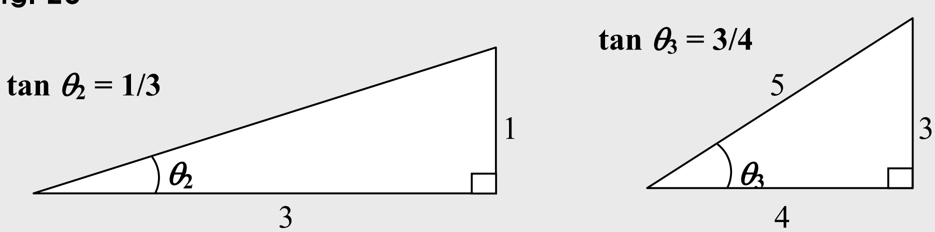
For instance, as shown in Fig. 19, if the tangent is 0.5, the opposite is half the adjacent.

Fig. 19



If the tangent is $\frac{1}{3}$ as shown in Fig. 20, the adjacent is 3 times the opposite.

Fig. 20



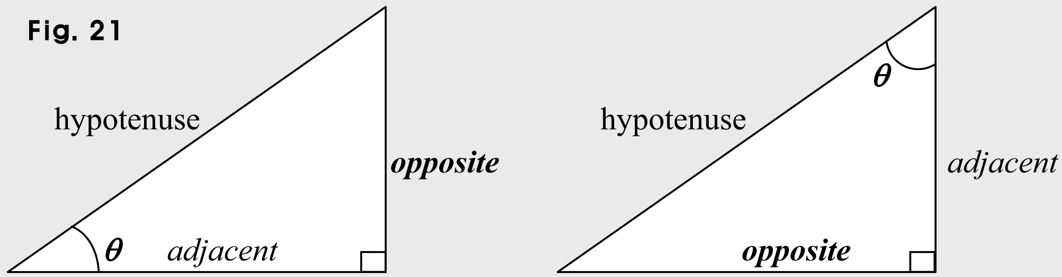
If as shown in Fig. 18 above, the opposite is 3 and the adjacent is 4, the tangent is $\frac{3}{4}$, and thus, the tangent is the ***opposite over*** the ***adjacent***.

$$\tan\theta = \frac{\textbf{\textit{opposite}}}{\textbf{\textit{adjacent}}}$$

How then can we use the tangent?

When using the tangent, we can also, multiply the tangent by a side in a right triangle.

Fig. 21



The side is the adjacent. When using the tangent, we want to get the opposite knowing the adjacent. The adjacent is adjacent to the governing angle.

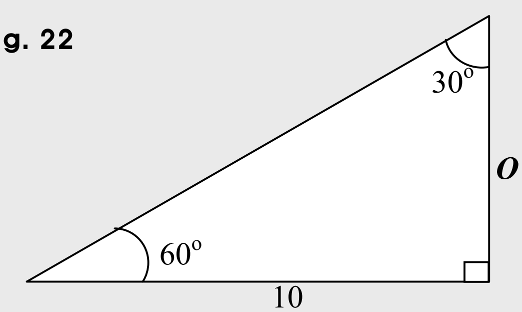
Suppose in a right triangle, we know an angle opposite to a leg. Then, the leg is the opposite, and the other leg is the adjacent, so if multiplying the adjacent by the tangent, what do we get?

We get the opposite.

$$\tan\theta = \frac{\boldsymbol{opposite}}{\boldsymbol{adjacent}}, \text{ so } \boldsymbol{adjacent}\cdot\tan\theta = \boldsymbol{adjacent}\cdot\frac{\boldsymbol{opposite}}{\boldsymbol{adjacent}} = \boldsymbol{opposite}$$

For instance, suppose in a right triangle, the adjacent is 10, the governing angle is 60°, and the opposite is $\boldsymbol{O}$, which is unknown. How then can we get the opposite $\boldsymbol{O}$?

Fig. 22



Multiplying the adjacent 10 by the tangent, we get $10\cdot\tan 60^{\circ} = 10\cdot\frac{\boldsymbol{O}}{10} = \boldsymbol{O}$.

And we have $\tan 60^{\circ} = \frac{\sqrt{3}}{3}$. So we get $10\cdot\tan 60^{\circ} = 10\cdot\frac{\sqrt{3}}{3} = \frac{10\sqrt{3}}{3}$.

Thus, we get $\boldsymbol{O} = \frac{10\sqrt{3}}{3}$. In sum, we get $\boldsymbol{O} = 10\cdot\tan 60^{\circ} = 10\cdot\frac{\sqrt{3}}{3} = \frac{10\sqrt{3}}{3}$.

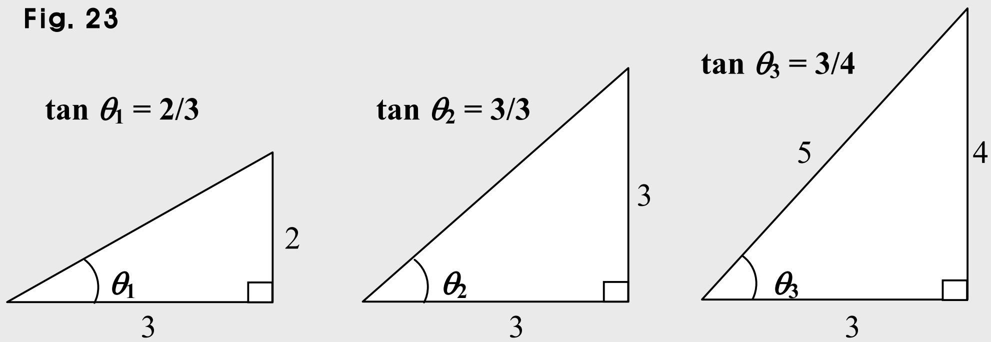
Now, we know 10 is the adjacent, and $\boldsymbol{O}$ is the opposite.
Using the tangent, we multiply by the tangent the known amount, which is the adjacent.
What then do we get using the tangent?

We get the opposite. So we can use the tangent to find the opposite.
Thus,, though the tangent is not about the opposite, we can easily get the opposite using the tangent.

Using the ***tangent***,
we multiply the adjacent by the tangent
to get the ***opposite***.

Now, if the tangent is not about the opposite, what then is it about?

Fig. 23



As shown in Fig. 23 above, the bigger the opposite, the bigger the tangent, and the hypotenuse gets steeper.
So the slope of the hypotenuse gets bigger as the tangent gets bigger. And in fact, the ***tangent*** is a ***slope***, and is about a rate of change, will be briefly covered shortly.
The tangent is the slope of the hypotenuse.
The slop of the hypotenuse is the ratio of the opposite to the adjacent.

The ratio is the opposite *compared to* the adjacent. For instance, if the opposite is 8 and the adjacent is 4, the opposite compared to the adjacent is 2, because it's $\frac{8}{4}$.

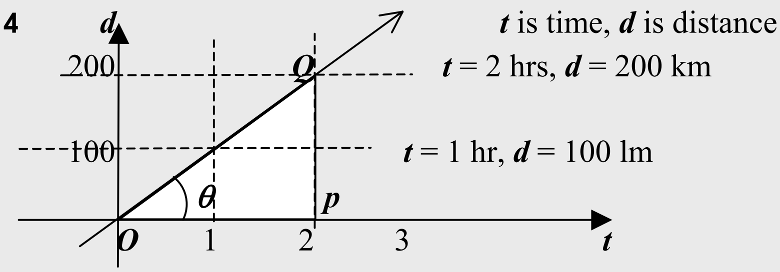
The tangent is ***a slope***.

The tangent is a slope and a slope can be a rate of change.
So the tangent can be a rate of change. If the slope is bigger, so is the tangent, and the change is faster.

A rate of change indicates how fast or slow an amount increases or decreases.

For instance, a time rate of change in distance is a speed as 100 Km/hour.

Fig. 24



The triangle ***OPQ*** is a right triangle, so ***OQ*** is the hypotenuse, and **tan** $\boldsymbol{\theta}$ is the slope of the hypotenuse ***OQ***, and is a rate of change, which is in this case, a speed, which is a time rate of change in distance.
The longer the opposite, the steeper the hypotenuse, so the slope is bigger.

Fig. 25

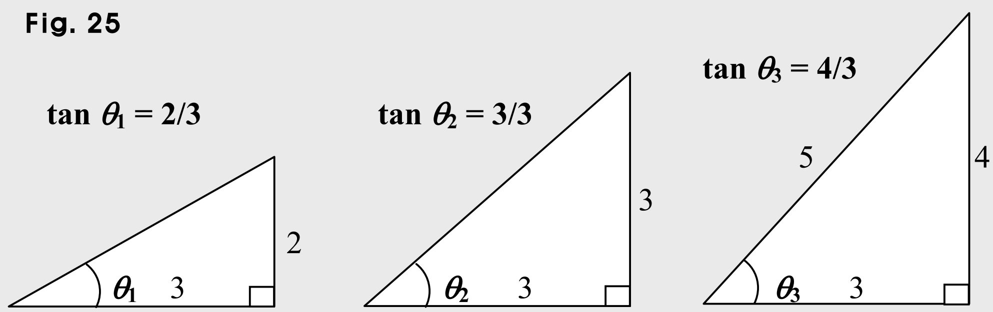
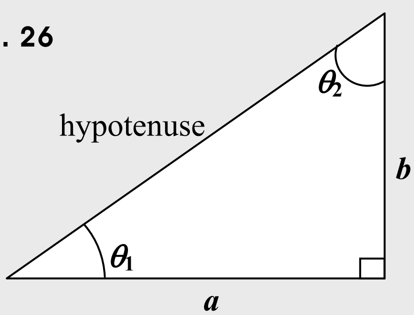


Fig. 26



So in Fig. 26, the ratio is the length of ***b*** *compared to* the length of ***a***.

In short, the ratio is ***b*** compared to ***a***, which means the ratio is $\frac{\boldsymbol{b}}{\boldsymbol{a}}$.

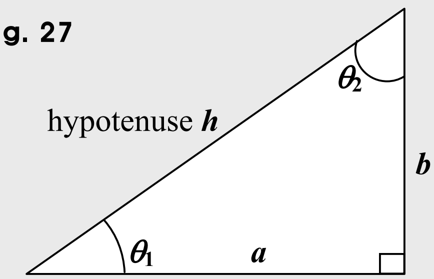
How then do we call the ratio?

It is the tangent, and is the slope, the slope of the hypotenuse, since it shows how steep the hypotenuse is. So the tangent is the slope that shows the steepness of the hypotenuse.

The ***longer*** the ***opposite***, the ***bigger*** the ***slope***, so is the tangent.
The ***longer*** the ***adjacent***, the ***smaller*** the ***slope***, so is the tangent.

Let's go over now, the idea of the tangent.
The tangent is a slope, which is a ratio. What ratio then?

Fig. 27



It is the ratio of the opposite to the adjacent.
So ***the tangent*** is ***the opposite over the adjacent***. We often call the opposite a rise, and call the adjacent a run. Thus, ***the tangent***, that is, ***a slope*** is often called ***rise over run***.

$$\textbf{tangent} = \textbf{slope} = \frac{\textit{opposite}}{\textit{adjacent}}$$

$$\textbf{tangent} = \textbf{slope} = \frac{\textit{rise}}{\textit{run}}$$

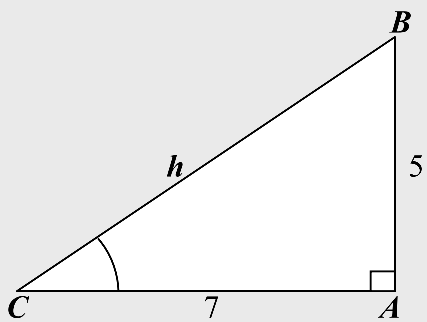
In Fig. 27, if the governing angle is θ_1, the tangent is $\frac{b}{a}$, since b is the opposite, and a is the adjacent. So the tangent of θ_1 is $\frac{b}{a}$, and we put it this way: $\tan \theta_1 = \frac{b}{a}$.

If however, θ_2 is the governing angle, the tangent is $\frac{a}{b}$, since a is the opposite, and b is the adjacent. And we put it this way: $\tan \theta_2 = \frac{a}{b}$.

For instance, if the opposite is 3 and the adjacent is 4, the tangent is $\frac{3}{4}$.

What is the tangent if the opposite is 5 and the adjacent is 7?

Fig. 28



In Fig. 28, the angel ***C*** is the governing angle, so 5 is the opposite, and 7 is the adjacent; thus, the tangent is $\frac{5}{7}$. And we can put it this way: **tan *C*** $= \frac{5}{7}$.

So what is the ***tangent***?

The ***slope*** of the hypotenuse.
The tangent is about ***a rate of change*** as speed.

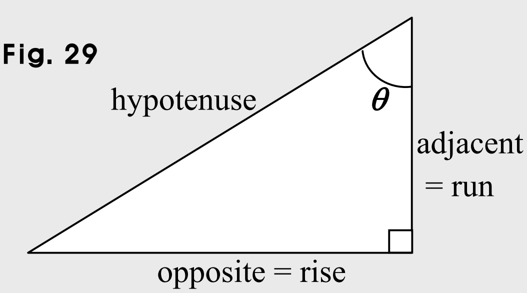
$$\textbf{tangent = slope} = \frac{\textit{opposite}}{\textit{adjacent}}$$

$$\textbf{tangent = slope} = \frac{\textit{rise}}{\textit{run}}$$

We now have covered, briefly, the three basic trig ratios, which are the sine, the cosine, and the tangent.

The ***sine*** is about the ***opposite***,
the ***cosine*** is about the ***adjacent***,
and the ***tangent*** is a ***slope***.
More specifically,
the sine is ***opposite over hypotenuse***,
the cosine is ***adjacent over hypotenuse***,
and the tangent is a slope, ***opposite over adjacent***,
often put ***rise over run***.
And the governing angle matters. So make sure where the governing angle is.

Fig. 29

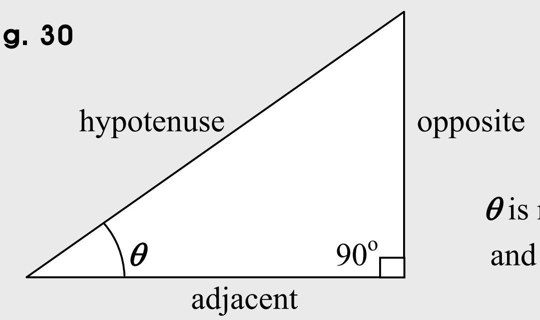


In fact, we don’t name the sides that way doing trigonometry.
Of the two legs, one is called the opposite, and the other is called the adjacent.
Why opposite?

It’s because the leg is opposite to the angle. And we just call it the opposite.
What then about the adjacent?

It’s the leg adjacent to the angle. And we just call it the adjacent.

Fig. 30



θ is read as theta,
and indicates an angle.

Thus, in trigonometry, the side called the hypotenuse connects the two legs called the opposite and the adjacent.   What then about the ratios?

In trigonometry, we work with three basic ratios, the sine, the cosine, and the tangent.
To get straight to the point, just keep in mind that

the sine is about the opposite,
the cosine is about the adjacent,
the tangent is about the slope of the hypotenuse,
and the angle matters.

We can even say that
the sine is the opposite, and the cosine is the adjacent.
And of course, the tangent is the slope.

What about the angle?

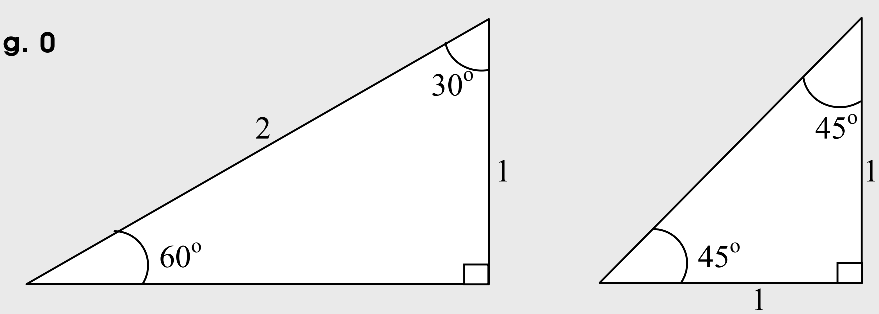
The angle is between the hypotenuse and the adjacent, and the ratios depend on it.

0.1. Review & Special Angles

There are three angles we often use when learning and working with trig ratios. The three are 30^o, 45^o, and 60^o. And we can call them special angles in trigonometry. It's probably because the angles are easy to create and express visually, and we can readily get and express the trig ratios for those angles. So we can get the ratios for the angles without looking the table of trig ratios and angles.

How then to create the angles and how to get the ratios?

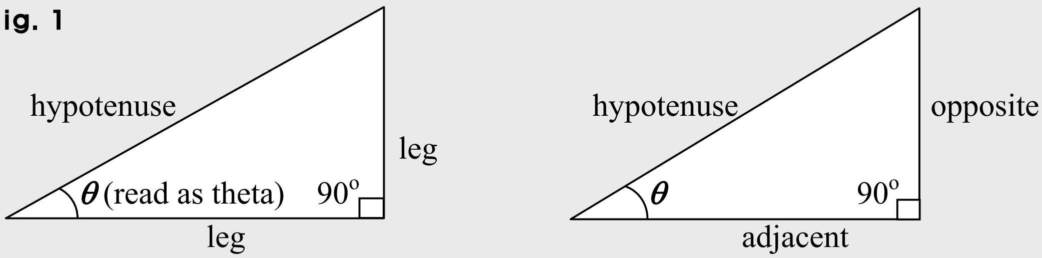
Fig. 0



To begin with, we may want to go over quickly the basic trig ratios.

A trig ratio is from a ***right triangle***. In a right triangle, an angle is 90^o, called a right angle. So a right triangle. One side is the hypotenuse, and the others are called the legs, and make a right angle, 90^o. Indicating a right angle, 90^o, we often use a small rectangle.

Fig. 1



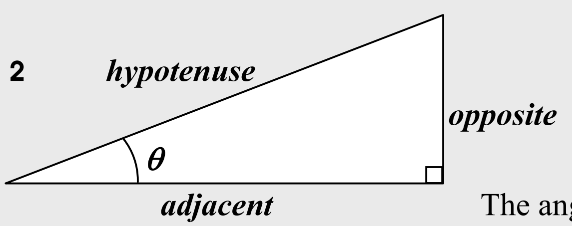
Of the two legs in a right triangle, the leg opposite to the angle θ is called ***the opposite***, and the leg adjacent to the angle θ is called ***the adjacent***. A trig ratio is made of two sides in a right triangle. And we have three basic trig ratios. What then, are the three?

The sine, the cosine, and the tangent. Getting a trig ratio, we pick two sides in a right triangle, and then, take the ratio between the two. We don't just pick two sides though. How then, do we choose the two?

Two sides make an important angle in a right triangle. And in this book, we call the angle the *governing angle*. That's because the angle governs the ratios.
What sides are those two then?

One is the adjacent, and the other is the hypotenuse. So the adjacent makes the governing angle with the hypotenuse.

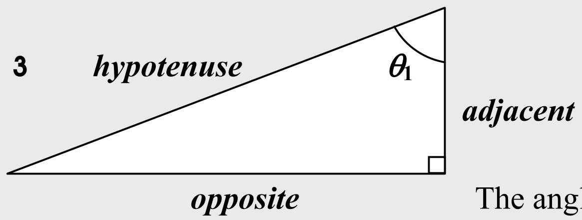
Fig. 2



The angle θ is the governing angle.

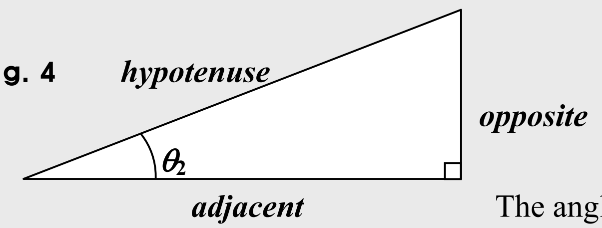
The governing angle is between the adjacent and the hypotenuse.
The adjacent is the leg next to, that is, adjacent to the governing angle.
How then do we get each ratio?

Fig. 3



The angle θ_1 is the governing angle.

Fig. 4



The angle θ_2 is the governing angle.

The sine is the ratio of the *opposite* to the hypotenuse, that is, the opposite over the hypotenuse; thus, $\sin\theta = \dfrac{\textbf{\textit{opposite}}}{\textbf{\textit{hypotenuse}}}$. And the sine is about the opposite.

The cosine is the ratio of the *adjacent* to the hypotenuse, so is the adjacent over the hypotenuse; thus, $\cos\theta = \dfrac{\textbf{\textit{adjacent}}}{\textbf{\textit{hypotenuse}}}$. And the cosine is about the adjacent.

And the tangent is the ratio of the *opposite* to the *adjacent*, so is the opposite over the adjacent; thus, $\tan\theta = \dfrac{\textbf{\textit{opposite}}}{\textbf{\textit{adjacent}}}$. And the tangent is a slope, and is often put $\dfrac{\textbf{\textit{rise}}}{\textbf{\textit{run}}}$, where ***rise*** is a ***vertical*** amount, and ***run*** is a ***horizontal*** amount

How then do we use the ratios?

If getting ***A*** by using the ratio of ***A*** to ***B***, we multiply ***B*** by the ratio.
So if multiplying ***B*** by the ratio of ***A*** to ***B***, we can get ***A***.
Thus, ***A*** is the product of ***B*** and the ratio of ***A*** to ***B***. So?

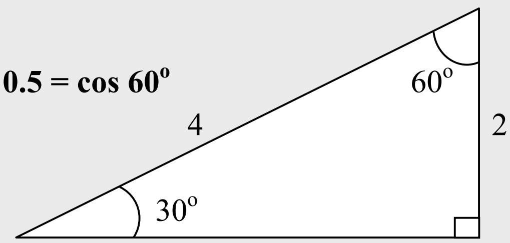
So getting the product of the hypotenuse and the sine, we get the opposite.
Getting the product of the hypotenuse and the cosine, we get the adjacent.
And getting the product of the adjacent and the tangent, we get the opposite.

Let's now take some examples.

So for instance, as shown in Fig. 5 below, in a right triangle where the governing angle is 30°, the hypotenuse is twice the opposite. Thus, the ratio of the opposite to the hypotenuse is $\frac{1}{2}$, which means, if the hypotenuse is 4, the opposite is 2.

Fig. 5

sin 30° = 2/4 = 0.5 = cos 60°



Thus, **sin 30°** $= \frac{2}{4} = \frac{1}{2} =$ **0.5**.

So if multiplying the hypotenuse by **sin 30°**, we get the opposite, that is,

$\boldsymbol{hypotenuse} \cdot \sin 30^{\circ} = 4 \cdot \frac{1}{2} = \frac{4}{2} = 2$, which is the opposite.

Next, in the right triangle above, if the governing angle is 60°, the adjacent is half the hypotenuse. So the ratio of the adjacent to the hypotenuse is $\frac{1}{2}$, also, that is, if the hypotenuse is 4, the adjacent is 2.

Thus, **cos 60°** $= \frac{2}{4} = \frac{1}{2} =$ **0.5**. So if multiplying the hypotenuse by **cos 60°**, we get the adjacent, that is, $\boldsymbol{hypotenuse} \cdot \cos 60^{\circ} = 4 \cdot \frac{1}{2} = \frac{4}{2} = 2$, which is the adjacent.

Notice that **sin 30° = cos 60°**, which is an example of a trig identity,
since 30° + 60° = 90°. Thus for instance, we get **cos 17° = sin 73°**.

So taking the product of a trig ratio and a relevant side, we can get the side we want.
If the ratio is the sine or the cosine, the relevant side is the hypotenuse.
And if the ratio is the tangent, the relevant side is the adjacent.
Thus:

We can get the opposite multiplying the hypotenuse by the sine, **sin** $\boldsymbol{\theta}$.

$$\textbf{\textit{hypotenuse}} \cdot \sin\theta = \textbf{\textit{hypotenuse}} \cdot \frac{\textbf{\textit{opposite}}}{\textbf{\textit{hypotenuse}}} = \textbf{\textit{opposite}}$$

We can get the adjacent multiplying the hypotenuse by the cosine, **cos** $\boldsymbol{\theta}$.

$$\textbf{\textit{hypotenuse}} \cdot \cos\theta = \textbf{\textit{hypotenuse}} \cdot \frac{\textbf{\textit{adjacent}}}{\textbf{\textit{hypotenuse}}} = \textbf{\textit{adjacent}}$$

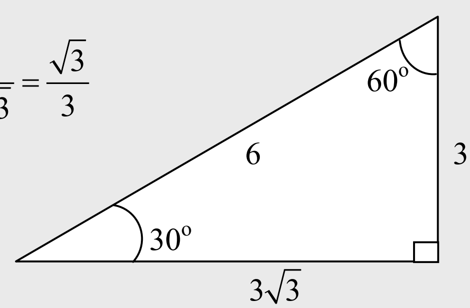
And we can get the opposite multiplying the adjacent by the tangent, tan θ.

$$\textbf{\textit{adjacent}} \cdot \tan\theta = \textbf{\textit{adjacent}} \cdot \frac{\textbf{\textit{opposite}}}{\textbf{\textit{adjacent}}} = \textbf{\textit{opposite}}$$

Let's next, move on to an example of the tangent.
In a right triangle in Fig. 6 below, if the governing angle is 30°, the opposite is 3, and the adjacent is $3\sqrt{3}$. the adjacent is $\sqrt{3}$ times the opposite. So the ratio of the opposite to the adjacent is $\frac{1}{\sqrt{3}}$, which means, if the adjacent is $3\sqrt{3}$, the opposite is 3.

Fig. 6

tan 30° = $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

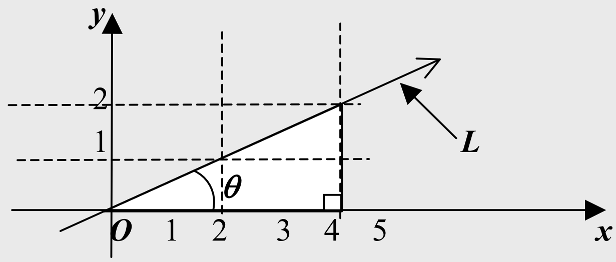


Thus, $\tan 30^{\circ} = \frac{3}{3\sqrt{3}} = \frac{1}{\sqrt{3}}$. So if multiplying the adjacent by **tan 30°**, we get the opposite. That is, $\textbf{\textit{adjacent}} \cdot \tan 30^{\circ} = 3\sqrt{3} \cdot \frac{1}{\sqrt{3}} = \frac{3\sqrt{3}}{\sqrt{3}} = 3$, which is the opposite.

Also, of the thee trig ratios, one has another important use, and is the tangent.
The tangent explains ***how much the hypotenuse is slanted***, and thus, is ***the slope*** of the hypotenuse in a right triangle.

So the tangent is a slope, and we use the tangent when we want to find the slope of a line.

Fig. 7



As shown in Fig. 7 above, the hypotenuse in the right triangle is a part of a line $\boldsymbol{L}$.

The equation of the line $\boldsymbol{L}$ is $\boldsymbol{y} = \frac{1}{2}\boldsymbol{x}$, where $\frac{1}{2}$ is the slope.

And $\tan\theta = \frac{\boldsymbol{opposite}}{\boldsymbol{adjacent}} = \frac{2}{4} = \frac{1}{2}$, which is the slope of the hypotenuse.

So we can put the equation of $\boldsymbol{L}$ this way, too: $\boldsymbol{y} = \tan\theta \cdot \boldsymbol{x}$, and can say that the tangent can be the slope of a line in the ***x-y*** plane.

There is one more idea, though, we can use when using the tangent.
The tangent, that is, the slope does not only explain how much the hypotenuse is slanted, that is, the steepness of the hypotenuse, but more importantly, specifies

a rate of change as a velocity or speed.

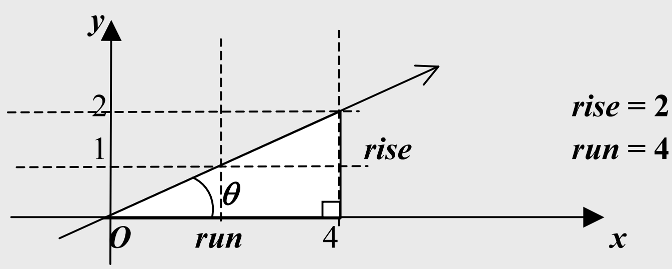
A ***velocity*** is a time ***rate of change*** in distance. And a speed is the magnitude of a velocity, since velocity is a vector, so has a direction, as well as a maginitude.

For instance, if a speed is 100 km/h, the distance traveled for every hour is 100 km. So in two hours, the distance covered is 200 km.

So the tangent, that is, the slope can indicate how fast or slow an amount changes (increases or decreases).

And the slope is often expressed as rise over run, i.e., $\frac{\boldsymbol{rise}}{\boldsymbol{run}}$, where ***rise*** is the ***opposite***, and ***run*** is the ***adjacent*** in a right triangle as in Fig. 8 below.

Fig. 8

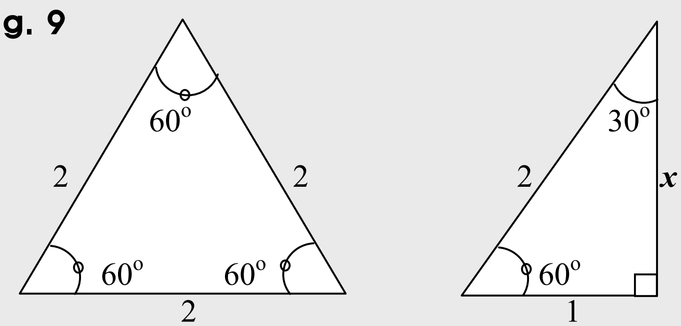


The rise is a vertical amount as a production or distance, so is usually an output, and the run is a horizontal amount as a number of people or time, so is usually an input. Actually in practice, each can be any that changes in amount as altitude, temperature, weight, pressure, volume, area, etc.

And we have three special angles, which are $\mathbf{30^o}$, $\mathbf{45^o}$, and $\mathbf{60^o}$.
We can easily get the trig ratios for those angles.
We can readily get them using a regular triangle and an isosceles right triangle.

In a regular (equilateral) triangle, every angle is equal, so is 60^o, and in an isosceles right triangle, two angles are equal, and thus, are 45^o each.
So to begin with, cutting in half a regular triangle, we can get a right triangle as in Fig. 9.

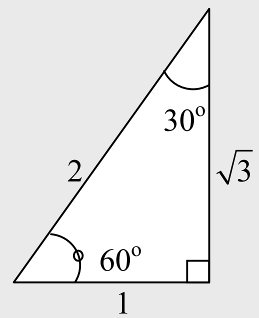
Fig. 9



Next, using the Pythagorean Theorem (the distance formula), we can get

$\boldsymbol{x^2 + 1^2 = 2^2 = 4 \Rightarrow x^2 = 3 \Rightarrow x = \sqrt{3}}$, since $\boldsymbol{x > 0}$. So we get this:

Fig. 10



Thus, taking the trig ratios of 60° and those of 30°, we get

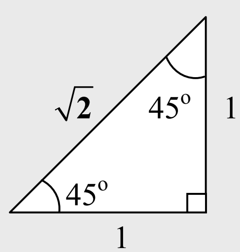
$$\sin 60^\circ = \frac{\textbf{\textit{opposite}}}{\textbf{\textit{hypotenuse}}} = \frac{\sqrt{3}}{2}, \text{ and } \sin 30^\circ = \frac{\textbf{\textit{opposite}}}{\textbf{\textit{hypotenuse}}} = \frac{1}{2}$$

$$\cos 60^\circ = \frac{\textbf{\textit{adjacent}}}{\textbf{\textit{hypotenuse}}} = \frac{1}{2}, \text{ and } \cos 30^\circ = \frac{\textbf{\textit{adjacent}}}{\textbf{\textit{hypotenuse}}} = \frac{\sqrt{3}}{2}$$

$$\tan 60^\circ = \frac{\textbf{\textit{opposite}}}{\textbf{\textit{adjacent}}} = \frac{\sqrt{3}}{1} = \sqrt{3}, \text{ and } \tan 30^\circ = \frac{\textbf{\textit{opposite}}}{\textbf{\textit{adjacent}}} = \frac{1}{\sqrt{3}}, \text{ which is } \frac{\sqrt{3}}{3}.$$

And the next is an isosceles right triangle as shown in Fig. 11 as follows.

Fig. 11



Taking the trig ratios of 45°, we get

$$\sin 45^{\circ} = \frac{\textbf{\textit{opposite}}}{\textbf{\textit{hypotenuse}}} = \frac{1}{\sqrt{2}}, \text{ which is } \frac{\sqrt{2}}{2}.$$

$$\cos 45^{\circ} = \frac{\textbf{\textit{adjacent}}}{\textbf{\textit{hypotenuse}}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan 45^{\circ} = \frac{\textbf{\textit{opposite}}}{\textbf{\textit{adjacent}}} = \frac{1}{1} = 1$$

And putting threads together, we have these:

$$\sin 30^{\circ} = \frac{\textbf{\textit{opposite}}}{\textbf{\textit{hypotenuse}}} = \frac{1}{2}, \quad \sin 45^{\circ} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}, \quad \sin 60^{\circ} = \frac{\sqrt{3}}{2}$$

$$\cos 30^{\circ} = \frac{\textbf{\textit{adjacent}}}{\textbf{\textit{hypotenuse}}} = \frac{\sqrt{3}}{2}, \quad \cos 45^{\circ} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}, \quad \cos 60^{\circ} = \frac{1}{2}$$

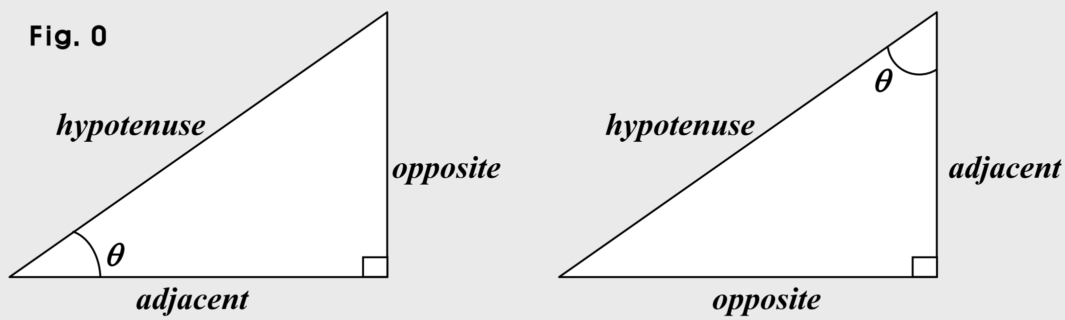
$$\tan 30^{\circ} = \frac{\textbf{\textit{opposite}}}{\textbf{\textit{adjacent}}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}, \quad \tan 45^{\circ} = \frac{1}{1} = 1, \quad \tan 60^{\circ} = \sqrt{3}$$

0.2. Reciprocals & Inverses

We have three basic trig ratios, which are the sine, the cosine, and the tangent, and are as follows.

$$\sin\theta = \frac{\textbf{\textit{opposite}}}{\textbf{\textit{hypotenuse}}}, \quad \cos\theta = \frac{\textbf{\textit{adjacent}}}{\textbf{\textit{hypotenuse}}}, \text{ and } \tan\theta = \frac{\textbf{\textit{opposite}}}{\textbf{\textit{adjacent}}}$$

Fig. 0



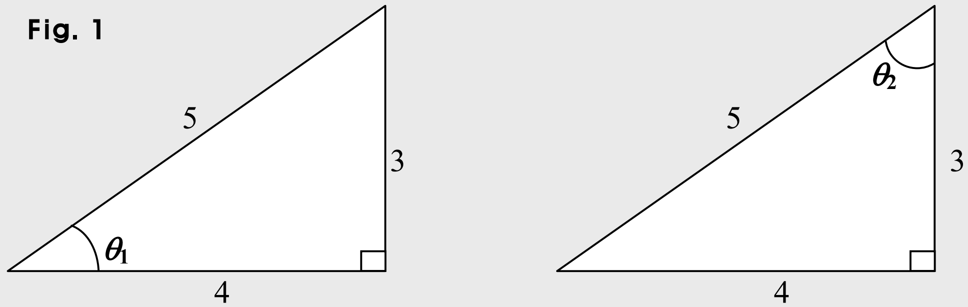
What then about the reciprocal of each?

We can flip the ratios so that the product of each ratio and its reciprocal is 1.

$$\frac{\textbf{\textit{opposite}}}{\textbf{\textit{hypotenuse}}}\cdot\frac{\textbf{\textit{hypotenuse}}}{\textbf{\textit{opposite}}} = 1 \quad \frac{\textbf{\textit{adjacent}}}{\textbf{\textit{hypotenuse}}}\cdot\frac{\textbf{\textit{hypotenuse}}}{\textbf{\textit{adjacent}}} = 1 \quad \frac{\textbf{\textit{opposite}}}{\textbf{\textit{adjacent}}}\cdot\frac{\textbf{\textit{adjacent}}}{\textbf{\textit{opposite}}} = 1$$

So what are the reciprocals?

Fig. 1



The reciprocal of the sine is called the *cosecant*, denoted by **csc**, and is the ratio of the hypotenuse to the opposite, since the sine is the ratio of the opposite to the hypotenuse.

So $\csc\theta = \dfrac{\boldsymbol{hypotenuse}}{\boldsymbol{opposit}} = \dfrac{1}{\sin\theta}$, and thus, $\sin\theta \cdot \csc\theta = \dfrac{\boldsymbol{opposite}}{\boldsymbol{hypotenuse}} \cdot \dfrac{\boldsymbol{hypotenuse}}{\boldsymbol{opposit}} = 1$

So in Fig. 1, $\csc\theta_1 = \dfrac{5}{3}$ and $\sin\theta_1 = \dfrac{3}{5}$, so $\sin\theta_1 \cdot \csc\theta_1 = \dfrac{3}{5} \cdot \dfrac{5}{3} = 1$

$\csc\theta_2 = \dfrac{5}{4}$ and $\sin\theta_2 = \dfrac{4}{5}$, so $\sin\theta_2 \cdot \csc\theta_2 = \dfrac{4}{5} \cdot \dfrac{5}{4} = 1$

The reciprocal of the cosine is called the *secant*, denoted by **sec**, and is the ratio of the hypotenuse to the adjacent, since the cosine is the ratio of the adjacent to the hypotenuse.

So $\sec\theta = \dfrac{\boldsymbol{hypotenuse}}{\boldsymbol{adjacent}} = \dfrac{1}{\cos\theta}$, and thus, $\cos\theta \cdot \sec\theta = \dfrac{\boldsymbol{adjacent}}{\boldsymbol{hypotenuse}} \cdot \dfrac{\boldsymbol{hypotenuse}}{\boldsymbol{adjacent}} = 1$

So in Fig. 1, $\sec\theta_1 = \dfrac{5}{4}$ and $\cos\theta_1 = \dfrac{4}{5}$, so $\cos\theta_1 \cdot \sec\theta_1 = \dfrac{4}{5} \cdot \dfrac{5}{4} = 1$

$\sec\theta_2 = \dfrac{5}{3}$ and $\cos\theta_2 = \dfrac{3}{5}$, so $\cos\theta_2 \cdot \sec\theta_2 = \dfrac{3}{5} \cdot \dfrac{5}{3} = 1$

And the reciprocal of the tangent is called the *cotangent*, denoted by **cot**, and is the ratio of the adjacent to the opposite, for the tangent is the ratio of the opposite to the adjacent.

So $\cot\theta = \dfrac{\textbf{\textit{adjacent}}}{\textbf{\textit{opposite}}} = \dfrac{1}{\tan\theta}$, and thus, $\tan\theta \cdot \cot\theta = \dfrac{\textbf{\textit{opposite}}}{\textbf{\textit{adjacent}}} \cdot \dfrac{\textbf{\textit{adjacent}}}{\textbf{\textit{opposite}}} = 1$

So in Fig. 1, $\cot\theta_1 = \dfrac{4}{3}$ and $\tan\theta_1 = \dfrac{3}{4}$, so $\tan\theta_1 \cdot \cot\theta_1 = \dfrac{3}{4} \cdot \dfrac{4}{3} = 1$

$\cot\theta_2 = \dfrac{3}{4}$ and $\tan\theta_2 = \dfrac{4}{3}$, so $\tan\theta_2 \cdot \cot\theta_2 = \dfrac{4}{3} \cdot \dfrac{3}{4} = 1$

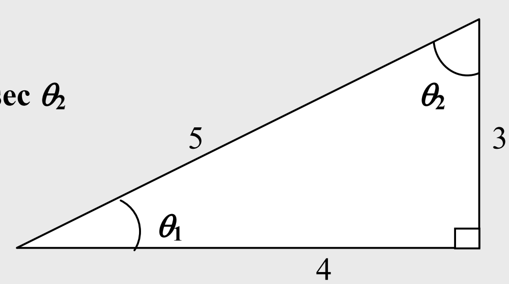
We read **csc** $\boldsymbol{\theta}$ as the cosecant of $\boldsymbol{\theta}$, or just cosecant $\boldsymbol{\theta}$, or cosec $\boldsymbol{\theta}$.
We read **sec** $\boldsymbol{\theta}$ as the secant of $\boldsymbol{\theta}$, or just secant $\boldsymbol{\theta}$, or sec $\boldsymbol{\theta}$.
And we read **cot** $\boldsymbol{\theta}$ as the cotangent of $\boldsymbol{\theta}$, or just cotangent $\boldsymbol{\theta}$, or cotan of $\boldsymbol{\theta}$, or cotan $\boldsymbol{\theta}$.
How then do we use the ratios?

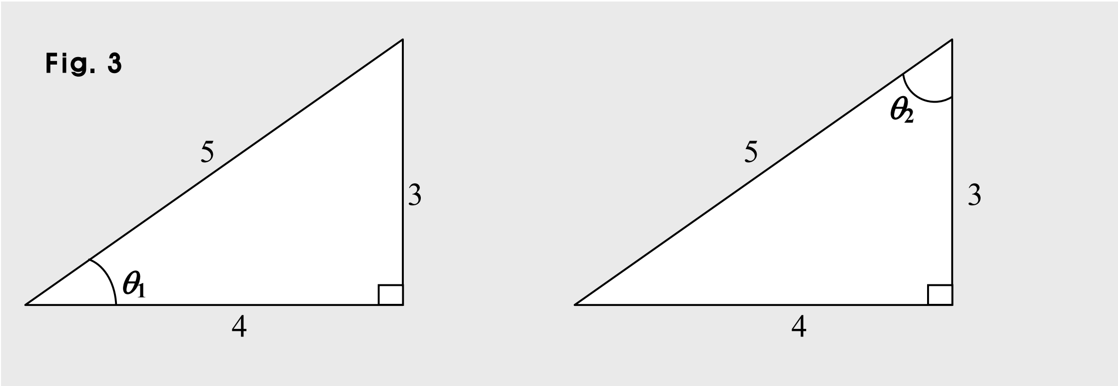
We have a fact that if getting ***A*** by using the ratio of ***A*** to ***B***, we multiply ***B*** by the ratio.
So if multiplying ***B*** by the ratio of ***A*** to ***B***, we can get ***A***.
Thus, ***A*** is the product of ***B*** and the ratio of ***A*** to ***B***.
So getting the product of a trig ratio and a relevant side, we can get the side we want.

So for instance, as shown in Fig. 2 below, in a right triangle where the governing angle is $\boldsymbol{\theta_1}$, the hypotenuse is $\dfrac{5}{3}$ of the opposite. So the ratio of the hypotenuse to the opposite is $\dfrac{5}{3}$, which means, if the opposite is 1, the hypotenuse is $\dfrac{5}{3}$.

Fig. 2

csc $\boldsymbol{\theta_1}$ = 5/3 = sec $\boldsymbol{\theta_2}$





Thus, **csc** $\boldsymbol{\theta_1} = \frac{5}{3}$. So if multiplying the opposite by the ratio, **csc** $\boldsymbol{\theta_1}$, we get the hypotenuse, since $\boldsymbol{opposite} \cdot \csc\theta_1 = \boldsymbol{opposite} \cdot \frac{5}{3} = 3 \cdot \frac{5}{3} = 5$, which is the hypotenuse.

Next, in the right triangle above, if the governing angle is $\boldsymbol{\theta_1}$, the hypotenuse is $\frac{5}{4}$ of the adjacent. So the ratio of the hypotenuse to the adjacent is $\frac{5}{4}$, that is, if the adjacent is 1, the hypotenuse is $\frac{5}{4}$.

Thus, **sec** $\boldsymbol{\theta_1} = \frac{5}{4}$. So if multiplying the adjacent by the ratio, **sec** $\boldsymbol{\theta_1}$, we get the hypotenuse, since $\boldsymbol{adjacent} \cdot \sec\theta_1 = \boldsymbol{adjacent} \cdot \frac{5}{4} = 4 \cdot \frac{5}{4} = 5$, which is the hypotenuse.

So taking the product of a trig ratio and a relevant side, we can get the side we want.
If the ratio is the cosecant, the relevant side is the opposite.
And if the ratio is the secant, the relevant side is the adjacent.

We can get the hypotenuse multiplying the opposite by the cosecant, **csc** $\boldsymbol{\theta}$.

$$\boldsymbol{opposite} \cdot \csc\theta = \boldsymbol{opposite} \cdot \frac{\boldsymbol{hypotenuse}}{\boldsymbol{opposite}} = \boldsymbol{hypotenuse}$$

Also, we can get the hypotenuse multiplying the adjacent by the secant, **sec** $\boldsymbol{\theta}$.

$$\boldsymbol{adjacent} \cdot \sec\theta = \boldsymbol{adjacent} \cdot \frac{\boldsymbol{hypotenuse}}{\boldsymbol{adjacent}} = \boldsymbol{hypotenuse}$$

And next, let's move on to the cotangent.

In a right triangle in Fig. 3 above, if the governing angle is $\boldsymbol{\theta_2}$, the opposite is 4, and the adjacent is 3. So the ratio of the adjacent to the opposite is $\frac{3}{4}$, which means, if the opposite is 1, the adjacent is $\frac{3}{4}$.

Thus, $\cot\theta_2 = \frac{3}{4}$. So if multiplying the opposite by **cot** $\boldsymbol{\theta_2}$, we get the adjacent, that is,

$\boldsymbol{opposite}\cdot\cot\theta_2 == 4\cdot\frac{3}{4} = 3$, which is the adjacent.

So taking the product of a trig ratio and a relevant side, we can get the side we want.
If the ratio is the cotangent, the relevant side is the opposite.

And we can get the adjacent multiplying the opposite by the cotangent, **cot** $\boldsymbol{\theta}$.

$$\boldsymbol{opposite}\cdot\cot\theta = \boldsymbol{opposite}\cdot\frac{\boldsymbol{adjacent}}{\boldsymbol{opposite}} = \boldsymbol{adjacent}$$

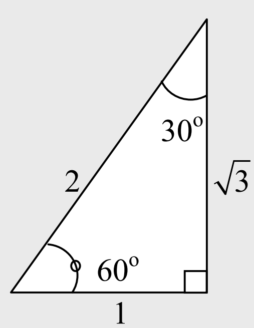
Let's now move on to the inverses.

The multiplicative inverses of the basic trig-ratios are the reciprocals of the basic ones.
So the multiplicative inverse of **sin** $\boldsymbol{\theta}$ is **csc** $\boldsymbol{\theta}$, which is the reciprocal of **sin** $\boldsymbol{\theta}$.

Saying however, just ***the inverse of a trig ratio***, we mean ***an angle***.

Thus for instance, the inverse of **sin** $\frac{1}{2}$ is $\mathbf{30^o}$, and the inverse of **cos** $\frac{1}{2}$ is $\mathbf{60^o}$.

Fig. 4



And the inverse of the sine is denoted by $\mathbf{sin^{-1}}$, which can be read as the sine inverse. Usually though, it is read as the *arc sine*.
It's because arc means an angle, so we use it as a prefix; thus, $\mathbf{sin^{-1}}$ is often read as the arc sine instead of the sine inverse.

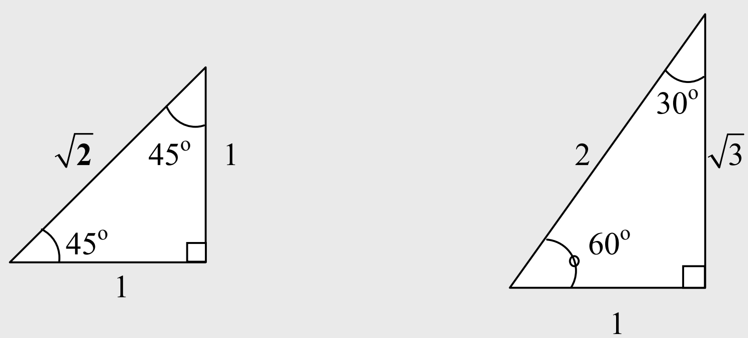
For instance, we know $\mathbf{sin\ 30^o}$ is $\frac{1}{2}$. So $\mathbf{sin^{-1}}\frac{1}{2}$ is $\mathbf{30^o}$.

And $\mathbf{sin^{-1}}\frac{1}{2}$ is read as the arc sine of $\frac{1}{2}$.

And of course, the same is true for the other trig ratios, too.

So the inverse of the cosine is denoted by $\mathbf{cos^{-1}}$, read as the *arc cosine*, and the inverse of the tangent is denoted by $\mathbf{tan^{-1}}$, read as the *arc tangent* or just the *arc tan*.

Fig. 5



For instance, the arc cosine of $\frac{1}{2}$ is $\mathbf{cos^{-1}}\frac{1}{2}$, which is 60^o, and the arc tan of 1 is $\mathbf{tan^{-1}\ 1}$, which is 45^o. And the same is true, also, for the reciprocals as follows. **csc**, **sec**, and **cot**.

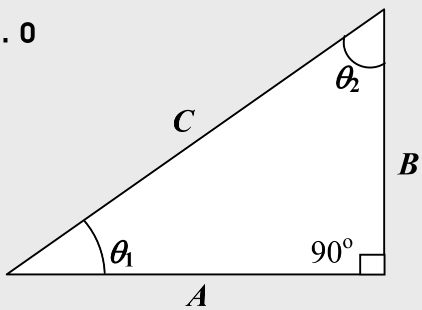
So the *arc cosecant* is $\mathbf{csc^{-1}}$, the *arc secant* is $\mathbf{sec^{-1}}$, and the *arc cotangent* is $\mathbf{cot^{-1}}$. And each of them is an angle, too, of course.

1. Circles and Angles

What is trigonometry about?

It’s about ratios, the trig ratios as the cosine, and such a ratio is between two sides in a right triangle. And the ratios depend on angles in right triangles. So the angles matter. We need to know which angle taking the ratio. The adjacent makes the angle with the hypotenuse. Wrong angle, wrong ratios. Angles matter in trigonometry. They do.

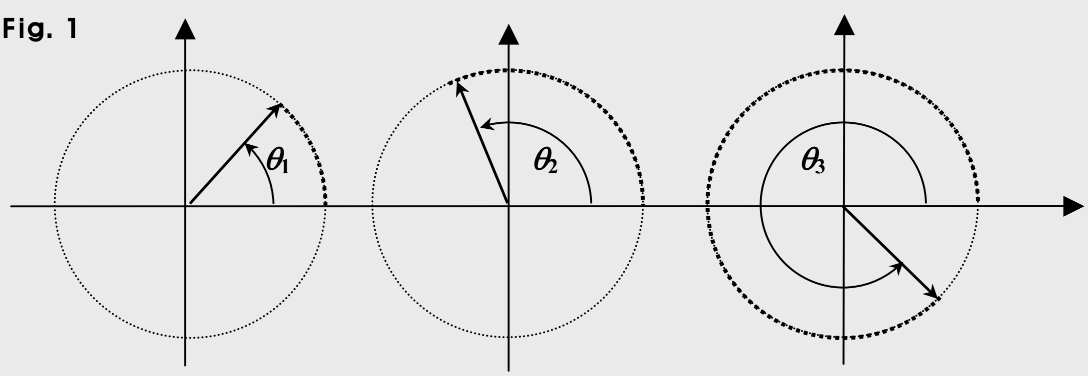
Fig. 0



In a right triangle, an angle is 90^o,
called a right angle.
And showing a right angle,
we use a small rectangle.
By the way, $\boldsymbol{\theta_1 + \theta_2 + 90^o = 180^o}$,
and $\boldsymbol{A + B > C}$.

Doing trigonometry in fact, we work with angles a lot. So we may want to know what angles are and how they work. So what is an angle?

Fig. 1

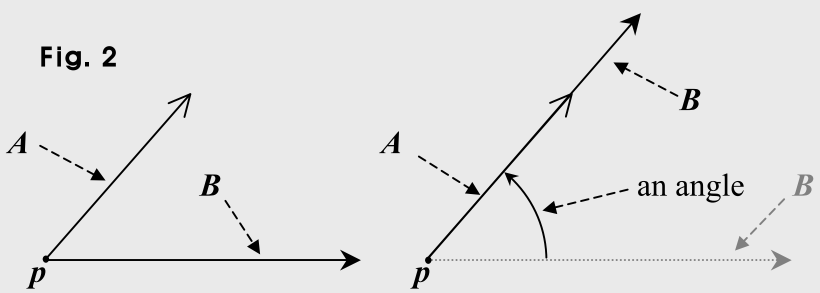


$\boldsymbol{\theta_3 > \theta_2 > \theta_3 > 0}$

Angles are from circles. An angle looks in fact, like a part of a circle.   How?

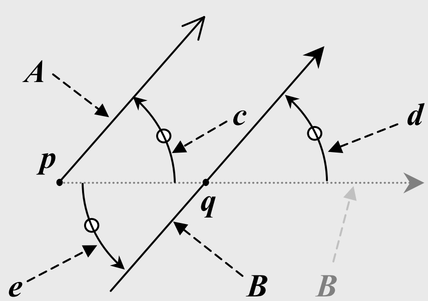
Let's see now first, what an angle is, and how it gets made.
As shown in Fig. 2 below, suppose two rays diverge from a point ***p***. Suppose also, one of the two ray turns about the point ***p*** towards the other until both rays cover each other. Then, the amount of turning is an angle.

Fig. 2



Suppose this time, as shown in Fig. 3 below, the ray ***B*** turns about a point ***q*** in the ray ***B*** itself until it gets parallel to the other ray ***A***. Then, the amount of turning is an angle, too, which is the same as the angle mentioned above.

Fig. 3



The three angles ***c***, ***d***, and ***e*** are the same.

Now, the ray ***B*** turned is parallel to the ray ***A***, and has the same direction as that of the ray ***A***. And we know the ray ***B*** before turning has a different direction.

What then can we say about an angle?

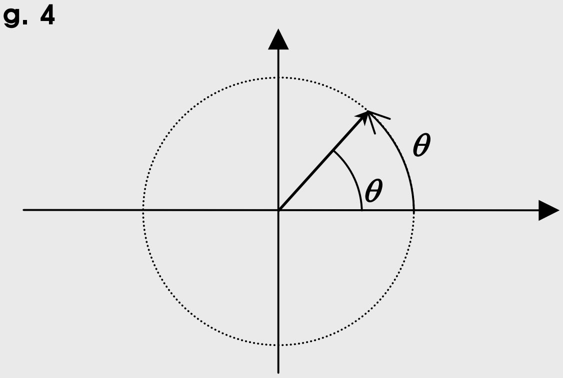
We can say that an angle is an amount of *difference in direction*, or an amount of *change in direction*. So in short, an angle can be called a change in direction, too.

Thus, we can use an angle specifying an amount of turning or a change in direction.

An angle is *an amount of turning*
or *a change in direction*.

A dictionary says an angle is the figure formed by two lines diverging from the point where the two lines meet, or is the figure formed by two planes diverging from the line where the two planes meet.
What figure then do we normally use as the figure stated above?

Fig. 4



We use as such a figure a part of a circle, which is called an arc.
And we can calculate an angle by means of an arc and a circle. How?

We can get an angle using a ratio between the arc and the circle. And it's the angle the arc has. A circle has 360^o. So we can get an angle the way as follows.

Take the ratio of an arc length to the circumference of the circle the arc belongs to.
Take the product of the ratio and 360^o, the angle in a circle.
Then, the product is the angle the arc has.

In short, ***an angle*** can be put in terms of ***a ratio of an arc to the circle*** that has the arc.

So for instance, assuming $\boldsymbol{\theta}$ is an angle, $\boldsymbol{k}$ is such a ratio, and $\boldsymbol{C}$ is a constant, we can set:
$\boldsymbol{\theta = Ck}$, where C is 360^o. Thus, we get $\boldsymbol{\theta = 360^o k}$.

So an arc is a part of a circle, and has an angle. What then is a circle?

A circle is in a plane, and is a collection of all the points that are the same distance away from a particular point called the center, and the same distance is the radius.

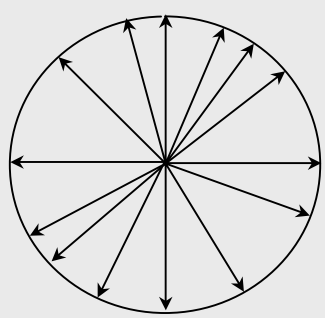
Thus, a circle is a 2-D object, and has its diameter, which is a line segment passing through the center and connecting two points facing each other in the circle, so is twice the radius of the circle.
What then about the circumference of a circle?

It is the length of the curved and closed line segment forming the circle. For instance, assuming the radius is $\boldsymbol{r}$, and $\boldsymbol{C}$ is the circumference, we get $\boldsymbol{C = 2\pi r}$, where π is the circular ratio, which is an irrational number, and is 3.141592…

Suppose now, as in Fig. 0 above, all the radii in a circle are vectors, and all the vectors begin at the center of the circle. A vector has its length and direction, so is made of a line segment and an arrowhead, called the terminal point.

Then, all their lengths are equal, so all their terminal points are in the circle. That is to say that all the terminal points form the circle. And we can say that the center emits such vectors in all directions.

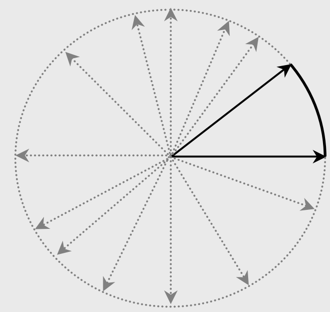
Fig. 5



Also, we can say that each of all the vectors has a different direction. So every radius in a circle has a different direction. What then do we call the difference between the directions of two vectors?

We call it an angle, so it's the angle between two vectors. Thus, we can say that a difference in direction is an angle.

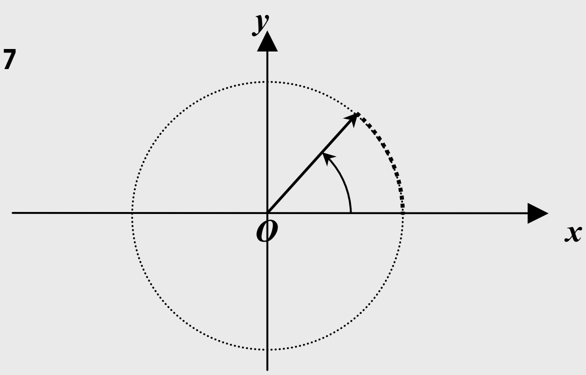
Fig. 6



As shown above, an angle can be pictured by a circular wedge made of two radii and an arc, a part of a circle. Thus, a pair of radii in a circle can form a geometric object called an angle. Is there anything else though, we can call an angle?

Suppose as in Fig. 7 below, a vector is turning about the origin in the ***x-y*** plane. What then can we call the amount of turning the vector makes?

Fig. 7



We call it an angle, too. So we can say that an angle is an amount of turning, also. Thus, we can use an angle to specify an amount of turning or a change in direction. In short, *an angle* is *an amount of turning* or *a change in direction*.

So an angle can tell us how much two radii in a circle are apart from each other. For instance, if in a circle, a radius is 30^{o} away from another radius in the circle, the angle between the two radii is 30^{o}.
And of course, expressing an amount of angle, we use a unit of measure as in the cases of other objects as lengths, areas, volumes, weights, etc. For instance, we use kilograms or pounds for weights. What then is the unit of measure for angles?

We have two kinds in such a unit. One is *degree*, and the other is *radian*, called *rad* for short. Thus, we can use degrees or radians for angles. So for instance, an angle can be 25 degrees, 1 degree, 3 radians or 3 rad, or 1 radian or 1 rad.

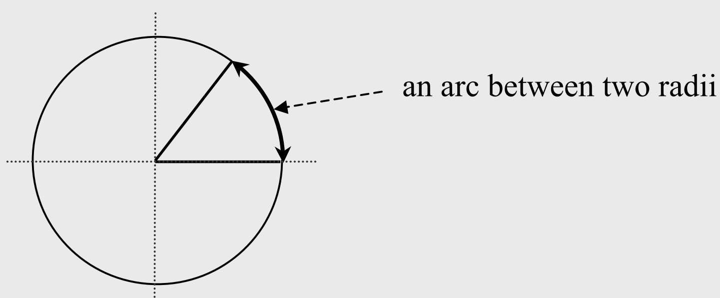
And putting an angle in writing using degrees, we use a small circle, and put it at the upper right-hand corner of the number indicating the amount of angle. So we use it as a superscript as in 25^{o}.   What then about radians?

Usually, we use no symbol for radians. So assuming for instance, ***A*** is an angle in radian, we just put it in writing this way: ***A* = 2**, which means ***A*** is 2 radians.
Having to clarify though, an angle is in radians, we put 'rad' after the number. So for instance, if an angle ***B*** is 3 radians, we can put it this way: ***B* = 3 rad**.

Thus, we have two systems of measurement for angles. And one system can get converted to the other. So two different amounts in angle can mean the same angle. For instance, we have $\mathbf{180^{o} = \pi}$ **rad** where π is the circular ratio, 3.141592… And briefly, we put it this way: $\mathbf{180^{o} = \pi}$.   How can we calculate though, an amount of angle?

As explained above, an angle is a geometric object, which is a wedge formed by two radii in a circle and an arc, a figure that can show not only a difference (or a change) in direction but an amount of turning, too.

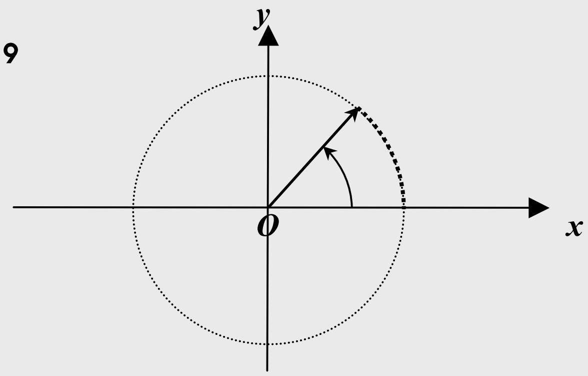
Fig. 8



And we can get an amount of angle using a ratio of an arc length to the circumference of the circle the arc belongs to. In short, an angle can be put in terms of a ratio of an arc to the circle that has the arc.   How though?

Let's get back to the vector turning about the origin in the ***x-y*** plane.

Fig. 9



Then, first, an angle is an amount of turning. And an arc has an angle.

Next, as the vector turns, the arrowhead traces an arc, and if the vector makes a complete turn, the arrowhead traces a circle. So if the vector makes a complete turn, the arc made is the circumference of the circle, that is, the arc is the circle itself.

And if half a complete turn is made, the arc is half the circle. If a third of a complete turn is made, the arc is a third of the circle.

Thus, an amount of turning, that is, the angle an arc has is proportional to the ratio of the length of the arc to the circumference of the circle the arc belongs to. So in short,

an angle is proportional to the ratio of an arc to its circle,

which is the case if the angle is in degrees.
In radians, an angle is simply, the ratio of an arc to its circle.

Thus, in degrees, assuming $\boldsymbol{\theta}$ is an angle, $\boldsymbol{k}$ is such a ratio, and $\boldsymbol{C}$ is a constant, we can set the equation as follows. $\boldsymbol{\theta = Ck}$. What then is the constant $\boldsymbol{C}$?

It is a constant of proportionality, and is 360°. Why though?

If the vector makes a complete turn, that is, if the arc made is a circle, the angle made is 360° by definition. In short, a complete turn is 360°. So the constant is 360°, and the angle a part of a circle has is a part of 360°.

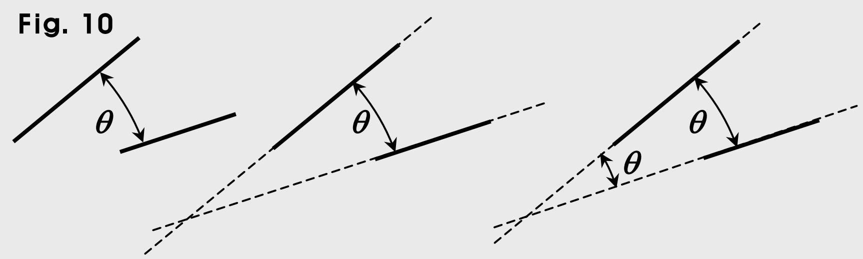
So for instance, if the length of an arc is 2, and 12 is the circumference of the circle the arc belongs to, the ratio of the arc to its circle is $\frac{2}{12} = \frac{1}{6}$. Then, the angle is $\frac{1}{6}$ of 360°, which is 60°.

So assuming $\boldsymbol{\theta}$ is an angle, and $\boldsymbol{k}$ is such a ratio, we can set $\boldsymbol{\theta = 360^\circ k}$.

And if two lines are parallel to each other, the angle between the two is 0.
What if two line segments not parallel are away from each other, and do not meet at a point?

A line segment is in a line, so is a part of a line, and two lines not parallel meet at a point. So the angle between the two lines that the two line segments belong to is the angle between the two line segments as shown in Fig. 10.

Fig. 10



How small is 1°?

Dividing a circle into 360 equal parts around the center, we get 360 equal wedges. So taking one of the wedges, we get $\frac{360^\circ}{360} = 1^\circ$. So if the vector tuning counterclockwise makes a three hundred and sixtieth of a complete turn, we say that the angle made is 1°.

And again, dividing one wedge worth 1° into 60 equal parts, we get 60 equal smaller wedges, each of which has an angle called 1 *minute* denoted by **1'**.

So we get $\frac{1^\circ}{60} = \mathbf{1'}$, which is pretty small.
Thus, if the vector tuning counterclockwise makes a twenty one thousand and six hundredth of a complete turn, the angle made is 1'.

Then again, dividing 1’ by 60, we get an angle called 1 *second* denoted by **1”**.

So we get $\frac{1^{\circ}}{360} = \frac{1'}{60} =$ **1”**, which is very small.

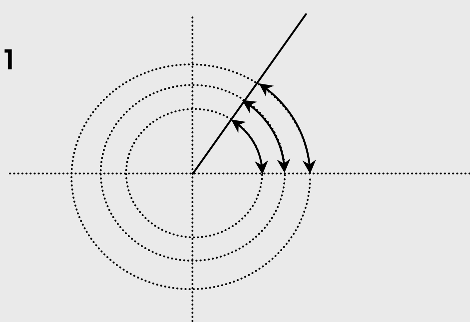
And we can keep going taking the smaller wedges. We do not normally though, get a wedge worth less than 1 second, 1”. What then about 0.78°?

It is an angle that is 78% of 1°, so $0.78^{\circ} = \frac{78 \cdot 1^{\circ}}{100}$. Thus, we get $0.1^{\circ} = \frac{1^{\circ}}{10}$.

So the bigger the wedge is, is the bigger the angle, in other words, the bigger the arc is, is the bigger the angle?

Not necessarily. Many different arcs can have the same angle.

Fig. 11



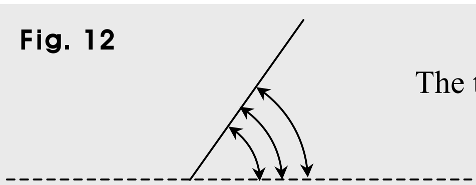
So what matters in an angle is not an arc length itself but the ratio of an arc to the circle the arc belongs to.

That’s because every circle has 360°, but the radius of each can be different. So what?

First, many arcs with different lengths can have the same angle. All those arcs belong to circles concentric. Concentric circles have the same center as shown above.

Next, holding an angle constant, we can see the larger the radius, the larger the arc. That is, for a particular angle, as the radius grows, the arc grows, too.

Fig. 12



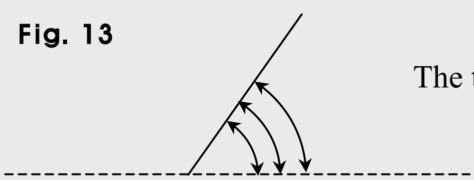
The three arcs are similar to each other.

So the ratio between the radii is the same as the ratio between the arcs.

For instance, assuming in Fig. 12 above, $\boldsymbol{R_1}$, $\boldsymbol{R_2}$, and $\boldsymbol{R_3}$ are the radii, and $\boldsymbol{A_1}$, $\boldsymbol{A_2}$, and $\boldsymbol{A_3}$ are the arcs, and also, assuming $\boldsymbol{R_1 < R_2 < R_3}$, and $\boldsymbol{A_1 < A_2 < A_3}$, since the arcs are similar, we get $\boldsymbol{R_1 : R_2 : R_3 = A_1 : A_2 : A_3}$.

Thus, we can get $\dfrac{\boldsymbol{A}_1}{\boldsymbol{R}_1} = \dfrac{\boldsymbol{A}_2}{\boldsymbol{R}_2} = \dfrac{\boldsymbol{A}_3}{\boldsymbol{R}_3}$. What then is the value of $\dfrac{\boldsymbol{A}_1}{\boldsymbol{R}_1}$?

Fig. 13



The three arcs share the same angle.

In other words, if $\dfrac{\boldsymbol{A}_1}{\boldsymbol{R}_1} = \dfrac{\boldsymbol{A}_2}{\boldsymbol{R}_2} = \dfrac{\boldsymbol{A}_3}{\boldsymbol{R}_3} = \boldsymbol{D}$, what is $\boldsymbol{D}$?

We know $\boldsymbol{D}$ is the ratio of each arc to its corresponding radius, and is constant.

Also, the angle in each of the three wedges, that is, the angle of each arc is the same, and thus, is constant. So we can reasonably expect that the ratio is the angle.

And the ratio is in fact, the angle. So $\boldsymbol{D}$ is the angle.
We know however, a ratio is a number. How then can an angle be a number?

As mentioned earlier, we have two systems of measurement for angles. One is radian system, and the other is degree system.

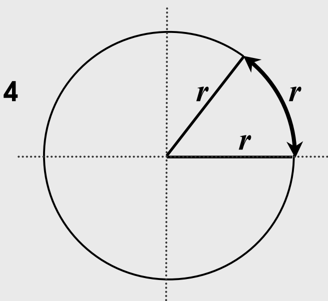
And using the radian system, we can use as angles all real numbers, which are therefore, angles in radians. And we will see now how we get angles in radians.

To begin with, stating angles, we can put angles in degree system.
We have two systems where we can put angles. What then is the other system?

It is the radian system, where we put angles in radians. So 1 radian is a unit of measure for angles. What is 1 radian though?

If the arc length is the radius of the circle that has the arc, the angle the arc has is 1 radian. That is, if an arc length is its radius, the angle of the arc is 1 radian.

Fig. 14



So in short, if an arc is its radius, the angle is 1 rad.
Thus, if a circular wedge is like a regular (equilateral) triangle where three sides are equal, the angle of the arc is 1 rad. So if an arc is its radius, we get 1 rad.

Thus, assuming $\boldsymbol{A}$ is the arc, $\boldsymbol{r}$ is its radius, and $\boldsymbol{A = r}$, we can get $\frac{A}{r} = \frac{r}{r} = 1$, which is called 1 radian.
So 1 radian is a **radi**us **an**gle, the angle of the arc the length of which is the radius.

Now, we know in a circle, the radius $\boldsymbol{r}$ is constant, and an arc $\boldsymbol{A}$ can change. So we can see that the larger the arc, the larger its angle.
Thus, we can see that $\boldsymbol{A}$ is proportional to its angle $\boldsymbol{\theta}$, and that $\boldsymbol{r}$ is the proportionality constant.
So we can have $\frac{A}{r} = \theta$ where $\boldsymbol{A}$ is the arc, $\boldsymbol{r}$ is its radius, and $\boldsymbol{\theta}$ is the angle $\boldsymbol{A}$ has.
And getting angles the way above, we call the angles radians.
Also, getting an arc length, we can use $\boldsymbol{A = r\theta}$., because we have $\frac{A}{r} = \theta$.
For instance, the circumference $\boldsymbol{C}$ of a circle of radius $\boldsymbol{r}$ is $2\pi\boldsymbol{r}$. And the angle in a circle is 2π. So in the equation $\frac{A}{r} = \theta$, we have $\boldsymbol{A = C}$, and $\boldsymbol{\theta = 2\pi}$.

Then, we get $\dfrac{\boldsymbol{A}}{\boldsymbol{r}} = \theta \Rightarrow \dfrac{\boldsymbol{C}}{\boldsymbol{r}} = 2\pi \Rightarrow \boldsymbol{C = 2\pi r}$.

What angle in the degree system then is 1 radian? That is, what degree is 1 rad?

We have $60^{\text{o}} = \dfrac{\pi}{3}$ rad, $90^{\text{o}} = \dfrac{\pi}{2}$ rad, $45^{\text{o}} = \dfrac{\pi}{4}$ rad, $180^{\text{o}} = \pi$ rad, $360^{\text{o}} = 2\pi$, etc.

And using any of the ones above, we can get the angle in degree equivalent to 1 rad.

Using the fourth one above, we get $180^{\text{o}} = \pi$ rad $\Rightarrow$ 1 rad $= \dfrac{180^{\text{o}}}{\pi} = 57.29657\ldots^{\text{o}}$.

So we get 1 rad $\approx 57^{\text{o}}$. What angle then in radians is 1^{o}?

We have $45^{\text{o}} = \dfrac{\pi}{4}$ rad. So we get $1^{\text{o}} = \dfrac{\pi}{4 \cdot 45} = \dfrac{\pi}{180}$ rad = 0.01745… rad.

So we get: $1^{\text{o}} \approx 0.0175$ rad. Why angles in degrees or radians though?

We can use either system working with angles. Working with angles though, we can say in large that we use angles in two different cases. In one, we use angles that do not change, and in the other, we use angles that can change.

And angles that do not change can be said to be static, and angles that can change can be said to be dynamic. And using angles static, we are likely to use the degree system, and using angles dynamic, we are likely to use the radian system. In either case though, we can still use either system.

Using angles in degrees, we have to use a symbol, which is a small circle as in 90^{o}.

Using angles in radians however, we just use real numbers as the angles. Usually, using angles in radians, we just use numbers only, and do not put rad or radian to the numbers.

So for instance, assuming $\boldsymbol{A}$ is π radian, we just put it this way: $\boldsymbol{A = \pi}$, which is a number 3.141592…, which is an irrational number, which is a real number.

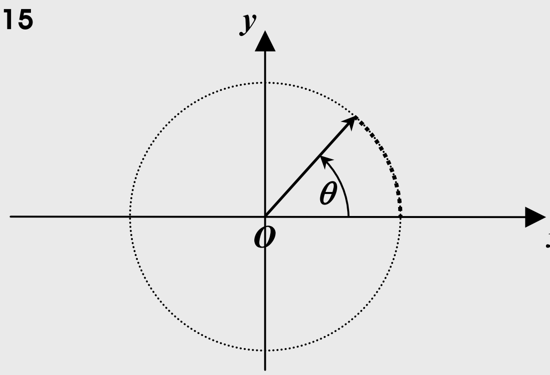
And for another instance, assuming $\boldsymbol{B}$ is 7 rad, we simply put it this way: $\boldsymbol{B = 7}$.

What then, can be the advantage of using angles in radians?

Suppose we work with a function, where the inputs are angles. Then, the input variable is an angle that can change, and thus, is dynamic (characterized by continuous change). And it's convenient to use real numbers as angles. So in such cases, we normally use angles in radians rather than degrees.

Let's now get back again to the vector turning about the origin in the ***x-y*** plane, and look at angles from a bit different perspective.

Fig. 15



Then, we can see that the vector keeps changing its direction, and that the angle between the vector and the ***x***-axis changes as the vector turns. We know that the ***x***-axis is fixed.

So what makes an angle is the vector turning; thus, we can say for simplicity that the angle between the vector and the ***x***-axis is the angle the vector makes. When the vector is at rest sitting on the ***x***-axis on the right of the origin, its angle is assumed to be 0^o.

Suppose now, the vector starts turning counterclockwise.
Then, as the vector keeps turning, it keeps changing its direction, and its angle keeps changing. When tuning counterclockwise, the angle is positive.

Technically, a circle is an arc, too. And by definition, a circle has 360°, which is 2π, and a half circle has 180^o, which is π. So if the vector tuning counterclockwise makes a complete turn about the origin, its arrowhead traces a circle; thus, the angle made is 360^o, that is, 2π. And if two complete turns are made, the arrowhead traces a circle twice, so the angle made is twice 360^o, and is 720^o, that is, 4π. Why not though, 360^o but 720^o?

The angle made in this case is the amount of turning, which is in this case, twice a complete turn. So the angle can be more than 2π, and can be 3π, 10.7, 19, 27π etc. And if a quarter of a turn is made, the arrowhead traces a quarter of a circle, so the angle made is a quarter of 360^o, and is 90^o, that is, $\frac{\pi}{2}$. What if the vector is turning clockwise?

Then, its angle is negative. So for instance, if the vector makes a complete turn clockwise, its angle is -360^o, that is, -2π, and if a quarter of a complete turn is made clockwise, the angle made is -90^o or $-\frac{\pi}{2}$.

Thus, tuning the vector counterclockwise or clockwise, we can get all angles. And indicating all such angles, we can use all real numbers.   So what?

We can come up with functions where inputs are angles. And it's convenient to use real numbers as inputs. So in such a function, using as inputs angles in radians, we can use all real numbers as such inputs. And such a function is called a trigonometric function, often just called a trig function.   What are angles for though?

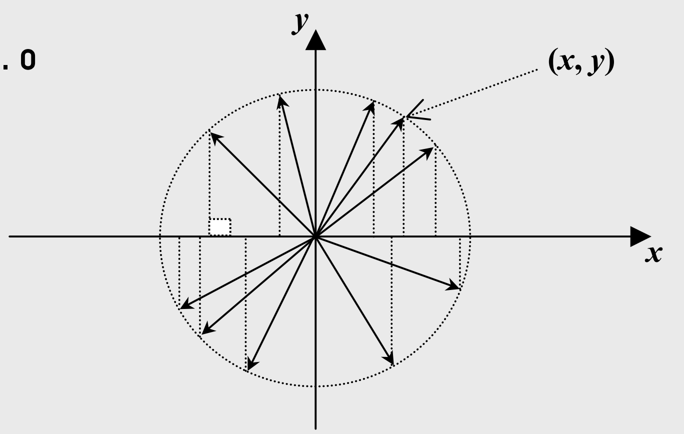
Working with objects that have directions and can change their amounts if changing their directions, we need to work with angles. We don't just work with angles though. The angle we get to work with in such a case is one of the angles in a triangle. It's not just any triangle though.   What triangle then?

Of all kinds in triangles, the simplest and the most basic, and thus, the most fundamental is a triangle where one angle is 90^o, called a right angle. So such a triangle is called a right triangle. And a right triangle is the place where a special geometry called trigonometry begins.

2.0. Coordinate Trigonometry 1

What is coordinate trigonometry?

Fig. 0



It makes trigonometry more interesting, and is trigonometry on the coordinate plane as the ***x-y*** coordinate plane, the ***x-y*** plane, for short. So we do it if using *coordinates* when doing trigonometry. Why though? What do we do it for?

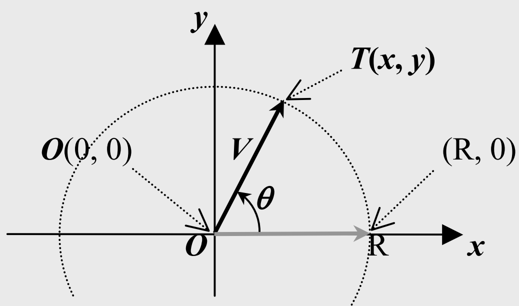
It's for more tools in math, and allows us to use *all kinds of angles*, as well as those between 0^o and 90^o.

So angles can be 760^o, 127^o, $\sqrt{2}^o$, -30^o, even 0^o, and can even be all real numbers.

Thus, more importantly, we can use functions using it, i.e., we can use *trigonometric functions*, trig functions, for short. We can do coordinate trigonometry if putting right triangles in the ***x-y*** plane. How to put them in the plane?

We are going to use a math object called a vector. A vector has both a magnitude and a direction. The magnitude is indicated by its length, along with a line segment, and the direction is indicated by an angle, along with an arrowhead. So it looks like an arrow, where the nock is called the initial point or the starting point, and the arrowhead is called the terminal point, called the endpoint, too.

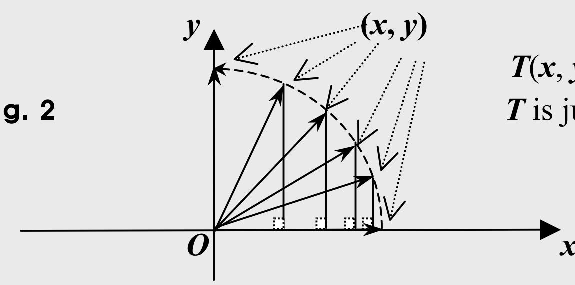
Fig. 1



In Fig. 1 above, for the vector $\boldsymbol{V}$, the origin $\boldsymbol{O}(0, 0)$ is the initial point, $\boldsymbol{T}(\boldsymbol{x}, \boldsymbol{y})$ is the terminal point, called the endpoint, too, and R is the length, and the angle $\boldsymbol{\theta}$ indicates the direction. In this book, $|\boldsymbol{V}|$ is taken as the length, that is, the magnitude of the vector $\boldsymbol{V}$. So in Fig. 1, $|\boldsymbol{V}| = \mathrm{R}$.
And we call such a vector as $\boldsymbol{V}$ a radius vector, also called a position vector, just called a vector for short in this book. Also in this book, |vector| is taken as such a vector's length.

Now, doing coordinate trigonometry, we take the |vector| or $|\boldsymbol{V}|$ as the hypotenuse, take $\boldsymbol{x}$ at $\boldsymbol{T}(\boldsymbol{x}, \boldsymbol{y})$ as one leg, and take $\boldsymbol{y}$ at $\boldsymbol{T}(\boldsymbol{x}, \boldsymbol{y})$ as the other leg.

Fig. 2



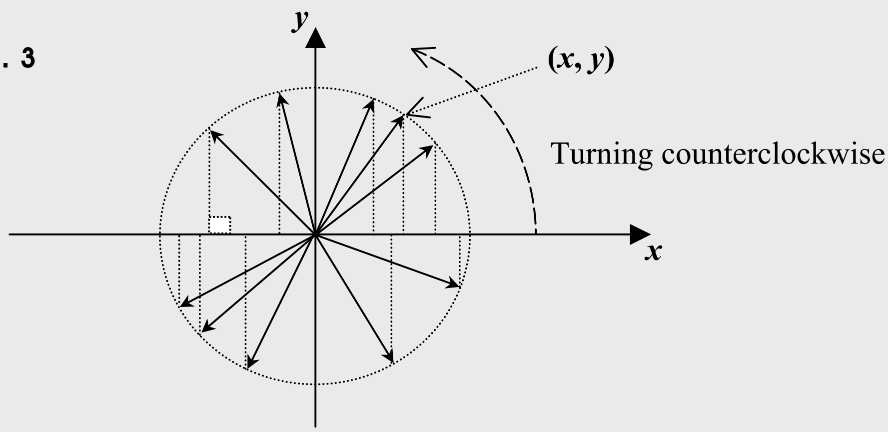
$\boldsymbol{T}(\boldsymbol{x}, \boldsymbol{y})$ is often put $(\boldsymbol{x}, \boldsymbol{y})$ for short.
$\boldsymbol{T}$ is just the name of the point $(\boldsymbol{x}, \boldsymbol{y})$.

Let's next, put in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane, as in Fig. 2 above, a (radius) vector turning about the origin keeping the initial point at the origin. The terminal point is at $(\boldsymbol{x}, \boldsymbol{y})$. So the initial point is $\boldsymbol{O}(0, 0)$ and the terminal point is $(\boldsymbol{x}, \boldsymbol{y})$.
Then, we can see many right triangles in the plane, since the vector keeps turning. Also, we can say that a right triangle changes as the vector turns.

Now, one leg is the adjacent, the other is the opposite, and in coordinate trigonometry, each leg has its value, and we can say that the value of each leg can be negative and 0, as well as positive. Understanding it, we can easily get into trig functions.

Suppose now, as in Fig. 3, a vector keeps turning about the origin in the ***x-y*** plane.

Fig. 3



Then, at every moment, taking the |vector| as a hypotenuse, we can see a different right triangle. The terminal point (arrowhead) of the vector is **(*x*, *y*)**, so in each triangle, we use ***x*** at **(*x*, *y*)** as the adjacent. What then is the opposite?

We use ***y*** at **(*x*, *y*)** as the opposite. So each leg has its value which can be negative and 0, as well as positive.
And the |vector| stays the same while the vector is turning. So the terminal point traces a circle, the radius is the |vector|, and of course, the center is the origin.

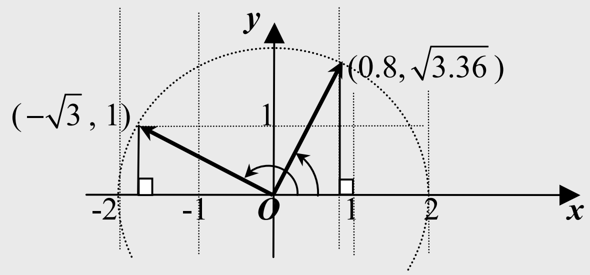
Thus, taking the |vector| as the hypotenuse, the ***x***-coordinate at the terminal point as the adjacent, and the ***y***-coordinate as the opposite, we can see a right triangle changing as the vector keeps turning.

Now, when the vector is on the coordinate axes, or is in a quadrant other than the first one, it's not the case where both coordinates at the terminal point are positive. So it's not the case where both the adjacent and the opposite are positive. How come?

It's because as shown in Fig. 3, each right triangle is a part of the coordinate plane, where we use all real numbers in each axis for the coordinates of the terminal point. Among real numbers, we have 0 and negative ones, as well as positive ones.

So each right triangle can be defined by the origin and the ***x*** and ***y***-coordinates of the terminal point **(*x*, *y*)**. Thus, we can take the |vector| as the hypotenuse, take ***x*** at **(*x*, *y*)** as the adjacent, and take ***y*** at **(*x*, *y*)** as the opposite.

Fig. 4



So for instance, in Fig. 4, in the right triangle on the left hand side, 2 is the hypotenuse, $-\sqrt{3}$ is the adjacent, and 1 is the opposite.
In coordinate trigonometry, no matter what the angle may be, we use the ***x***-coordinate as the adjacent, and use the ***y***-coordinate as the opposite. So for instance, even if the angle is 200° or -500°, we use as the legs the ***x*** and ***y***-coordinates at the terminal point of the vector.

In coordinate trigonometry,
the ***x***-coordinate at the terminal point is the adjacent,
the ***y***-coordinate is the opposite,
the |vector| is the hypotenuse,
and the vector turned makes the angle.

So if |***V***| is the length of the vector ***V***. we get the trig-ratios the way as follows.

$$\sin\theta = \frac{\boldsymbol{oppsite}}{\boldsymbol{hypotenuse}} = \frac{\boldsymbol{y - coordinate}}{\boldsymbol{hypotenuse}} = \frac{\boldsymbol{y}}{|\boldsymbol{vector}|}. \quad \text{So} \quad \sin\theta = \frac{\boldsymbol{y}}{|\boldsymbol{V}|}$$

$$\cos\theta = \frac{\boldsymbol{adjacent}}{\boldsymbol{hypotenuse}} = \frac{\boldsymbol{x - coordinate}}{\boldsymbol{hypotenuse}} = \frac{\boldsymbol{x}}{|\boldsymbol{vector}|}. \quad \text{So} \quad \cos\theta = \frac{\boldsymbol{x}}{|\boldsymbol{V}|}$$

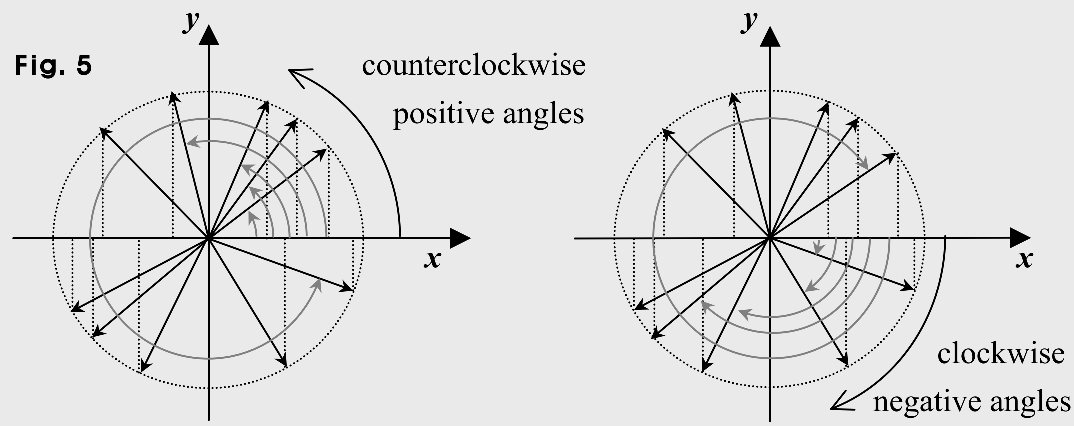
$$\tan\theta = \frac{\boldsymbol{opposite}}{\boldsymbol{adjacent}} = \frac{\boldsymbol{y - ccodiante}}{\boldsymbol{x - coordinate}} = \frac{\boldsymbol{y}}{\boldsymbol{x}}. \quad \text{So} \quad \tan\theta = \frac{\boldsymbol{y}}{\boldsymbol{x}}$$

What then is the purpose of the idea of a right triangle with a side negative or zero?

The purpose is trig functions. A trig function is a function of an angle, and we want the angle to get all real numbers, negative and 0, as well as positive.
How does the angle get all real numbers?

We know an angle is an amount of turning. So the vector turning the way described above makes angles. And if it turns counterclockwise, the angles are positive. If clockwise, it makes negative angles. And of course, no turning means that the angle is 0^o.

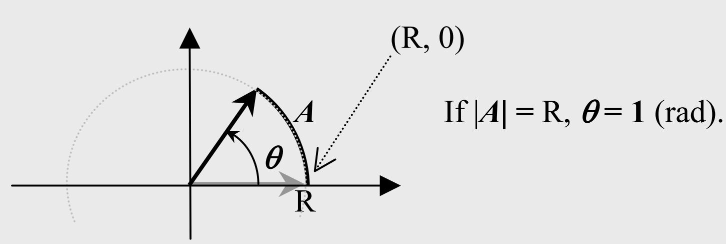
Fig. 5



And we can put all angles in degrees in a different system where we can use real numbers to specify angles. The system is a metrology called *radian* system, where we use 2π as the basis. 2π is a real number, and is equivalent to 360^o.
It's because every complete turn, 2π repeats. For a complete turn, the vector turned makes 2π, so two complete turns, it makes twice 2π.

In Fig. 6, suppose $\boldsymbol{A}$ is an arc which the arrowhead traced as the vector turned, and the length of $\boldsymbol{A}$ is R, which is the radius of the circle the arc belongs to. Then, the angle $\boldsymbol{\theta}$ is 1 radian, 1 rad for short. Usually though, if the system is understood, we omit rad, so for instance, π means 180^o, $\frac{\pi}{3} = 60^o$, and we can just say $\frac{\pi}{2}$ to mean 90^o.

Fig. 6



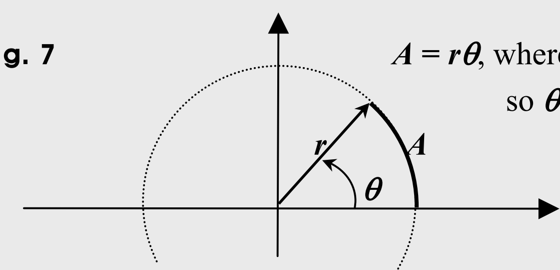
An angle is an amount of turning, so the angle $\boldsymbol{\theta}$ is the amount the vector turned.
In short, $\boldsymbol{\theta}$ is the vector turned.
Note that the radius of the circle the arc belongs to is simply the radius of the arc. Now, if the (length of the) arc is the radius, the angle the vector turned is 1 (rad).

So in short, if an arc is the radius, the angle is 1.
In radian system, an *angle* is the ***ratio*** *of the arc* to its *radius*.

So we have $\frac{\boldsymbol{A}}{\boldsymbol{r}} = \theta$, where $\boldsymbol{A}$ is the length of the arc $\boldsymbol{A}$, $\boldsymbol{r}$ is the radius, and $\boldsymbol{\theta}$ is the angle the arc $\boldsymbol{A}$ has. Thus, assuming $\boldsymbol{A = r}$, we get $\theta = \frac{\boldsymbol{A}}{\boldsymbol{r}} = \frac{\boldsymbol{r}}{\boldsymbol{r}} = 1$, which is 1 radian, 1 for short.

And since $\frac{\boldsymbol{A}}{\boldsymbol{r}} = \theta$, we get $\boldsymbol{A = r\theta}$, where $\boldsymbol{\theta}$ is in radians.

Fig. 7



$\boldsymbol{A = r\theta}$, where $\boldsymbol{\theta}$ is in radians,
so $\boldsymbol{\theta}$ is a real number.

So knowing the angle $\boldsymbol{\theta}$ and the radius $\boldsymbol{r}$, we can get the arc length, the length of $\boldsymbol{A}$.
What angle then in the degree system is 1 radian?   That is, what degree is 1?

An angle 2π is equivalent to 360°, that is, $360^\circ = 2\pi$,
so we get $1^\circ = \frac{2\pi}{360} = \frac{2 \cdot 3.141592...}{360} \approx 0.017453293$, which is a real number.

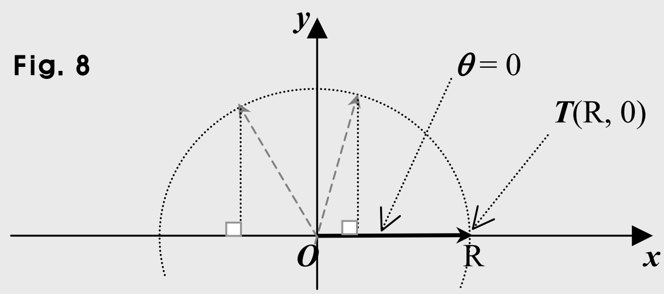
Also, we get $1 \text{ rad} = \frac{360^\circ}{2\pi} = \frac{360^\circ}{2 \cdot 3.141592...} \approx 57.29577951^\circ$.
Note that an angle in radians is the ratio of the arc to its radius. So such an angle is a ratio, which is a number, and is a real number.
Now, what's important is that using the radian system in coordinate trigonometry, we can use all real numbers as angles.

Let's now make sure what angle we use to get the trig ratios. In a right triangle not in a coordinate plane, the adjacent and the hypotenuse make the angle we use to get trig ratios. In coordinate trigonometry though, what makes that angle is the vector that is turning. That is to say that the vector turned makes the angle we use to get trig ratios.

So let's see now how the angles get made, and how we get the trig ratios for the angles.

To begin with, before turning, as in Fig. 8, the vector is assumed to be on the ***x***-axis to the right of the origin, and the angle made by the vector is 0.

Fig. 8

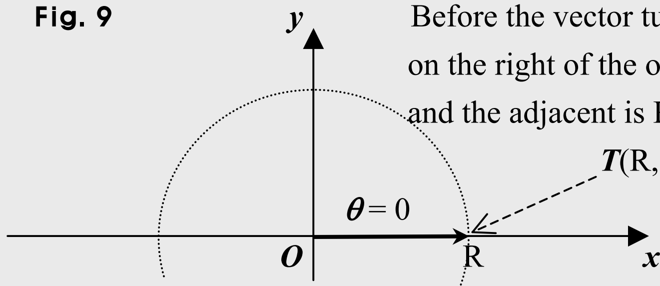


What then is the right triangle we can work with?

Though it looks like just a vector, it's mathematically, a right triangle with one leg and a hypotenuse, the hypotenuse is R, and the adjacent is R, too, since the ***x***-coordinate at the terminal point ***T*** is R, and the opposite is 0, since the ***y***-coordinate at ***T*** is 0.

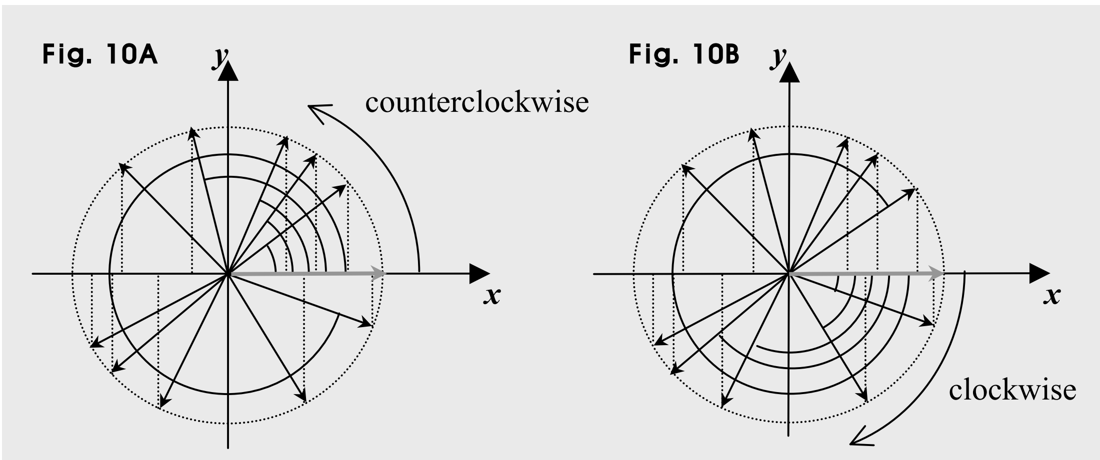
It's because as shown in Fig. 8, each right triangle is a part of the coordinate plane, where we can use all real numbers in each axis for the coordinates of the terminal point.

Fig. 9



Before the vector turns, the vector is on the ***x***-axis on the right of the origin, the opposite is 0, and the adjacent is R.

Suppose now that in Fig. 10A, the vector in gray is not turning, and is at rest sitting on the ***x***-axis on the right of the origin. In coordinate trigonometry, the angle made by the vector turning is the angle we use to get trig ratios. So the angle in this case is 0. Normally, we omit rad.



As soon as the vector starts turning, the opposite starts growing. Up to the length of the vector, |vector|. Then, it shrinks, down to 0, and then, it grows, up to the |vector|. The growth and the shrinkage continue. And the same is true for the adjacent, too. Starting with the shrinkage though.

It's because the vector keeps turning and the arrowhead traces circles over and over. If the turning continues ***counterclockwise***, the angle made by the vector is ***positive*** and keeps ***increasing***.

So the angle begins with 0, becomes … 0.1 … 1, … 2, … 2.4, …, 100, …, and gets larger and larger. That's the way the angle covers 0 and all the positive real numbers. What about the negative real numbers?

Suppose now in Fig. 10B above, the vector in gray is at rest again sitting on the ***x***-axis on the right of the origin. The angle made is 0. As soon as the vector starts turning, similar activities happen, so the opposite starts growing. Up to the –|vector|. Note that it's negative. Then, it shrinks down to 0, and then, it grows up to the |vector|. The growth and the shrinkage continue. And the same is true for the adjacent, too. Starting with the shrinkage though.

It's because the vector keeps turning and the arrowhead traces circles over and over. If the turning continues ***clockwise***, the angle made by the vector is ***negative***, and keeps ***decreasing*** in value (but increasing in magnitude). So the angle can begin with 0, and it becomes …,-0.5,… -1,… -2.5,… -3.2, …, -100, …, and gets smaller and smaller. That's the way the angle covers 0 and all the negative real numbers.

So putting threads together, the vector turning can cover all the real numbers. Let's now see how trig ratios get made as the vector turns.

Suppose now again, as in Fig. 11, the vector is at rest sitting on the $\boldsymbol{x}$-axis on the right of the origin.

We know in coordinate trigonometry, in a right triangle, the hypotenuse is the |vector|, the adjacent is the $\boldsymbol{x}$-coordinate of the terminal point, and the opposite is the $\boldsymbol{y}$-coordinate.

And we know the sine is the opposite over the hypotenuse. So the sine is 0, because the opposite is 0, which is the $\boldsymbol{y}$-coordinate at the terminal point of the vector.

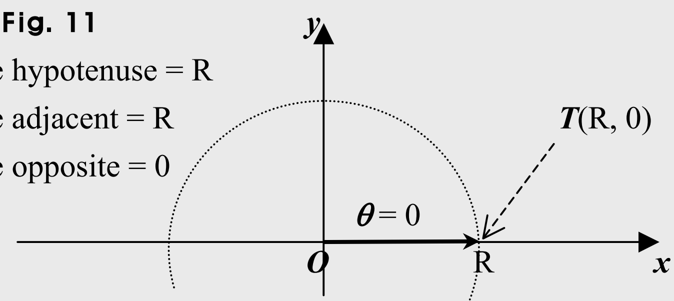
Thus, the sine of an angle 0 is $\dfrac{\boldsymbol{opposite}}{\boldsymbol{hypotenuse}} = \dfrac{0}{\mathrm{R}} = 0$, that is, **sin 0** is 0.

Fig. 11

The hypotenuse = R

The adjacent = R

The opposite = 0



Thus, though it does not look like any triangle at all, we can take it as a right triangle with a hypotenuse and one leg, which is the adjacent, and is overlapped with the hypotenuse, which is the |vector|. What then about the cosine?

We know the cosine is the adjacent over the hypotenuse. So the cosine is 1, because the hypotenuse is R, and the adjacent is R, too, since R is the $\boldsymbol{x}$-coordinate at $\boldsymbol{T}$(R, 0).

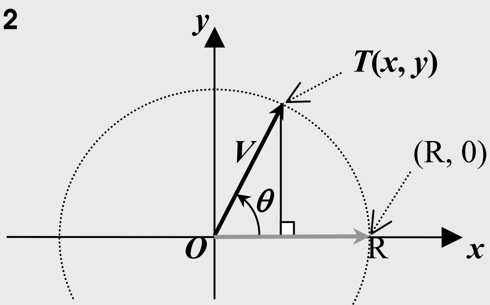
Thus, the cosine of an angle 0 is $\dfrac{\boldsymbol{adjacent}}{\boldsymbol{hypotenuse}} = \dfrac{\mathrm{R}}{\mathrm{R}} = 1$, so **cos 0** is 1.

And next, what about the tangent?

The tangent is the opposite over the adjacent. So the tangent is 0, for the opposite is 0, which is the ***y***-coordinate at ***T***(R, 0), and the adjacent is R, the ***x***-coordinate at ***T***(R, 0). Thus, the tangent of an angle 0 is $\dfrac{\textbf{\textit{opposite}}}{\textbf{\textit{adjacent}}} = \dfrac{0}{\mathrm{R}} = 0$, so **tan 0** is 0.

Next, when the vector is in the first quadrant, the vector is between both coordinate axes. Suppose now, as in Fig. 12, the |vector| is R and the angle made is $\boldsymbol{\theta}$.

Fig. 12

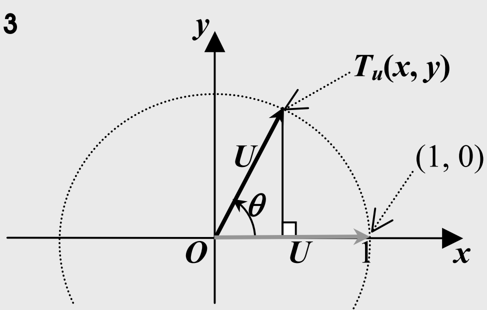


Since ***T***(***x***, ***y***) is the terminal point of the vector ***V***, we get

$$\textbf{sin } \theta = \frac{\textbf{\textit{opposite}}}{\textbf{\textit{hyoitenuse}}} = \frac{y}{\mathrm{R}}, \ \textbf{cos } \theta = \frac{\textbf{\textit{adjacent}}}{\textbf{\textit{hypotenuse}}} = \frac{x}{\mathrm{R}}, \text{ and } \textbf{tan } \theta = \frac{\textbf{\textit{opposite}}}{\textbf{\textit{adjacent}}} = \frac{y}{x}.$$

Suppose this time, as in Fig. 13, the length of a vector ***U*** is 1.

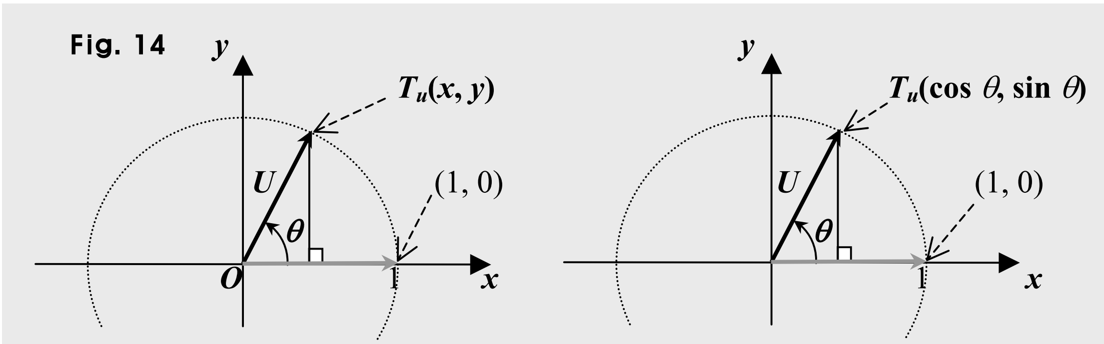
Fig. 13



Since $\boldsymbol{T_u}$(***x***, ***y***) is the terminal point of the vector ***U*** of length 1, we get

$$\textbf{sin } \theta = \frac{\textbf{\textit{opposite}}}{\textbf{\textit{hyoitenuse}}} = \frac{y}{1}, \ \textbf{cos } \theta = \frac{\textbf{\textit{adjacent}}}{\textbf{\textit{hypotenuse}}} = \frac{x}{1}, \text{ and } \textbf{tan } \theta = \frac{\textbf{\textit{opposite}}}{\textbf{\textit{adjacent}}} = \frac{y}{x}.$$

Thus, we get **sin** $\boldsymbol{\theta = y}$, **cos** $\boldsymbol{\theta = x}$, and **tan** $\boldsymbol{\theta = \frac{y}{x}}$.



And we have $\boldsymbol{x^2 + y^2 = 1}$. So we get $\mathbf{sin}^2\,\boldsymbol{\theta} + \mathbf{cos}^2\,\boldsymbol{\theta} = \mathbf{1}$, which is a trig identity.
How then do we get $\boldsymbol{x^2 + y^2 = 1}$?

We know the vector ***U*** is of length 1, and is turning about the origin.
So if the vector ***U*** makes a complete turn about the origin, its terminal point $\boldsymbol{T_u}$ traces a circle of radius 1 centered at the origin, a unit circle centered at the origin. And $\boldsymbol{T_u(x, y)}$ is the terminal point, and thus, is in the unit circle. So the equation of such a unit circle is

$$\boldsymbol{x^2 + y^2 = 1},$$

which is the equation we can get using Pythagorean Theorem.

The equation explains that 1 is the distance from the point $\boldsymbol{(x, y)}$ to the origin $\boldsymbol{O}(0, 0)$.
What then about the ratios as follows? **sin** $\boldsymbol{\theta} = \frac{\boldsymbol{y}}{\mathrm{R}}$, **cos** $\boldsymbol{\theta} = \frac{\boldsymbol{x}}{\mathrm{R}}$, and **tan** $\boldsymbol{\theta = \frac{y}{x}}$.

We know $\mathbf{sin}^2\,\boldsymbol{\theta} + \mathbf{cos}^2\,\boldsymbol{\theta} = \mathbf{1}$. So we can get

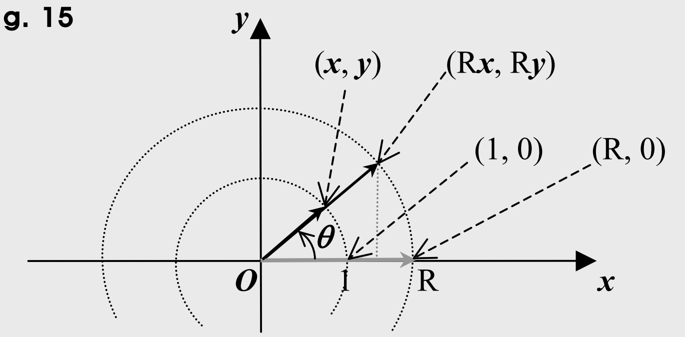
$$\mathbf{sin}^2\,\boldsymbol{\theta} + \mathbf{cos}^2\,\boldsymbol{\theta} = \left(\frac{\boldsymbol{y}}{\mathrm{R}}\right)^2 + \left(\frac{\boldsymbol{x}}{\mathrm{R}}\right)^2 = 1 \Rightarrow \frac{\boldsymbol{y}^2}{\mathrm{R}^2} + \frac{\boldsymbol{x}^2}{\mathrm{R}^2} = 1 \Rightarrow \boldsymbol{x}^2 + \boldsymbol{y}^2 = \mathrm{R}^2,$$

which is the equation of the circle of radius R centered at the origin. And the tangent has nothing to do with the length of the vector. What then about the sine and the cosine?

The length of the vector does not matter.
As well as the tangent, both the sine and the cosine have nothing to do with the length of the vector.

Let's now put the two cases together, and see what happens to the trig ratios if the length of the vector is different.

Fig. 15



Note that R$\boldsymbol{x}$ = R · $\boldsymbol{x}$, so $\boldsymbol{x}$ is not a subscript.

To begin with, why is (R$\boldsymbol{x}$, R$\boldsymbol{y}$) the terminal point of the longer vector?

$(\mathrm{R}x)^2 + (\mathrm{R}y)^2 = \mathrm{R}^2(x^2 + y^2) = \mathrm{R}^2$, since $x^2 + y^2 = 1$. So $(\mathrm{R}x)^2 + (\mathrm{R}y)^2 = \mathrm{R}^2$. Thus, putting the point (R$\boldsymbol{x}$, R$\boldsymbol{y}$) into $x^2 + y^2 = \mathrm{R}^2$, which is the equation of the circle of radius R centered at the origin, we get $(\mathrm{R}x)^2 + (\mathrm{R}y)^2 = \mathrm{R}^2$. So the length of the longer vector is R, and thus, (R$\boldsymbol{x}$, R$\boldsymbol{y}$) is the terminal point of the longer vector.

Now, in Fig. 15, we can see that from the inner circle we can get

$$\textbf{sin } \theta = \frac{\textit{opposite}}{\textit{hyoitenuse}} = \frac{y}{1} = y,\ \textbf{cos } \theta = \frac{\textit{adjacent}}{\textit{hypotenuse}} = \frac{x}{1} = x, \text{ and } \textbf{tan } \theta = \frac{\textit{opposite}}{\textit{adjacent}} = \frac{y}{x}.$$

And from the outer circle, we can get

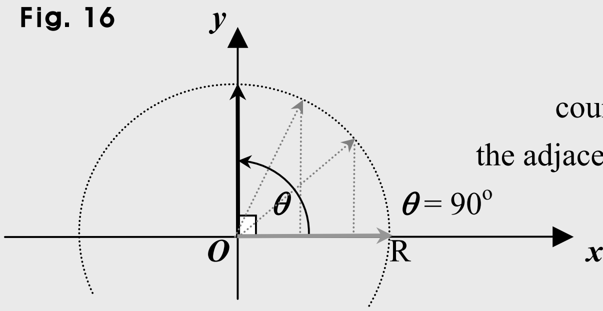
$$\textbf{sin } \theta = \frac{\textit{opposite}}{\textit{hyoitenuse}} = \frac{\mathrm{R}y}{\mathrm{R}} = y,\ \textbf{cos } \theta = \frac{\textit{adjacent}}{\textit{hypotenuse}} = \frac{\mathrm{R}x}{\mathrm{R}} = x,$$

$$\text{and } \textbf{tan } \theta = \frac{\textit{opposite}}{\textit{adjacent}} = \frac{\mathrm{R}y}{\mathrm{R}x} = \frac{y}{x}$$

So trig ratios have nothing to do with the length of the vector, that is, those ratios have noting to do with the radius of the circle that the terminal point traces.

Let's get back to the vector turning, and suppose the vector is now, on the y-axis above the origin as in Fig. 16. Then, we get again, a right triangle with a hypotenuse and a leg.

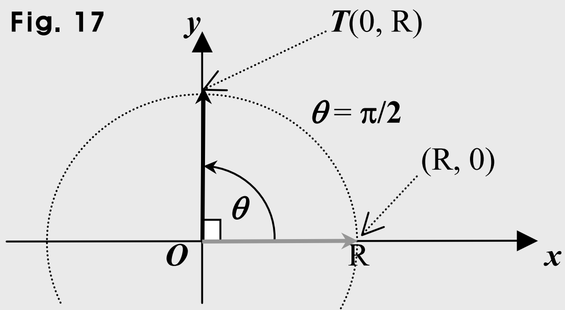
Fig. 16



While the vector is turning counterclockwise in the first quadrant, the adjacent shrinks, but the opposite grows.

In such a right triangle, the hypotenuse is the |vector|, R, and the opposite is R, too, but the adjacent is 0, which is possible in coordinate trigonometry.
And the angle we use to get trig ratios in coordinate trigonometry can be any real number as $0, \frac{\pi}{2}, \frac{7\pi}{2}$, 12, 15, or 9π.

Fig. 17



And we know the sine is the opposite over the hypotenuse. So the sine is 1, for the hypotenuse is R, and the opposite is R, too, since R is the y-coordinate at T.

Thus, the sine of the angle $\frac{\pi}{2}$ is $\frac{\textit{\textbf{opposite}}}{\textit{\textbf{hypotenuse}}} = \frac{\mathrm{R}}{\mathrm{R}} = 1$, so $\mathbf{sin}\frac{\pi}{2} = 1$.

What then about the cosine?

We know the cosine is the adjacent over the hypotenuse. So the cosine is 0, since the adjacent is 0. Thus, the cosine of the angle $\frac{\pi}{2}$ is $\frac{\textit{\textbf{adjacent}}}{\textit{\textbf{hypotenuse}}} = \frac{0}{\mathrm{R}} = 0$ so $\mathbf{cos}\frac{\pi}{2} = 0$.

And next what about the tangent?

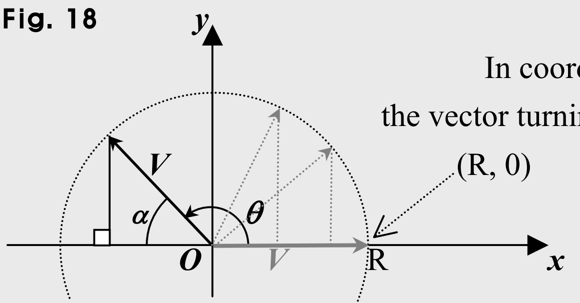
We know the tangent is the opposite over the adjacent, so it is $\dfrac{\boldsymbol{opposite}}{\boldsymbol{adjacent}} = \dfrac{\text{R}}{0}$, which is undefined, since no denominator can be 0. So the tangent cannot be defined if $\boldsymbol{\theta} = \dfrac{\pi}{2}$.

That is, $\mathbf{tan}\dfrac{\pi}{2}$ is not defined. In other words, $\mathbf{tan}\dfrac{\pi}{2}$ does not exist.

We can see, as in Fig. 18, that as the vector ***V*** approaches the ***y***-axis, the tangent gets bigger monotonically, since the adjacent gets smaller, and the opposite gets larger, and the tangent is said to go to *infinity* when the vector ***V*** gets extremely close to the ***y***-axis, since the adjacent is nearly 0, but not 0 yet, whereas the opposite is almost R, the length of the vector ***V***.

Let's next, move on to the case where the vector is in the second quadrant as in Fig. 18

Fig. 18



In coordinate trigonometry, the angle made by the vector turning is the angle we use to get trig ratios. So in this case, too, $\boldsymbol{\theta}$ is the angle.

Then, taking the |vector| as a hypotenuse, we can see a right triangle. In the triangle, we use an ***x***-coordinate as the adjacent, and use a ***y***-coordinate as the opposite. And the point that has the ***x*** and ***y***-coordinates is the terminal point of the vector.

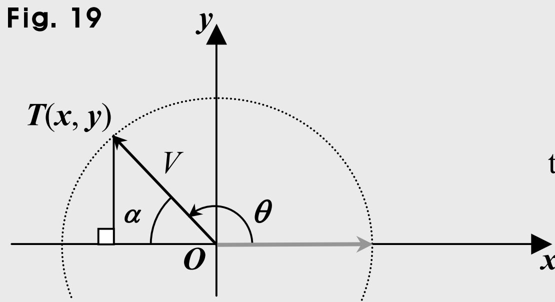
In this case though, the ***x***-coordinate is negative, so the adjacent is negative.

Besides, the angle made is not between 0 and $\dfrac{\pi}{2}$, but is greater than $\dfrac{\pi}{2}$.

So in this case, also, we get to work with a bit weird right triangle.
The right triangle has two legs, one is positive, and is the opposite, but the other is negative, and is the adjacent. And the |vector| is the hypotenuse, of course.
How in this case then can we get trig ratios?

To begin with, we want to note that in a right triangle in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane, the angle we use is the angle made by the vector turning.

Fig. 19



In this case, also,
the |vector| is the hypotenuse,
the $\boldsymbol{x}$-coordinate at $\boldsymbol{T}(\boldsymbol{x}, \boldsymbol{y})$ is the adjacent,
and the $\boldsymbol{y}$-coordinate is the opposite.

So not the angle $\boldsymbol{\alpha}$ but the angle $\boldsymbol{\theta}$ is the angle we use for the trig ratios.

In coordinate trigonometry, we use as the adjacent the $\boldsymbol{x}$-coordinate at the terminal point, and use the $\boldsymbol{y}$-coordinate as the opposite no matter what the angle may be.

So for instance, even if the angle is 400°, or -670°, we need to use as the legs the $\boldsymbol{x}$ and $\boldsymbol{y}$-coordinates at the terminal point of the vector.

In coordinate trigonometry,
the $\boldsymbol{x}$-coordinate at the terminal point is the adjacent,
the $\boldsymbol{y}$-coordinate is the opposite,
the |vector| is the hypotenuse,
and the vector turned makes the angle.

So we need to get the trig-ratios the way as follows.

$$\sin\theta = \frac{\boldsymbol{oppsite}}{\boldsymbol{hypotenuse}} = \frac{\boldsymbol{y - coordinate}}{\boldsymbol{hypotenuse}} = \frac{\boldsymbol{y}}{|\,\boldsymbol{vector}\,|}, \text{ so } \sin\theta = \frac{\boldsymbol{y}}{|\,\boldsymbol{V}\,|}$$

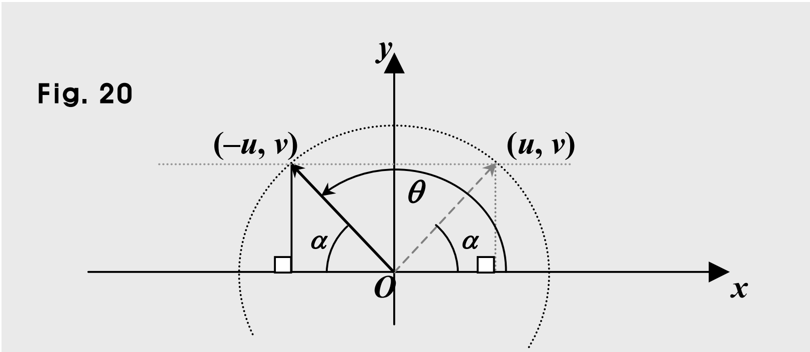
$$\cos\theta = \frac{\boldsymbol{adjacent}}{\boldsymbol{hypotenuse}} = \frac{\boldsymbol{x - coordinate}}{\boldsymbol{hypotenuse}} = \frac{\boldsymbol{x}}{|\,\boldsymbol{vector}\,|}, \text{ so } \cos\theta = \frac{\boldsymbol{x}}{|\,\boldsymbol{V}\,|}$$

$$\tan\theta = \frac{\boldsymbol{opposite}}{\boldsymbol{adjacent}} = \frac{\boldsymbol{y - ccodiante}}{\boldsymbol{x - coordinate}} = \frac{\boldsymbol{y}}{\boldsymbol{x}}. \quad \text{So } \tan\theta = \frac{\boldsymbol{y}}{\boldsymbol{x}}$$

What about the angle $\boldsymbol{\alpha}$ in Fig. 19 above?

In fact, getting the ratios for the angle $\boldsymbol{\theta}$, we use the right triangle with the angle $\boldsymbol{\alpha}$.

Fig. 20



In Fig. 20 above, the angle $\boldsymbol{\alpha}$ is the angle between the adjacent and the hypotenuse in each of the two right triangles symmetric about the $\boldsymbol{y}$-axis. So the two triangles themselves are identical. Thus, taking the ratios for the angle α, we can use them for the ***absolute*** values of the ratios for the angle $\boldsymbol{\theta}$. Why absolute value?

It's because the trig ratios in coordinate trigonometry can be negative.

In coordinate trigonometry, coordinates matter.
The sign of a trig ratio depends on the quadrant where the vector is.
Thus, the sign depends on the quadrant where the terminal point is.

If the terminal point is in the second quadrant, the $\boldsymbol{x}$-coordinate is negative, so the adjacent is negative, but the $\boldsymbol{y}$-coordinate is positive, so the opposite is positive.

Now, as in Fig. 20, if $\boldsymbol{(u, v)}$ is the terminal point of the vector that made the angle $\boldsymbol{\alpha}$, the terminal point of the vector that made the angle $\boldsymbol{\theta}$ is $\boldsymbol{(-u, v)}$.

So first, we know the sine is $\dfrac{\textit{\textbf{opposite}}}{\textit{\textbf{hypotenuse}}}$, so $\sin\theta = \dfrac{\boldsymbol{v}}{|\,\textit{\textbf{vector}}\,|}$, and $\sin\alpha = \dfrac{\boldsymbol{v}}{|\,\textit{\textbf{vector}}\,|}$.

Thus, we get $\mathbf{sin}\ \boldsymbol{\theta} = \mathbf{sin}\ \boldsymbol{\alpha}$.

Next, we know the cosine is $\dfrac{\textit{\textbf{adjacent}}}{\textit{\textbf{hypotenuse}}}$, so $\cos\theta = \dfrac{-\boldsymbol{u}}{|\,\textit{\textbf{vector}}\,|}$, whereas $\cos\alpha = \dfrac{\boldsymbol{u}}{|\,\textit{\textbf{vector}}\,|}$.

Thus, we get $\mathbf{cos}\ \boldsymbol{\theta} = -\mathbf{cos}\ \boldsymbol{\alpha}$.

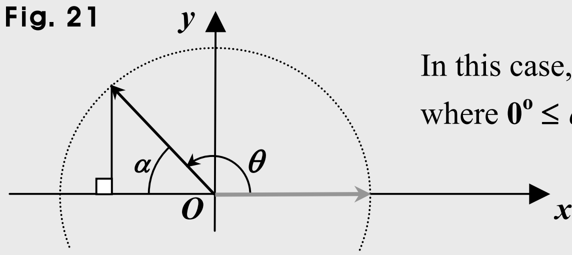
What then about the tangent?

We know the tangent is $\dfrac{\textbf{\textit{opposite}}}{\textbf{\textit{adjacent}}}$, so $\tan\theta = \dfrac{v}{-u} = -\dfrac{v}{u}$, whereas $\tan\alpha = \dfrac{v}{u}$.

Thus, we get **tan θ = –tan α**.

So taking the trig ratios in coordinate trigonometry, we take the ratios for the ***acute*** angle such as α in this case, and then, take the sign (+ or –) considering the quadrant where the terminal point is located when the angle θ is made. It's because the signs of the coordinates depend on the quadrant.

Fig. 21



In this case, $\boldsymbol{\theta = 180^{\circ} - \alpha = \pi - \alpha}$,
where $\boldsymbol{0^{\circ} \le \alpha \le 90^{\circ}}$ **or** $\boldsymbol{0 \le \alpha \le \pi/2}$

So we get **sin θ = sin α** in this case. And in Fig. 21 above, we can see that $\boldsymbol{\theta = \pi - \alpha}$. So we can say that if $\boldsymbol{\theta = \pi - \alpha}$, we get **sin θ = sin α**.

Thus, $\boldsymbol{\sin(\pi - \alpha) = \sin\alpha}$, where $\boldsymbol{0 \le \alpha \le \dfrac{\pi}{2}}$, which can be taken as a trig identity.

For instance, $\sin 135^{\circ} = \sin 45^{\circ} = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$, which means $\sin\dfrac{3\pi}{4} = \sin\dfrac{\pi}{4} = \dfrac{\sqrt{2}}{2}$.

What then about the cosine?

We know **cos θ = –cos α**, and know that $\boldsymbol{\theta = \pi - \alpha}$.
So we can say that if $\boldsymbol{\theta = \pi - \alpha}$, we get **cos θ = –cos α**.

Thus, $\boldsymbol{\cos(\pi - \alpha) = -\cos\alpha}$, where $\boldsymbol{0 \le \alpha \le \dfrac{\pi}{2}}$, which can be taken as a trig identity.

For instance, $\cos 150^{\circ} = -\cos 30^{\circ} = -\dfrac{1}{2}$, which means $\cos\dfrac{5\pi}{6} = -\cos\dfrac{\pi}{6} = -\dfrac{1}{2}$.

And next, what about the tangent?

We know **tan** $\boldsymbol{\theta}$ **= –tan** $\boldsymbol{\alpha}$, and know that $\boldsymbol{\theta} = \pi - \boldsymbol{\alpha}$.
So we can say that if $\boldsymbol{\theta} = \pi - \boldsymbol{\alpha}$, we get **tan** $\boldsymbol{\theta}$ **= –tan** $\boldsymbol{\alpha}$.

Thus, **tan (π – α) = –tan α**, where $\mathbf{0} \le \boldsymbol{\alpha} \le \frac{\pi}{2}$, which can be taken as a trig identity, too.

For instance, tan 120° = –tan 60° $= -\frac{\sqrt{3}}{2}$, which means $\tan\frac{2\pi}{3} = -\tan\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$.

sin (π – α) = sin α,

cos (π – α) = –cos α,

tan (π – α) = –tan α,

where $\mathbf{0} \le \boldsymbol{\alpha} \le \frac{\pi}{2}$.

In coordinate trigonometry,
the ***x***-coordinate at the terminal point is the adjacent,
the ***y***-coordinate is the opposite,
the |vector| is the hypotenuse,
and the angle is made by the vector turned.

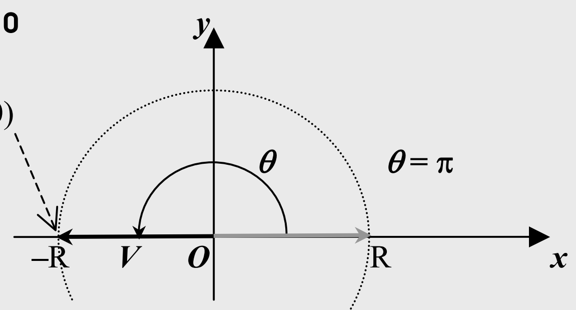
sin 0 = 0, cos 0 = 1, tan 0 = 0,

$\sin\frac{\pi}{2} = 1$, $\cos\frac{\pi}{2} = 0$, and $\tan\frac{\pi}{2}$ does not exist.

2.1. Coordinate Trigonometry 2

Suppose now, the vector $\boldsymbol{V}$ is on the $\boldsymbol{x}$-axis, this time, to the left of the origin. Then, the triangle we get is again, a right triangle with one leg, along with a hypotenuse.

Fig. 0



In the triangle described above, the hypotenuse is the |vector|, so is R, the adjacent is –R, since –R is the $\boldsymbol{x}$-coordinate at the terminal point $\boldsymbol{T}$, and the $\boldsymbol{y}$-coordinate is 0, so the opposite is 0. And the angle made by the vector $\boldsymbol{V}$ is π.

We know the sine is the opposite over the hypotenuse, which is R in this case. So the sine is 0, for the opposite is 0 as stated above. Thus, the sine of the angle π is $\frac{0}{\mathrm{R}} = 0$, so we get **sin π** = 0. What then about the cosine?

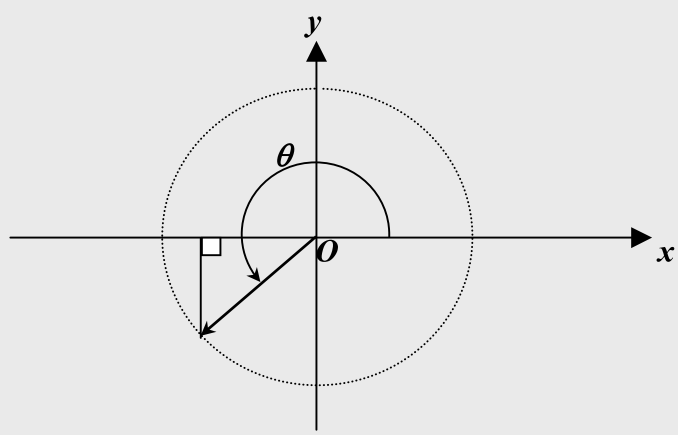
We know the cosine is the adjacent over the hypotenuse. The hypotenuse is R, the length of the vector $\boldsymbol{V}$, and the adjacent is –R, as –R is the $\boldsymbol{x}$-coordinate at the terminal point $\boldsymbol{T}$. Thus, the cosine of the angle π is $\frac{-\mathrm{R}}{\mathrm{R}} = -1$, so **cos π** = –1.

And next, what about the tangent?

The tangent is the opposite over the adjacent. So the tangent is 0 if the angle is π, for the opposite is 0, and the adjacent is –R. Thus, the tangent of the angle π is $\frac{0}{-R} = 0$, that is, we get **tan** $\pi = 0$.

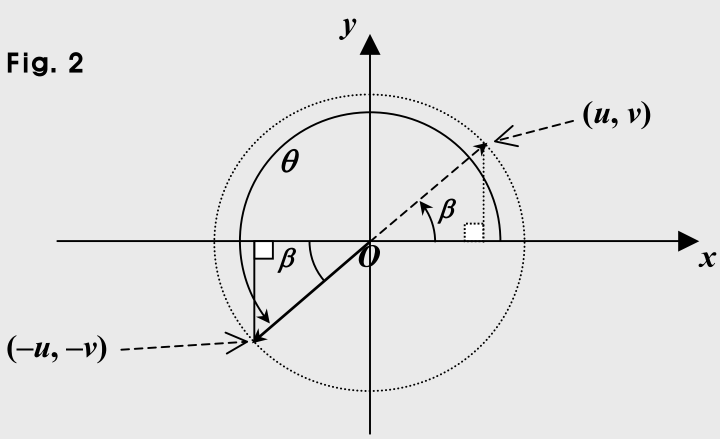
Let's next, move on to the case where the vector is in the *third quadrant* as in Fig. 1.

Fig. 1



To begin with, we want to note again that in a right triangle in the ***x-y*** plane, the angle we use is the angle made by the vector turning. So in this case, also, θ is the angle we use to get trig ratios.

Fig. 2



Getting the ratios for the angle θ, $\pi + \beta$, we use the right triangle with the angle β.

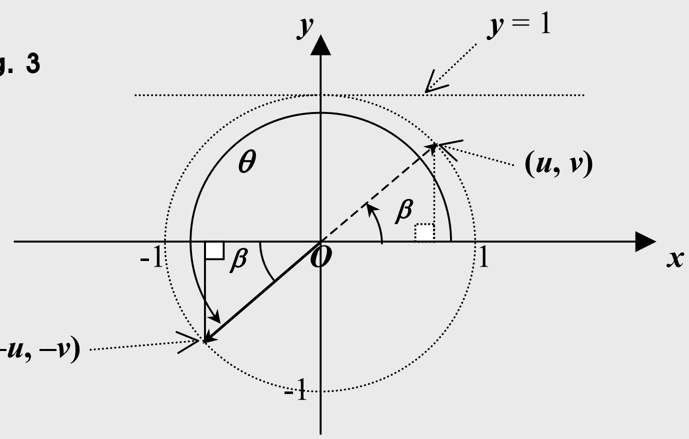
In Fig. 2 above, the angle β is the angle between the adjacent and the hypotenuse in each of the two right triangles symmetric about the origin O. So the two triangles themselves are identical. Thus, if taking the ratios for the angle β, we can use them for the ***absolute*** values of the ratios for the angle θ. Why absolute?

In coordinate trigonometry, coordinates matter. The sign of a trig ratio depends on the quadrant with the vector, because the signs of the coordinates at the terminal point can be negative, and thus, the signs determine the sign of the ratio.

If the terminal point is in the third quadrant, the x-coordinate of the terminal point is negative, so the adjacent is negative, and the y-coordinate is negative, so the opposite is negative, too.

Now, as in Fig. 2, if (u, v) is the terminal point of the vector that made the angle β, the terminal point of the vector that made the angle θ is $(-u, -v)$.
Let's now take the ratios.

Fig. 3

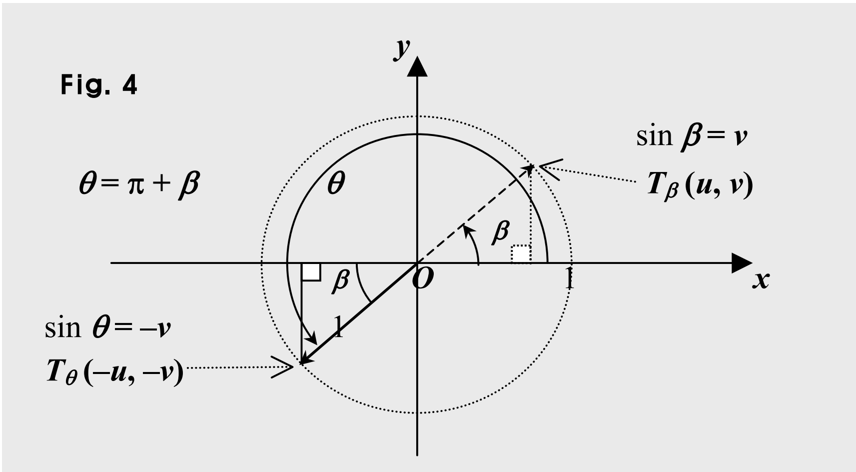


To begin with, we know the sine is the opposite over the hypotenuse.

So as in Fig. 3, suppose now the |vector| is 1, so the hypotenuse is 1.

Then, we can see that the sine is $\frac{-v}{1}$, for the opposite is $-v$, and the hypotenuse is 1.

Thus, if the vector is in the third quadrant, the sine of the angle θ is $-v$, that is, $\sin\theta = -v$, so we get $\sin\theta = -\sin\beta$, since $\sin\beta = v$.



So we get **sin** $\boldsymbol{\theta}$ **= –sin** $\boldsymbol{\beta}$. And in Fig. 4 above, we can see that $\boldsymbol{\theta = \pi + \beta}$.

Thus, **sin (π +** $\boldsymbol{\beta}$**) = –sin** $\boldsymbol{\beta}$, where $\boldsymbol{0 \le \beta \le \frac{\pi}{2}}$**,** which can be taken as a trig identity.

For instance, sin 210° = sin (180° + 30°) = –sin 30° $= -\frac{1}{2}$, that is,

$$\sin\frac{7\pi}{6} = \sin\left(\pi + \frac{\pi}{6}\right) = -\sin\frac{\pi}{6} = -\frac{1}{2}.$$ What then about the cosine?

We know the cosine is the adjacent over the hypotenuse. So assuming again, the hypotenuse is 1, we can see the cosine of $\boldsymbol{\theta}$ is $\boldsymbol{-u}$, so we get **cos** $\boldsymbol{\theta = -u}$, because the adjacent is $\boldsymbol{-u}$, since $\boldsymbol{-u}$ is the $\boldsymbol{x}$-coordinate at the terminal point $\boldsymbol{T_\theta(-u, -v)}$.
What then about **cos** $\boldsymbol{\beta}$?

The $\boldsymbol{x}$-coordinate at the terminal point $\boldsymbol{T_\beta(u, v)}$ is $\boldsymbol{u}$, and the hypotenuse is 1, so **cos** $\boldsymbol{\beta}$ is $\boldsymbol{u}$. Now, we have $\boldsymbol{\theta = \pi + \beta}$, and **cos** $\boldsymbol{\theta = -u}$.

So we can see that if $\boldsymbol{\theta = \pi + \beta}$, we get **cos** $\boldsymbol{\theta}$ **= –cos** $\boldsymbol{\beta}$.

Thus, **cos (π +** $\boldsymbol{\beta}$**) = –cos** $\boldsymbol{\beta}$, where $\boldsymbol{0 \le \beta \le \frac{\pi}{2}}$**,** which can be taken as another trig identity.

For instance, cos 240° = –cos 60° $= -\frac{1}{2}$, that is, $\cos\frac{4\pi}{3} = -\cos\frac{\pi}{3} = -\frac{1}{2}$

And next, what about the tangent?

We know the tangent is the opposite over the adjacent.

So the tangent of θ is $\frac{-v}{-u} = \frac{v}{u}$, since the opposite is $-v$, and the adjacent is $-u$, because of the terminal point $T_\theta(-u, -v)$. What then about **tan** β?

The x-coordinate at the terminal point $T_\beta(u, v)$ is u, and the y-coordinate at the terminal point $T_\beta(u, v)$ is v, so $\tan\beta = \frac{v}{u}$. Now, we have $\theta = \pi + \beta$, and $\tan\theta = \frac{v}{u}$.

So we can see that if $\theta = \pi + \beta$, we get $\tan\theta = \tan\beta$.

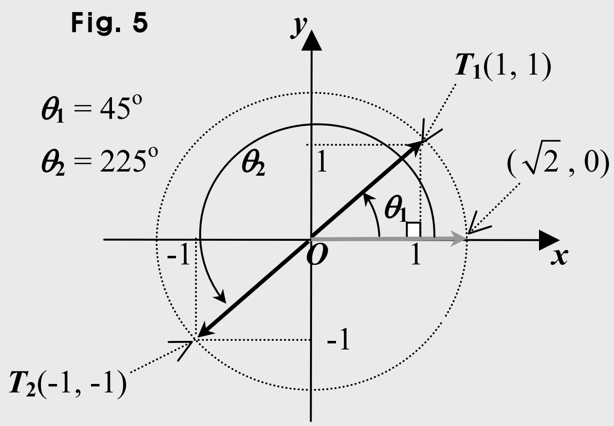
Thus, $\tan(\pi + \beta) = \tan\beta$, where $0 \le \beta \le \frac{\pi}{2}$, which can be taken as a trig identity, too.

For instance, tan 225° = tan 45° $= \frac{1}{1} = 1$, that is, $\tan\frac{5\pi}{4} = \tan\frac{\pi}{4} = \frac{1}{1} = 1$

Fig. 5

$\theta_1 = 45^\circ$

$\theta_2 = 225^\circ$



$$\sin(\pi + \beta) = -\sin\beta$$

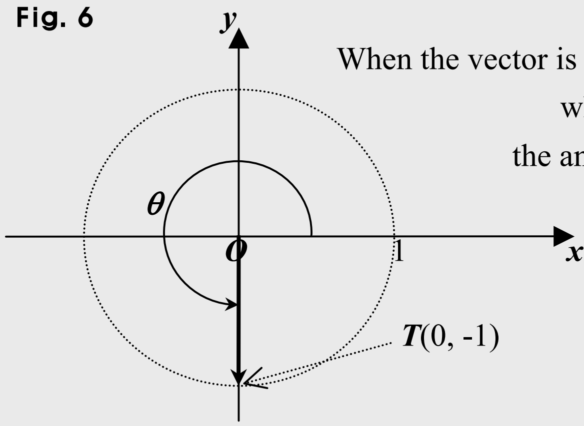
$$\cos(\pi + \beta) = -\cos\beta$$

$$\tan(\pi + \beta) = \tan\beta$$

$$0 \le \beta \le \frac{\pi}{2}$$

Let's get back to the vector turning, and suppose the vector is now, on the ***y***-axis below the origin. Then, we get again, a right triangle with a hypotenuse and one leg.

Fig. 6



When the vector is on the ***y***-axis below the origin while turning counterclockwise, the angle made by the vector is **3π/2**.

In such a right triangle, assuming again, the |vector| is 1, we can say that the hypotenuse is 1, the opposite is –1, and the adjacent is 0, since the terminal point ***T*** is at (0, –1).

And the angle we use to get trig ratios is $\frac{3\pi}{2}$.

Now, we are ready to get the ratios.

So to begin with, we know the sine is the opposite over the hypotenuse.
Thus, since the hypotenuse is 1, and the opposite is –1, the sine of $\frac{3\pi}{2}$ is $\frac{-1}{1}$,

so we get $\sin\frac{3\pi}{2} = -1$.
What then about the cosine?

We know the cosine is the adjacent over the hypotenuse. So the cosine is 0, since the adjacent is 0, and the hypotenuse is 1. Thus, the cosine of the angle $\frac{3\pi}{2}$ is 0.

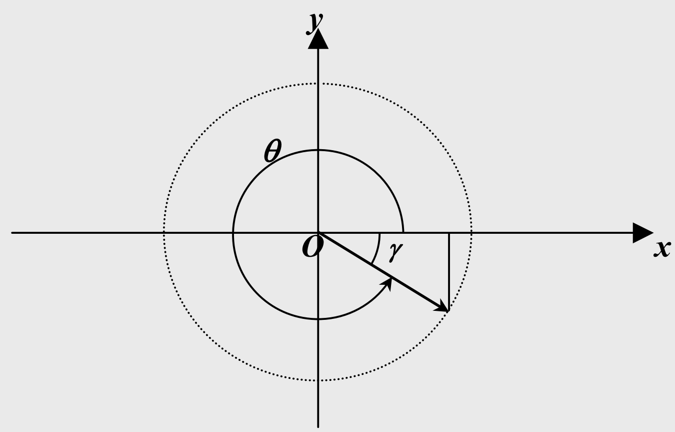
That is, $\mathbf{\cos\frac{3\pi}{2} = 0}$. And next, what about the tangent?

We know the tangent is the opposite over the adjacent.
The adjacent is however, 0 in this case, and thus, causes the tangent to be undefined, because no denominator can be 0. So the tangent cannot be defined in this case.

That is, $\mathbf{tan}\frac{3\pi}{2}$ is not defined. In other words, $\mathbf{tan}\frac{3\pi}{2}$ does not exist.
By the way, as the vector approaches the $\boldsymbol{y}$-axis very closely below the origin, the tangent is said to go to infinity, since the adjacent is nearly 0, but the opposite is getting very close to the |vector|.

Let's now, move on to the case where the vector is in the fourth quadrant as shown in Fig. 7 below.

Fig. 7

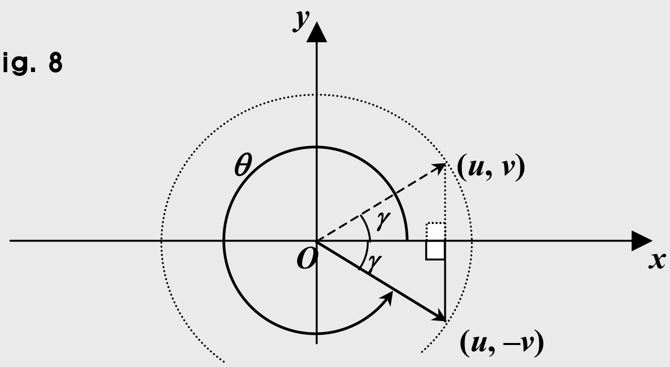


To begin with, the angle we use to get trig ratios is the angle made by the vector turning.

The angle is between $\frac{3\pi}{2}$ and 2π.
This is a case similar to the case where the vector is in the second quadrant. So we can use the similar method we used with the vector in the second quadrant.

So taking trig ratios with the vector in the fourth quadrant, we take the ratios for the ***acute*** angle γ first, and then, apply the negative sign to both the sine and the tangent, but the positive sign to the cosine. It's because only the opposite is negative, since only the $\boldsymbol{y}$-coordinate at the terminal point is negative if the vector is in the fourth quadrant.
Let's now take the trig ratios.

Fig. 8



First, in Fig. 8, assuming the terminal point of the vector in the first quadrant is $(\boldsymbol{u}, \boldsymbol{v})$, we can say that $(\boldsymbol{u}, -\boldsymbol{v})$ is the terminal point of the vector in the fourth quadrant, because the two right triangles with the angle γ are symmetric about the $\boldsymbol{x}$-axis. Why symmetric?

It's because γ is the angle between the adjacent and the hypotenuse in each of the two right triangles, the two triangles share the same adjacent, and each of the two has the same hypotenuse.
So the two triangles themselves are identical. Thus, taking the ratios for the angle γ, we can use them for the *absolute* values of the ratios for the angle θ.

So to begin with, we know the sine is the opposite over the hypotenuse.
Thus, assuming the hypotenuse is 1, we can see that the sine of γ is $\boldsymbol{v}$, for the opposite is $\boldsymbol{v}$, and the hypotenuse is 1. So we get $\sin \gamma = v$.

And $\sin \theta = -v$, since the opposite is $-v$, and the hypotenuse is 1.

Thus, we can see that $\sin \theta = -\sin \gamma$, where $\theta = 2\pi - \gamma$, and $0 \le \gamma \le \frac{\pi}{2}$.

So we can say that if $\theta = 2\pi - \gamma$ and $0 \le \gamma \le \frac{\pi}{2}$, we get $\sin \theta = -\sin \gamma$.

Thus, $\sin (2\pi - \gamma) = -\sin \gamma$, where $0 \le \gamma \le \frac{\pi}{2}$, which can be taken as a trig identity.

What about the cosine?

We know the cosine is the adjacent over the hypotenuse.
So assuming again, the hypotenuse is 1, we can see that $\cos \gamma = u$, because the adjacent is $\boldsymbol{u}$, since the terminal point is $(\boldsymbol{u}, -\boldsymbol{v})$.

And $\cos\theta = u$, also, because the adjacent is u, since the terminal point is $(u, -v)$.

Thus, we have $\cos\theta = u$, where $\theta = 2\pi - \gamma$, and $0 \le \gamma \le \frac{\pi}{2}$,

So we can see that if $\theta = 2\pi - \gamma$ and $0 \le \gamma \le \frac{\pi}{2}$, we get $\cos\theta = \cos\gamma$.

Thus, $\cos(2\pi - \gamma) = \cos\gamma$, where $0 \le \gamma \le \frac{\pi}{2}$, which can be taken as another trig identity.

And next, what about the tangent?

We know the tangent is the opposite over the adjacent.

So $\tan\gamma = \frac{v}{u}$, since the opposite is v, and the adjacent is u.

And $\tan\theta = \frac{-v}{u} = -\frac{v}{u}$, where $\theta = 2\pi - \gamma$, and $0 \le \gamma \le \frac{\pi}{2}$, since the opposite is $-v$, and the adjacent is u.

So we can see that if $\theta = 2\pi - \gamma$ and $0 \le \gamma \le \frac{\pi}{2}$, we get $\tan\theta = -\tan\gamma$.

Thus, $\tan(2\pi - \gamma) = -\tan\gamma$, where $0 \le \gamma \le \frac{\pi}{2}$, which can be taken as a trig identity, too.

$$\sin(2\pi - \gamma) = -\sin\gamma$$

$$\cos(2\pi - \gamma) = \cos\gamma$$

$$\tan(2\pi - \gamma) = -\tan\gamma$$

$$0 \le \gamma \le \frac{\pi}{2}$$

$$\sin\pi = 0,\ \cos\pi = -1,\ \tan\pi = 0$$

$$\sin\frac{3\pi}{2} = -1,\ \cos\frac{3\pi}{2} = 0,\ \tan\frac{3\pi}{2} \text{ does not exist.}$$

In short:

$\sin 0 = 0$, $\cos 0 = 1$, $\tan 0 = 0$, $\sin\frac{\pi}{2} = -1$, $\cos\frac{\pi}{2} = 0$, and $\tan\frac{\pi}{2}$ does not exist.

$\mathbf{\sin(\pi - \alpha) = \sin \alpha}$, $\mathbf{\cos(\pi - \alpha) = -\cos \alpha}$, $\mathbf{\tan(\pi - \alpha) = -\tan \alpha}$, where $\mathbf{0 \le \alpha \le}\frac{\pi}{2}$.

$\sin \pi = 0$, $\cos \pi = -1$, and $\tan \pi = 0$

$\mathbf{\sin(\pi + \beta) = -\sin \beta}$, $\mathbf{\cos(\pi + \beta) = -\cos \beta}$, $\mathbf{\tan(\pi + \beta) = \tan \beta}$, where $\mathbf{0 \le \beta \le}\frac{\pi}{2}$.

$\sin\frac{3\pi}{2} = -1$, $\cos\frac{3\pi}{2} = 0$, and $\tan\frac{3\pi}{2}$ does not exist.

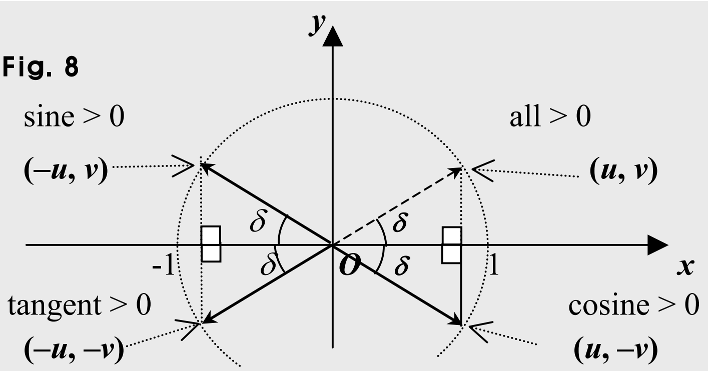
$\mathbf{\sin(2\pi - \gamma) = -\sin \gamma}$, $\mathbf{\cos(2\pi - \gamma) = \cos \gamma}$, $\mathbf{\tan(2\pi - \gamma) = -\tan \gamma}$, where $\mathbf{0 \le \gamma \le}\frac{\pi}{2}$.

In coordinate trigonometry,
the ***x***-coordinate at the terminal point is the adjacent,
the ***y***-coordinate is the opposite,
the |vector| is the hypotenuse,
and the angle is made by the vector turned.

If $(\boldsymbol{x}, \boldsymbol{y})$ is the terminal point, the tangent is $\frac{\boldsymbol{y}}{\boldsymbol{x}}$.

If $(\boldsymbol{x}, \boldsymbol{y})$ is the terminal point, and |vector| is 1, the sine is $\boldsymbol{y}$, and the cosine is $\boldsymbol{x}$.

Fig. 8



All we need is the ratios for acute angles as 30° or $\frac{\pi}{6}$. And then, choose the sign.

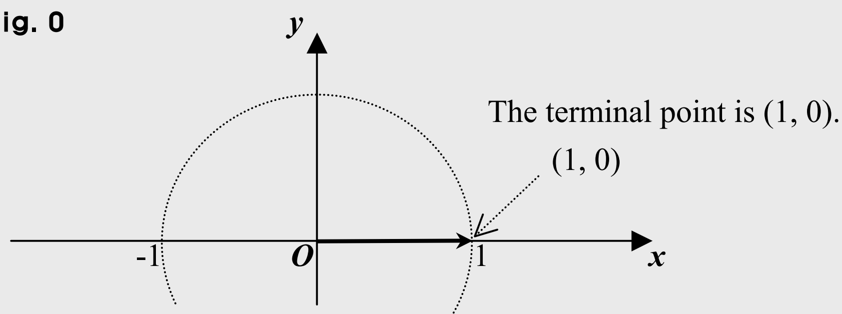
The signs of the coordinates at the terminal point show the sign of the trig ratio.
So if the vector is in the first quadrant, all the three are positive.
In the second, the sine is +; in the third, the tangent is +, and in the fourth, the cosine is +.

2.2. Coordinate Trigonometry 3

What if the angle is greater than 2π? So what if the angle is 5π or $\frac{17\pi}{3}$?

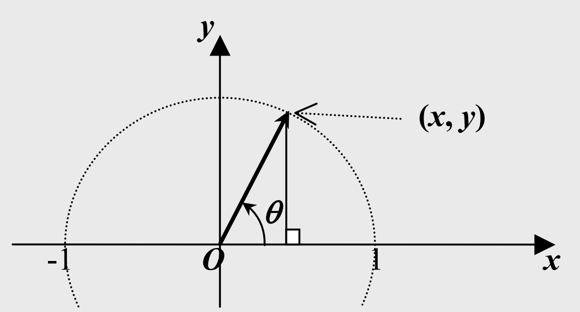
We are now going to cover angles in general.
Let's first go over briefly, the trig ratios when the vector is in each quadrant assuming the |vector| turning is 1.
To begin with, the angle for the ratios is made by the vector turning. Suppose first, as in Fig. 0 below, the vector is now at rest sitting on the x-axis to the right of the origin. Then, the angle made by the vector is 0. And we get $\mathbf{\sin 0 = 0}$, $\mathbf{\cos 0 = 1}$, and $\mathbf{\tan 0 = 0}$.

Fig. 0



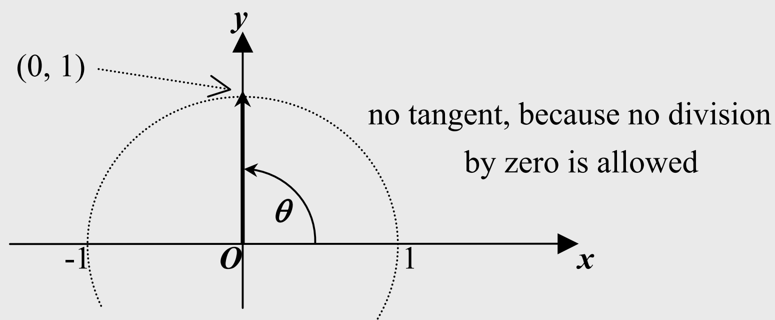
Next, as in Fig. 1 below, when the vector is in the first quadrant, so if $\mathbf{0} < \boldsymbol{\theta} < \frac{\pi}{2}$, and if the terminal point is $(\boldsymbol{x}, \boldsymbol{y})$, we get $\mathbf{\sin}\ \boldsymbol{\theta} = \boldsymbol{y}$, $\mathbf{\cos}\ \boldsymbol{\theta} = \boldsymbol{x}$, and $\mathbf{\tan}\ \boldsymbol{\theta} = \frac{\boldsymbol{y}}{\boldsymbol{x}}$.

Fig. 1



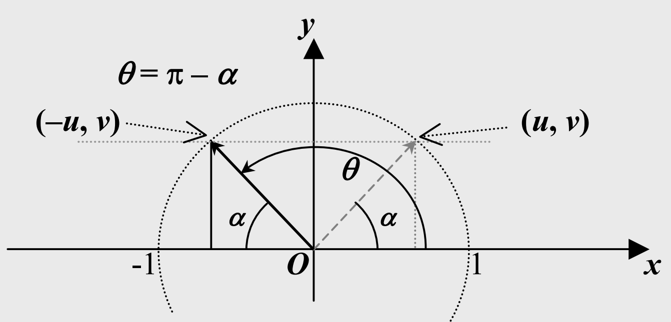
Next, as in Fig. 2 below, when the vector is on the $\boldsymbol{y}$-axis above the origin, the angle made is $\frac{\pi}{2}$, and we get $\mathbf{sin}\,\frac{\pi}{2} = \mathbf{1}$, and $\mathbf{cos}\,\frac{\pi}{2} = \mathbf{0}$, but no $\mathbf{tan}\,\frac{\pi}{2}$, for the adjacent is 0, so $\mathbf{tan}\,\frac{\pi}{2}$ is not defined, since the tangent is the opposite over the adjacent.

Fig. 2



Next, as in Fig. 3 below, when the vector is in the second quadrant, so if $\frac{\pi}{2} < \boldsymbol{\theta} < \pi$, the angle $\boldsymbol{\theta}$ is $\pi - \boldsymbol{\alpha}$, where $\mathbf{0} < \boldsymbol{\alpha} < \frac{\pi}{2}$, and we get $\mathbf{sin}\,(\pi - \boldsymbol{\alpha}) = \mathbf{sin}\,\boldsymbol{\alpha}$, $\mathbf{cos}\,(\pi - \boldsymbol{\alpha}) = -\mathbf{cos}\,\boldsymbol{\alpha}$, and $\mathbf{tan}\,(\pi - \boldsymbol{\alpha}) = -\mathbf{tan}\,\boldsymbol{\alpha}$, where $\mathbf{0} < \boldsymbol{\alpha} < \frac{\pi}{2}$.

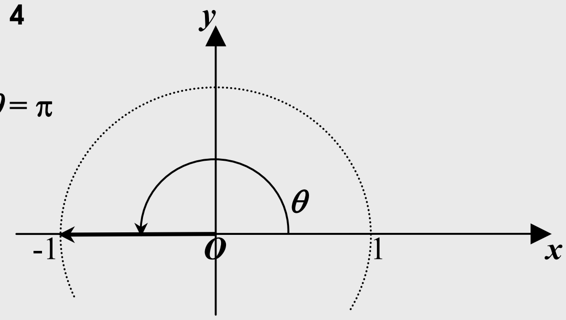
Fig. 3



It's because as in Fig. 3, if $\boldsymbol{\theta} = \pi - \boldsymbol{\alpha}$, we get $\mathbf{sin}\,\boldsymbol{\theta} = \boldsymbol{v}$, $\mathbf{cos}\,\boldsymbol{\theta} = -\boldsymbol{u}$, and $\mathbf{tan}\,\boldsymbol{\theta} = -\frac{\boldsymbol{v}}{\boldsymbol{u}}$, since the terminal point is $(-\boldsymbol{u}, \boldsymbol{v})$. And if the angle is $\boldsymbol{\alpha}$, since the terminal point is $(\boldsymbol{u}, \boldsymbol{v})$, we get $\mathbf{sin}\,\boldsymbol{\alpha} = \boldsymbol{v}$, $\mathbf{cos}\,\boldsymbol{\alpha} = \boldsymbol{u}$, and $\mathbf{tan}\,\boldsymbol{\alpha} = \frac{\boldsymbol{v}}{\boldsymbol{u}}$. So $\mathbf{sin}\,(\pi - \boldsymbol{\alpha}) = \mathbf{sin}\,\boldsymbol{\alpha}$, $\mathbf{cos}\,(\pi - \boldsymbol{\alpha}) = -\mathbf{cos}\,\boldsymbol{\alpha}$, and $\mathbf{tan}\,(\pi - \boldsymbol{\alpha}) = -\mathbf{tan}\,\boldsymbol{\alpha}$, where $\mathbf{0} < \boldsymbol{\alpha} < \frac{\pi}{2}$.

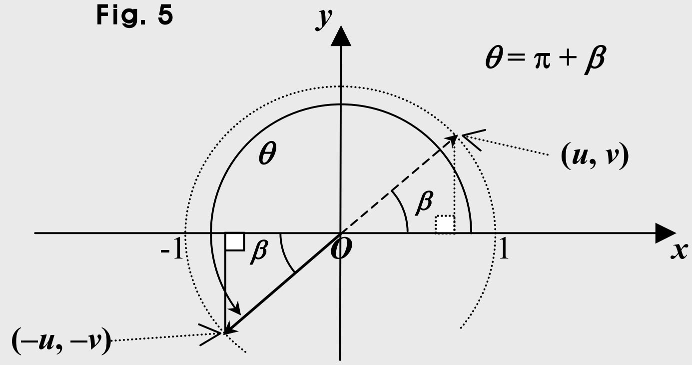
Next, as in Fig. 4 below, when the vector is on the $\boldsymbol{x}$-axis to the left of the origin, we get $\mathbf{sin}\ \pi = \mathbf{0}$, $\mathbf{cos}\ \pi = \mathbf{-1}$, and $\mathbf{tan}\ \pi = \mathbf{0}$, since the terminal point is (–1, 0).

Fig. 4



Next, as in Fig. 5 below, when the vector is in the third quadrant, so if $\pi < \theta < \dfrac{3\pi}{2}$, the angle θ is $\pi + \beta$, where $\mathbf{0} < \beta < \dfrac{\pi}{2}$, and we get $\mathbf{sin}\ (\pi + \beta) = -\mathbf{sin}\ \beta$, $\mathbf{cos}\ (\pi + \beta) = -\mathbf{cos}\ \beta$, and $\mathbf{tan}\ (\pi + \beta) = \mathbf{tan}\ \beta$, where $\mathbf{0} < \beta < \dfrac{\pi}{2}$.

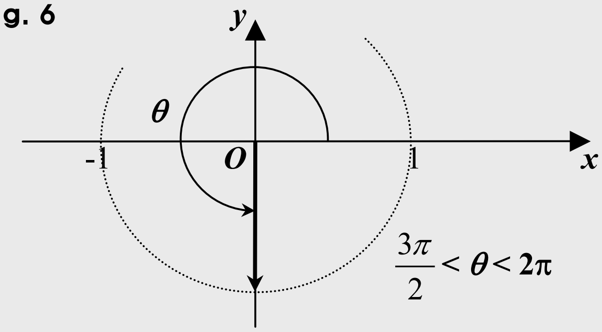
Fig. 5



It's because as in Fig. 5, if $\theta = \pi + \beta$, we get $\mathbf{sin}\ \theta = -\boldsymbol{v}$, $\mathbf{cos}\ \theta = -\boldsymbol{u}$, and $\mathbf{tan}\ \theta = \dfrac{\boldsymbol{v}}{\boldsymbol{u}}$, since the terminal point is $(-\boldsymbol{u}, -\boldsymbol{v})$, and if the angle is β, since the terminal point is $(\boldsymbol{u}, \boldsymbol{v})$, we get $\mathbf{sin}\ \beta = \boldsymbol{v}$, $\mathbf{cos}\ \beta = \boldsymbol{u}$, and $\mathbf{tan}\ \beta = \dfrac{\boldsymbol{v}}{\boldsymbol{u}}$. So $\mathbf{sin}\ (\pi + \beta) = -\mathbf{sin}\ \beta$, $\mathbf{cos}\ (\pi + \beta) = -\mathbf{cos}\ \beta$, and $\mathbf{tan}\ (\pi + \beta) = \mathbf{tan}\ \beta$, where $\mathbf{0} < \beta < \dfrac{\pi}{2}$.

Next, as in Fig. 6 below, when the vector is on the $\boldsymbol{y}$-axis below the origin, the angle is $\frac{3\pi}{2}$, and we get $\mathbf{sin}\,\frac{3\pi}{2} = \mathbf{-1}$, and $\mathbf{cos}\,\frac{3\pi}{2} = \mathbf{0}$, but no $\mathbf{tan}\,\frac{3\pi}{2}$, for the adjacent is 0, so $\mathbf{tan}\,\frac{3\pi}{2}$ is not defined, since the tangent is the opposite over the adjacent. And no $\mathbf{tan}\,\frac{\pi}{2}$ is defined, either, for the same reason. Thus, if the vector is on the $\boldsymbol{y}$-axis, no tangent.

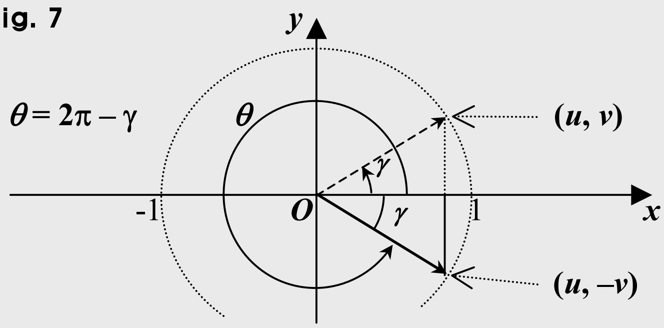
Fig. 6



And next, as in Fig. 7 below, when the vector is in the fourth quadrant, the angle made is $\mathbf{2\pi} - \boldsymbol{\gamma}$, where $\mathbf{0} < \boldsymbol{\gamma} < \frac{\pi}{2}$, and we get $\mathbf{sin\,(2\pi} - \boldsymbol{\gamma}) = \mathbf{-sin}\,\boldsymbol{\beta}$, $\mathbf{cos\,(2\pi} - \boldsymbol{\gamma}) = \mathbf{cos}\,\boldsymbol{\gamma}$,

and $\mathbf{tan\,(2\pi} - \boldsymbol{\gamma}) = \mathbf{-tan}\,\boldsymbol{\gamma}$, where $\mathbf{0} < \boldsymbol{\gamma} < \frac{\pi}{2}$.

Fig. 7



It's because as in Fig. 7, if $\boldsymbol{\theta} = \mathbf{2\pi} - \boldsymbol{\gamma}$, and $\mathbf{0} < \boldsymbol{\gamma} < \frac{\pi}{2}$, we get $\mathbf{sin}\,\boldsymbol{\theta} = -\boldsymbol{v}$, $\mathbf{cos}\,\boldsymbol{\theta} = \boldsymbol{u}$, and $\mathbf{tan}\,\boldsymbol{\theta} = -\frac{\boldsymbol{v}}{\boldsymbol{u}}$, since the terminal point is $(\boldsymbol{u}, -\boldsymbol{v})$, and if the angle is $\boldsymbol{\gamma}$, since the terminal point is $(\boldsymbol{u}, \boldsymbol{v})$, we get $\mathbf{sin}\,\boldsymbol{\gamma} = \boldsymbol{v}$, $\mathbf{cos}\,\boldsymbol{\gamma} = \boldsymbol{u}$, and $\mathbf{tan}\,\boldsymbol{\gamma} = \frac{\boldsymbol{v}}{\boldsymbol{u}}$. So we get $\mathbf{sin\,(2\pi} - \boldsymbol{\gamma}) = \mathbf{-sin}\,\boldsymbol{\gamma}$, $\mathbf{cos\,(2\pi} - \boldsymbol{\gamma}) = \mathbf{cos}\,\boldsymbol{\gamma}$, and $\mathbf{tan\,(2\pi} - \boldsymbol{\gamma}) = \mathbf{-tan}\,\boldsymbol{\gamma}$, where $\mathbf{0} < \boldsymbol{\gamma} < \frac{\pi}{2}$.

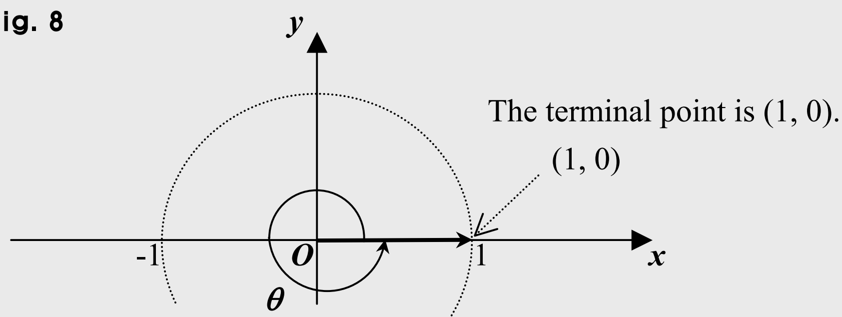
So we have covered eight cases where the angle $\boldsymbol{\theta}$ can be made by the vector turning. Suppose now the vector is placed back on the ***x***-axis. What then is the angle $\boldsymbol{\theta}$?

The angle $\boldsymbol{\theta}$ is 2π, because the vector has just made a complete turn. What will happen then to the trig ratios if the vector keeps turning counterclockwise?

The eight cases described above will repeat themselves in the same sequence.
So going back to the first case, we can say that if the |vector| is 1, and is at rest sitting on the ***x***-axis on the right of the origin, the terminal point is (1, 0), and the angle made is 0.
Thus, when $\boldsymbol{\theta = 0}$, we get $\mathbf{\sin 0 = 0}$, $\mathbf{\cos 0 = 1}$, and $\mathbf{\tan 0 = 0}$.

After a complete turn, the total angle made is $\mathbf{2\pi}$, the vector is again on the ***x***-axis on the right of the origin, and the terminal point is (1, 0), again.
Thus, when $\boldsymbol{\theta} = \mathbf{2\pi}$, we get $\mathbf{\sin 2\pi = 0}$, $\mathbf{\cos 2\pi = 1}$, and $\mathbf{\tan 2\pi = 0}$.

Fig. 8



After another complete turn, the total angle made is $\mathbf{4\pi}$, the vector is again on the ***x***-axis on the right of the origin, and the terminal point is (1, 0), again.
Thus, we get $\mathbf{\sin 4\pi = 0}$, $\mathbf{\cos 4\pi = 1}$, and $\mathbf{\tan 4\pi = 0}$.

Every complete turn, the vector makes 2π more. So after ***n*** turns, $\boldsymbol{\theta} = \mathbf{2}\boldsymbol{n}\boldsymbol{\pi}$, $\boldsymbol{n} = 0, 1, 2\ldots$, and the vector is in the same place. So the terminal point is (1, 0) again, and we get
$\mathbf{\sin 2}\boldsymbol{n}\mathbf{\pi = 0}$, $\mathbf{\cos 2}\boldsymbol{n}\mathbf{\pi = 1}$, and $\mathbf{\tan 2}\boldsymbol{n}\mathbf{\pi = 0}$ for $\boldsymbol{n} \geq \mathbf{0}$, and ***n*** is an integer.

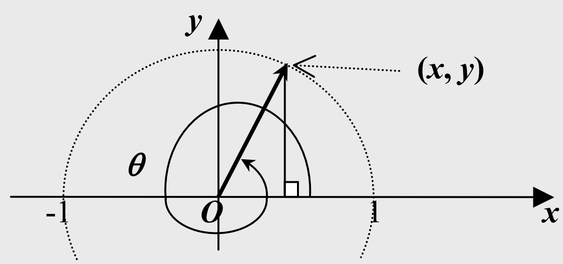
Let's next, move on to the second case.

Then, we can say that if the vector is in the first quadrant, the terminal point is ($\boldsymbol{x}$, $\boldsymbol{y}$), and the angle made is θ, we get $\mathbf{sin}\ \boldsymbol{\theta} = \boldsymbol{y}$, $\mathbf{cos}\ \boldsymbol{\theta} = \boldsymbol{x}$, and $\mathbf{tan}\ \boldsymbol{\theta} = \frac{\boldsymbol{y}}{\boldsymbol{x}}$, where $\mathbf{0} < \boldsymbol{\theta} < \frac{\pi}{2}$.

Next, after the vector turns 2π more, the total angle made by the vector is $\mathbf{2\pi} + \boldsymbol{\alpha}$, where $\mathbf{0} < \boldsymbol{\alpha} < \frac{\pi}{2}$, and as in Fig. 9, the vector is back to the same location, so the terminal point is again ($\boldsymbol{x}$, $\boldsymbol{y}$). Thus, when $\boldsymbol{\theta} = \mathbf{2\pi} + \boldsymbol{\alpha}$, where $\mathbf{0} < \boldsymbol{\alpha} < \frac{\pi}{2}$, we get again

$\mathbf{sin}\ \boldsymbol{\theta} = \boldsymbol{y}$, $\mathbf{cos}\ \boldsymbol{\theta} = \boldsymbol{x}$, and $\mathbf{tan}\ \boldsymbol{\theta} = \frac{\boldsymbol{y}}{\boldsymbol{x}}$.

Fig. 9



And after another complete turn, the total angle made is $\mathbf{4\pi} + \boldsymbol{\alpha}$, where $\mathbf{0} < \boldsymbol{\alpha} < \frac{\pi}{2}$, the vector is back in the same position, and the terminal point is again **($\boldsymbol{x}$, $\boldsymbol{y}$)**.
Thus, when $\boldsymbol{\theta} = \mathbf{4\pi} + \boldsymbol{\alpha}$, we get also, $\mathbf{sin}\ \boldsymbol{\theta} = \boldsymbol{y}$, $\mathbf{cos}\ \boldsymbol{\theta} = \boldsymbol{x}$, and $\mathbf{tan}\ \boldsymbol{\theta} = \frac{\boldsymbol{y}}{\boldsymbol{x}}$.

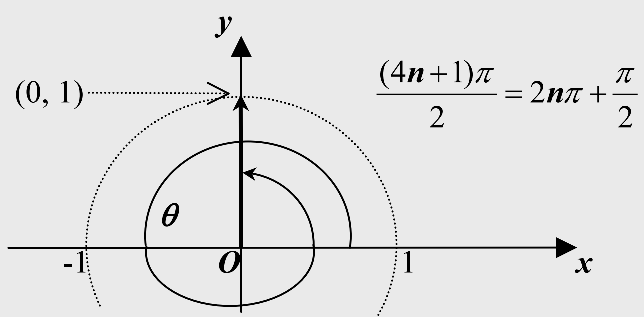
Every complete turn, the vector adds 2π. So after $\boldsymbol{n}$ turns, $\boldsymbol{\theta} = \mathbf{2}\boldsymbol{n}\pi + \boldsymbol{\alpha}$, $\boldsymbol{n} = 0, 1, 2\ldots$, and the vector is in the same place. So the terminal point is again **($\boldsymbol{x}$, $\boldsymbol{y}$)**, and we get

$\mathbf{sin}\ (\mathbf{2}\boldsymbol{n}\pi + \boldsymbol{\alpha}) = \boldsymbol{y}$, $\mathbf{cos}\ (\mathbf{2}\boldsymbol{n}\pi + \boldsymbol{\alpha}) = \boldsymbol{x}$, and $\mathbf{tan}\ (\mathbf{2}\boldsymbol{n}\pi + \boldsymbol{\alpha}) = \frac{\boldsymbol{y}}{\boldsymbol{x}}$, where $\mathbf{0} < \boldsymbol{\alpha} < \frac{\pi}{2}$ for $\boldsymbol{n} \geq \mathbf{0}$, where $\boldsymbol{n}$ is an integer.

Next, moving on to the third case, we can say that if the vector is on the ***y***-axis above the origin, then the angle made is $\frac{\pi}{2}$, and the terminal point is (0, 1). So we get $\mathbf{sin}\frac{\pi}{2} = \mathbf{1}$, and $\mathbf{cos}\frac{\pi}{2} = \mathbf{0}$, but no $\mathbf{tan}\frac{\pi}{2}$, since the adjacent is 0, and the tangent is the opposite over the adjacent. No division by 0 is allowed, so $\mathbf{tan}\frac{\pi}{2}$ is not defined.

And after the vector makes 2π more, the total angle made is $\mathbf{2\pi} + \frac{\pi}{2}$, and as in Fig. 10, the vector is back in the same position. So the terminal point is (0, 1) again, and thus, when the angle $\boldsymbol{\theta} = \mathbf{2\pi} + \frac{\pi}{2}$, we get again $\mathbf{sin}\ \boldsymbol{\theta} = \mathbf{1}$, $\mathbf{cos}\ \boldsymbol{\theta} = \mathbf{0}$, but no $\mathbf{tan}\ \boldsymbol{\theta}$.

Fig. 10



Also, after another complete turn, $\boldsymbol{\theta}$ is $\mathbf{4\pi} + \frac{\pi}{2}$, and the vector is in the same location. So the terminal point is (0, 1) again. Thus, if $\boldsymbol{\theta} = \mathbf{4\pi} + \frac{\pi}{2}$, then again $\mathbf{sin}\ \boldsymbol{\theta} = \mathbf{1}$, and $\mathbf{cos}\ \boldsymbol{\theta} = \mathbf{0}$, but no $\mathbf{tan}\ \boldsymbol{\theta}$.

Every complete turn, the vector adds 2π. So after ***n*** turns, $\boldsymbol{\theta} = \mathbf{2}\boldsymbol{n}\mathbf{\pi} + \frac{\pi}{2}$, $\boldsymbol{n} = 0, 1, 2\ldots$, and the vector is in the same place. Thus, the terminal point is (0, 1) again.

So for any integer $\boldsymbol{n} \geq \mathbf{0}$, when $\boldsymbol{\theta} = \mathbf{2}\boldsymbol{n}\mathbf{\pi} + \frac{\pi}{2}$, we get $\mathbf{sin}\ \boldsymbol{\theta} = \mathbf{1}$, $\mathbf{cos}\ \boldsymbol{\theta} = \mathbf{0}$, but no $\mathbf{tan}\ \boldsymbol{\theta}$.

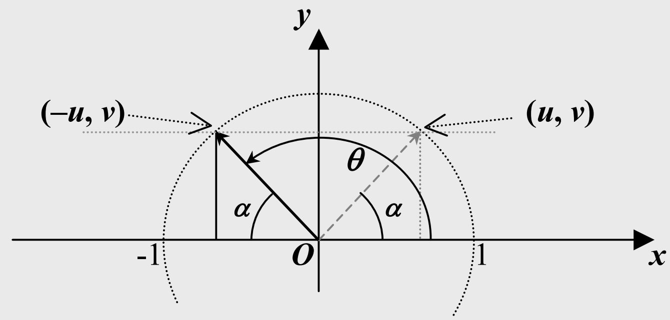
In other words, we get

$\mathbf{sin}\frac{(4\boldsymbol{n}+1)\pi}{2} = \mathbf{1}$, $\mathbf{cos}\frac{(4\boldsymbol{n}+1)\pi}{2} = \mathbf{0}$, and no $\mathbf{tan}\frac{(4\boldsymbol{n}+1)\pi}{2}$, where ***n*** is an integer $\geq \mathbf{0}$.

Next, let's move on to the fourth case.

Then, as in Fig. 11, if the vector is in the second quadrant, then the angle θ is $\pi - \alpha$ for $\mathbf{0 < \alpha < \frac{\pi}{2}}$. and $\mathbf{\sin(\pi - \alpha) = \sin\alpha}$, $\mathbf{\cos(\pi - \alpha) = -\cos\alpha}$, and $\mathbf{\tan(\pi - \alpha) = -\tan\alpha}$.

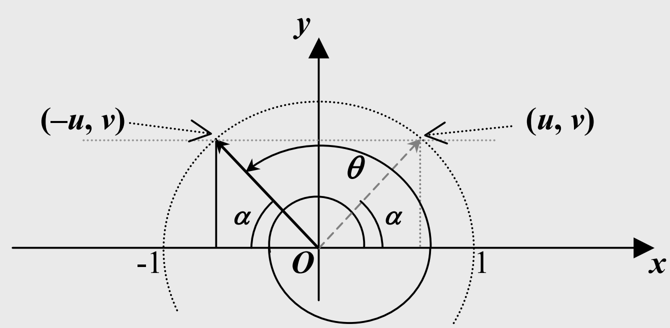
Fig. 11



After the vector makes 2π more, the total angle is $\mathbf{2\pi + \pi - \alpha}$, and as in Fig. 12, the vector is back in the same position. So when $\theta = \mathbf{2\pi + \pi - \alpha}$, we get also, for $\mathbf{0 < \alpha < \frac{\pi}{2}}$,

$\mathbf{\sin(2\pi + \pi - \alpha) = \sin\alpha}$, $\mathbf{\cos(2\pi + \pi - \alpha) = -\cos\alpha}$, and $\mathbf{\tan(2\pi + \pi - \alpha) = -\tan\alpha}$,.

Fig. 12



And after another complete turn, that is, when $\theta = \mathbf{4\pi + \pi - \alpha}$, we get also,
$\mathbf{\sin(4\pi + \pi - \alpha) = \sin\alpha}$, $\mathbf{\cos(4\pi + \pi - \alpha) = -\cos\alpha}$, and $\mathbf{\tan(4\pi + \pi - \alpha) = -\tan\alpha}$,

where $\mathbf{0 < \alpha < \frac{\pi}{2}}$. Every complete turn, the vector adds 2π. So after n turns,

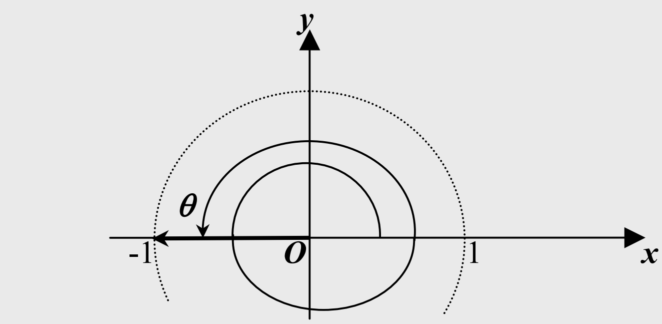
$\theta = \mathbf{2n\pi + \pi - \alpha}$, $n = 0, 1, 2\ldots$, and the vector is in the same place.
And we can have $\theta = \mathbf{2n\pi + \pi - \alpha = (2n + 1)\pi - \alpha}$, where n is an integer $\geq \mathbf{0}$.

So when $\theta = \mathbf{(2n + 1)\pi - \alpha}$, we get $\mathbf{\sin((2n + 1)\pi - \alpha) = \sin\alpha}$,

$\mathbf{\cos((2n + 1)\pi - \alpha) = -\cos\alpha}$, and $\mathbf{\tan((2n + 1)\pi - \alpha) = -\tan\alpha}$, where $\mathbf{0 < \alpha < \frac{\pi}{2}}$.

Next, we have **sin** π **= 0**, **cos** π **= –1**, and **tan** π **= 0**. After a complete turn, 2π is added, and the vector is in the same place, so we get **sin 3**π **= 0**, **cos 3**π **= –1**, and **tan 3**π **= 0**.

Fig. 13

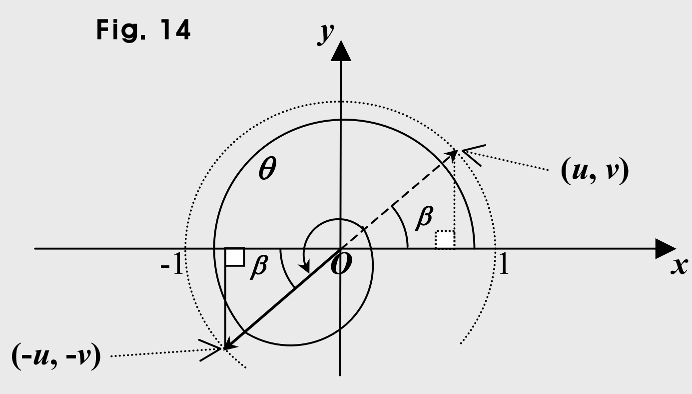


After another complete turn, the angle $\theta = 2\pi + 3\pi = 5\pi$, and the vector is in the same position, so we get **sin** $\theta = 0$, **cos** $\theta = -1$, and **tan** $\theta = 0$.
Every complete turn, the vector adds 2π. So after n turns, $\theta = (2n + 1)\pi$, $n = 0, 1, 2\ldots$
And the vector is in the same place. So we get
sin $(2n + 1)\pi = 0$, **cos** $(2n + 1)\pi = -1$, and **tan** $(2n + 1)\pi = 0$, where n is an integer ≥ 0,

Suppose now, the vector is in the third quadrant. Then, since the angle is $\pi + \beta$, where $0 < \beta < \frac{\pi}{2}$, we get $\sin(\pi + \beta) = -\sin\beta$, $\cos(\pi + \beta) = -\cos\beta$, and $\tan(\pi + \beta) = \tan\beta$.

After a complete turn, the total angle made is $2\pi + \pi + \beta = 3\pi + \beta$, and as in Fig. 14, the vector is in the same place, so we get $\sin(3\pi + \beta) = -\sin\beta$, $\cos(3\pi + \beta) = -\cos\beta$, and $\tan(3\pi + \beta) = \tan\beta$.

Fig. 14



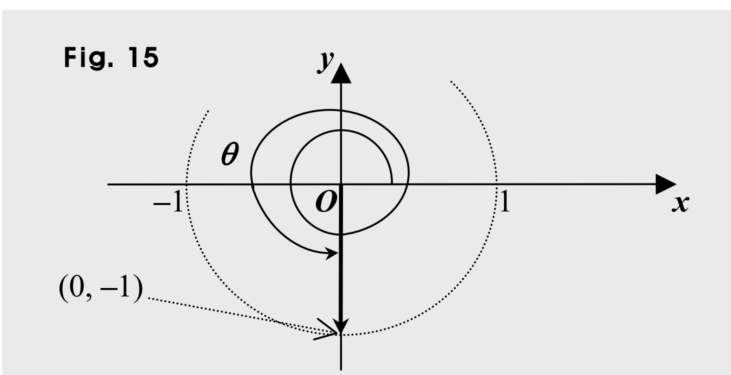
After another complete turn, the total angle is **2π + 3π + *β* = 5π + *β***, and the vector is in the same place, so we get
sin (5π + *β*) = –sin *β*, **cos (5π + *β*) = –cos *β***, and **tan (5π + *β*) = tan *β***.

Every complete turn, the vector adds 2π. So after ***n*** turns, ***θ* = 2*n*π** + π + ***β***, ***n*** = 0, 1, 2…, and the vector is in the same place.

And we can have ***θ* = 2*n*π** + π + ***β* = (2*n* + 1)π + *β***, where ***n*** is an integer ≥ **0**.

So we get **sin ((2*n* + 1)π + *β*) = –sin *β***, **cos ((2*n* + 1)π + *β*) = –cos *β***,
and **tan ((2*n* + 1)π + *β*) = tan *β***, where ***n*** is an integer ≥ **0**.

Next, when $\boldsymbol{\theta} = \frac{3\pi}{2}$, we get $\mathbf{sin}\,\frac{3\pi}{2} = \mathbf{-1}$, and $\mathbf{cos}\,\frac{3\pi}{2} = \mathbf{0}$, but no $\mathbf{tan}\,\frac{3\pi}{2}$.
As in Fig. 15, after a complete turn, the total angle made is $\mathbf{2\pi} + \frac{3\pi}{2}$, and the vector is in the same place, so if $\boldsymbol{\theta} = \mathbf{2\pi} + \frac{3\pi}{2}$, we get **sin *θ* = –1**, and **cos *θ* = 0**, but no **tan *θ***.



After another complete turn, the total angle made is $\mathbf{4\pi} + \frac{3\pi}{2}$, and the vector is in the same place, so if $\boldsymbol{\theta} = \mathbf{4\pi} + \frac{3\pi}{2}$, we get **sin *θ* = –1**, and **cos *θ* = 0**, but no **tan *θ***.

Every complete turn, the vector adds 2π. So after n turns, $\theta = 2n\pi + \frac{3\pi}{2}$, n = 0, 1, 2…, and the vector is in the same place.

And we can have $\theta = 2n\pi + \frac{3\pi}{2} = 2n\pi + \pi + \frac{\pi}{2} = (2n + 1)\pi + \frac{\pi}{2}$.

So we get $\sin(2n\pi + \frac{3\pi}{2}) = -1$, and $\cos(2n\pi + \frac{3\pi}{2}) = 0$, but no $\tan(2n\pi + \frac{3\pi}{2})$, where n is an integer ≥ 0,

In other words, we get $\sin((2n + 1)\pi + \frac{\pi}{2}) = -1$, and $\cos((2n + 1)\pi + \frac{\pi}{2}) = 0$,

but no $\tan((2n + 1)\pi + \frac{\pi}{2})$, where n is an integer ≥ 0.

And we can put it this way, too: $\sin(k\pi + \frac{\pi}{2}) = -1$, and $\cos(k\pi + \frac{\pi}{2}) = 0$,

but no $\tan(k\pi + \frac{\pi}{2})$, where k is an odd integer ≥ 1, so k = 1, 3, 5, …

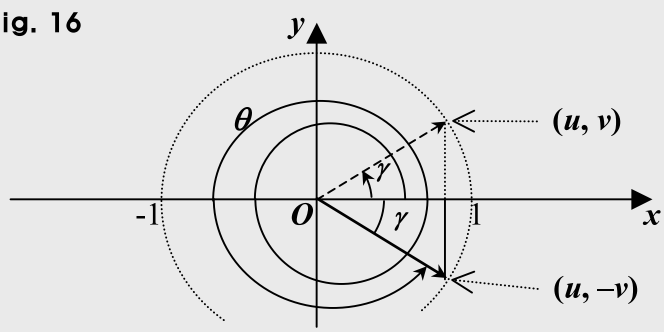
When the vector is on the y-axis, the tangent is not defined, because the adjacent is 0. So no tangent.

And next, when the vector is in the fourth quadrant, the angle is $2\pi - \gamma$, where $0 < \gamma < \frac{\pi}{2}$, so we get $\sin(2\pi - \gamma) = -\sin\beta$, $\cos(2\pi - \gamma) = \cos\gamma$, and $\tan(2\pi - \gamma) = -\tan\gamma$.

After a complete turn, the total angle is $2\pi + 2\pi - \gamma = 4\pi - \gamma$, and the vector is in the same place as in Fig. 16. So we get

$\sin(4\pi - \gamma) = -\sin\beta$, $\cos(4\pi - \gamma) = \cos\gamma$, and $\tan(4\pi - \gamma) = -\tan\gamma$, where $0 < \gamma < \frac{\pi}{2}$.

Fig. 16



After another complete turn, the total angle is $\mathbf{2\pi + 4\pi - \gamma = 6\pi - \gamma}$., and the vector is in the same place. So we get

$\mathbf{\sin (6\pi - \gamma) = -\sin \beta}$, $\mathbf{\cos (6\pi - \gamma) = \cos \gamma}$, and $\mathbf{\tan (6\pi - \gamma) = -\tan \gamma}$, where $\mathbf{0} < \gamma < \frac{\pi}{2}$.

Every complete turn, the vector adds 2π. So after $\boldsymbol{n}$ turns, $\boldsymbol{\theta} = \mathbf{2n\pi + 2\pi - \gamma}$, $\boldsymbol{n} = 0, 1, 2\ldots$, and the vector is in the same place.

And we can have $\boldsymbol{\theta} = \mathbf{2n\pi + 2\pi - \gamma = 2(n + 1)\pi - \gamma}$. So we get

$\mathbf{\sin (2(n + 1)\pi - \gamma) = -\sin \beta}$, $\mathbf{\cos (2(n + 1)\pi - \gamma) = \cos \gamma}$, and $\mathbf{\tan (2(n + 1)\pi - \gamma) = -\tan \gamma}$, where $\mathbf{0} < \gamma < \frac{\pi}{2}$.

And assuming $\boldsymbol{k}$ is an integer $\geq \mathbf{1}$,we can put it the way below, too.

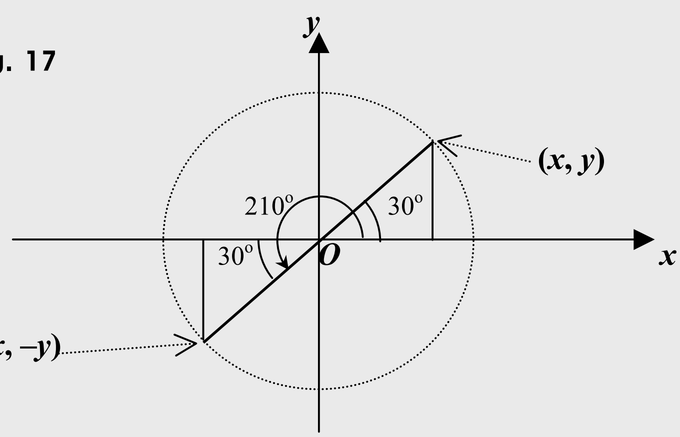
$\mathbf{\sin (2k\pi - \gamma) = -\sin \beta}$, $\mathbf{\cos (2k\pi - \gamma) = \cos \gamma}$, and $\mathbf{\tan (2k\pi - \gamma) = -\tan \gamma}$, where $\mathbf{0} < \gamma < \frac{\pi}{2}$.

How than do we use such facts as above?

Suppose for instance, we want to get the value of $\mathbf{\sin 210^o}$.

We know $\mathbf{\sin 30^o} = \frac{1}{2}$, and 210 = 180 + 30. So quickly drawing a graph, we can get this:

Fig. 17



Now, assuming the |vector| = 1, $\sin 30^o = \boldsymbol{y}$, and $\sin 210^o = -\boldsymbol{y}$.

So we get $\sin 210^o = -\sin 30^o = -\frac{1}{2}$

Now, comparing some of the eight cases above, we can simplify those.

To begin with, we have $\mathbf{sin\ 0 = 0}$, $\mathbf{cos\ 0 = 1}$, and $\mathbf{tan\ 0 = 0}$.

And we have $\mathbf{sin}\ \pi = \mathbf{0}$, $\mathbf{cos}\ \pi = \mathbf{-1}$, and $\mathbf{tan}\ \pi = \mathbf{0}$.

So assuming $\boldsymbol{n}$ is an integer $\geq \mathbf{0}$, we get $\mathbf{sin}\ \boldsymbol{n}\pi = \mathbf{0}$, $\mathbf{cos}\ \boldsymbol{n}\pi = (\mathbf{-1})^{\boldsymbol{n}}$, and $\mathbf{tan}\ \boldsymbol{n}\pi = \mathbf{0}$.

Next, we have $\mathbf{sin}\frac{\pi}{2} = \mathbf{1}$, and $\mathbf{cos}\frac{\pi}{2} = \mathbf{0}$, but no $\mathbf{tan}\frac{\pi}{2}$.

And we have $\mathbf{sin}\frac{3\pi}{2} = \mathbf{-1}$, and $\mathbf{cos}\frac{3\pi}{2} = \mathbf{0}$, but no $\mathbf{tan}\frac{3\pi}{2}$.

So assuming $\boldsymbol{n}$ is an integer $\geq \mathbf{0}$, we get

$\mathbf{sin\ (2\boldsymbol{n} + 1)}\frac{\pi}{2} = (\mathbf{-1})^{\boldsymbol{n}}$, $\mathbf{cos\ (2\boldsymbol{n} + 1)}\frac{\pi}{2} = \mathbf{0}$, but no $\mathbf{tan\ (2\boldsymbol{n} + 1)}\frac{\pi}{2}$.

What if the angles are negative?

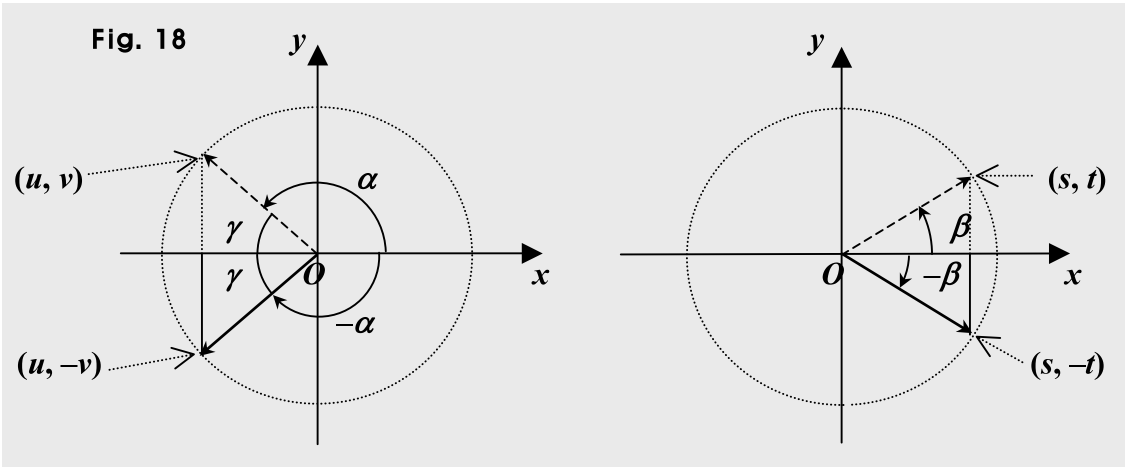
If the vector turns clockwise, the angle made is negative. And we get the trig ratios for the negative angles the way we get the trig ratios for positive angles.

In coordinate trigonometry,
the $\boldsymbol{x}$-coordinate at the terminal point is the adjacent,
the $\boldsymbol{y}$-coordinate is the opposite,
the |vector| is the hypotenuse,
and the angle is made by the vector turned.

We can use identities regarding signs of angles. And the identities are as follows.
$\mathbf{sin}\ (-\theta) = -\mathbf{sin}\ \theta$, $\mathbf{cos}\ (-\theta) = \mathbf{cos}\ \theta$, and $\mathbf{tan}\ (-\theta) = -\mathbf{tan}\ \theta$.

Let's see now how it is the case and some examples.

Fig. 18



Assuming the hypotenuse is 1 in Fig. 18 above, we get

$\sin \alpha = v$, $\sin(-\alpha) = -v$, $\cos \alpha = u$, $\cos(-\alpha) = u$, $\tan \alpha = \frac{v}{u}$, and $\tan(-\alpha) = -\frac{v}{u}$.

$\sin \beta = t$, $\sin(-\beta) = -t$, $\cos \beta = s$, $\cos(-\beta) = s$, $\tan \beta = \frac{t}{s}$, and $\tan(-\beta) = -\frac{t}{s}$.

Thus, we get $\sin(-\theta) = -\sin \theta$, $\cos(-\theta) = \cos \theta$, and $\tan(-\theta) = \tan -\theta$.

For instance, $\sin -30^\circ = -\sin 30^\circ = -\frac{1}{2}$, $\cos -330^\circ = \cos 330^\circ = \cos 30^\circ = \frac{\sqrt{3}}{2}$, and

$\tan -390^\circ = -\tan 390^\circ = -\tan(360^\circ + 30^\circ) = -\tan 30^\circ = -\frac{\sqrt{3}}{3}$

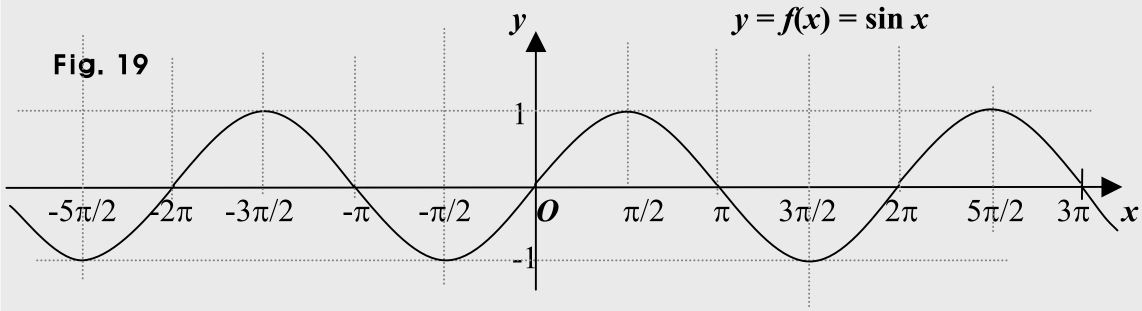
Now, what's more important is that working in coordinate trigonometry, we can come up with another set of tools, called *trigonometric functions*, called *trig functions* for short.

If not quite sure of functions, you may want to refer to the book **Basic Functions**.

A trig function is a function of an angle, and thus, takes an angle as an input.
So for instance, assuming f is such a function, and θ is the input variable,
we can set $f(\theta) = \sin \theta$.

And of course, we can use as the variable any of other letters as $\boldsymbol{x}$ and $\boldsymbol{t}$. So for instance, we can set $\boldsymbol{f(x) = \sin x}$, where $\boldsymbol{x}$ is an angle. And we can put in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane, the curve of the sine function $\boldsymbol{y = f(x) = \sin x}$ the way as follows.

Fig. 19



What angle then do we use?

It's an angle in radians, and is the amount of angle made by the vector turning about the origin in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane.
In short, in $\boldsymbol{f(x) = \sin x}$, $\boldsymbol{x}$ is an angle made by the vector turning.
What then about outputs?

Each output is a trig ratio, so $\boldsymbol{x}$ is an angle, and $\boldsymbol{\sin x}$ is a trig ratio.
So all the values of $\boldsymbol{x}$ are angles, and all the values of $\boldsymbol{f}$ are trig ratios.

Usually, we use real numbers as values of $\boldsymbol{x}$. That's because we use as inputs, angle in radians. And using angles in radians, we just use real numbers as angles. So for instance, 0^{o} is 0 radian, or 0 rad for short. Usually though, we omit rad, so we just use 0.
180^{o} is π rad, where π is the circular ratio, which is 3.141592… And we often omit rad.
So 360^{o} is 2π.

And the vector can make as many turns as necessary either counterclockwise or clockwise.

So we can put it this way: $\boldsymbol{-360^{o}\cdot n \le \theta \le 360^{o}\cdot n \Leftrightarrow -2n\pi \le \theta \le 2n\pi}$,
where $\boldsymbol{n}$ is an integer $\ge \boldsymbol{0}$.

That is, $-\infty < \boldsymbol{\theta} < \infty$, where ∞ indicates infinity. So $\boldsymbol{\theta}$ can be all real numbers.

And in trig functions, we have three basic ones.
One is a sine function, often written as **sin** $\boldsymbol{x}$. Another is a cosine function, often written as **cos** $\boldsymbol{x}$. And the other is a tangent function, often written as **tan** $\boldsymbol{x}$.
And assuming the domain is a set of all angles, and $\boldsymbol{f}$ is a sine function, we can put the trig function $\boldsymbol{f}$ the way as follows: $\boldsymbol{y = f(x) = \sin x}$ for $\boldsymbol{x}$ real.
What do we mean by '$\boldsymbol{x}$ real'?

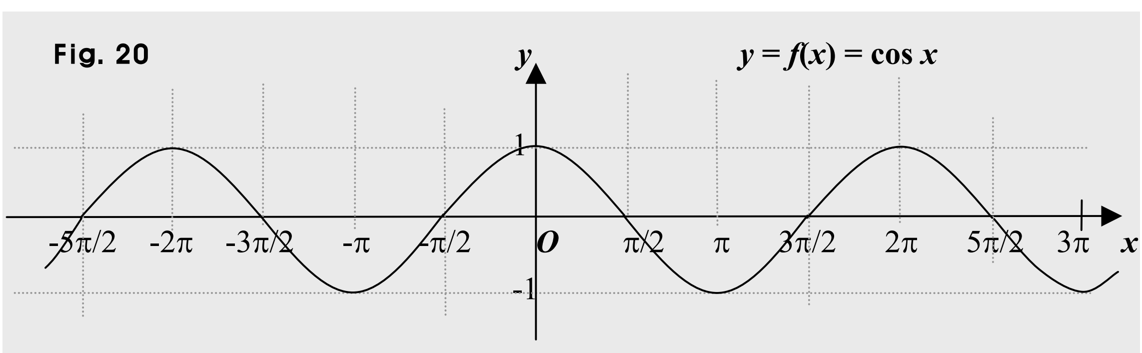
It means $\boldsymbol{x}$ takes all real numbers, i.e., any real number.
And putting angles in radian, we just use real numbers. For instance

$1^\circ = \dfrac{\pi}{180}$, $30^\circ = \dfrac{\pi}{6}$, $45^\circ = \dfrac{\pi}{4}$, $60^\circ = \dfrac{\pi}{3}$, $90^\circ = \dfrac{\pi}{2}$, $180^\circ = \pi$, etc. where $\pi = 3.141\ldots$

So '$\boldsymbol{x}$ real' means that $\boldsymbol{x}$ can be any real number meaning an angle.
Thus, the domain of $\boldsymbol{f}$ is a set of all real numbers, that is, all angles.
What then about the range?

The range can be a set of all numbers from -1 to 1, and each number is a trig ratio.
So the range of $\boldsymbol{f}$ can be put this way, too: $\boldsymbol{-1 \le y \le 1}$, or $\boldsymbol{|y| \le 1}$.

And the same is true for a cosine function, too.
So for instance, assuming the domain is a set of all angles, and $\boldsymbol{g}$ is a cosine function, we can put the trig function $\boldsymbol{g}$ the way as follows: $\boldsymbol{y = g(x) = \cos x}$ for $\boldsymbol{x}$ real.

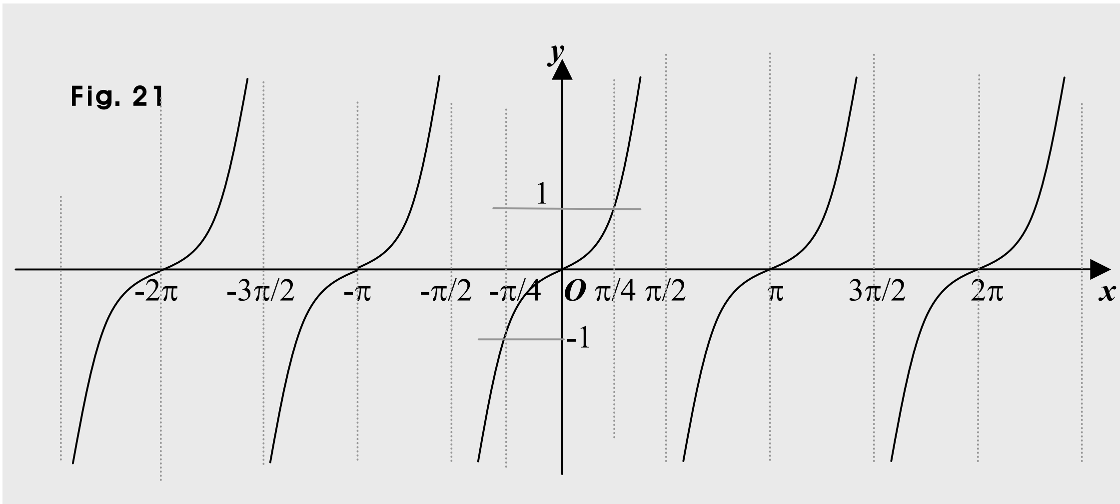


What then, about the range?

The range is a set of all numbers from -1 to 1, and all the number are trig ratios.
So the range of $\boldsymbol{g}$ can be put this way, too: $\boldsymbol{-1 \le y \le 1}$, or $\boldsymbol{|y| \le 1}$.
What then about a tangent function?

For instance, assuming the domain is a set of all angles, and ***h*** is a tangent function, we can put the trig function ***h*** the way as follows: ***y* = *h*(*x*) = tan *x*** for ***x*** real.

Fig. 21



What then about the range?

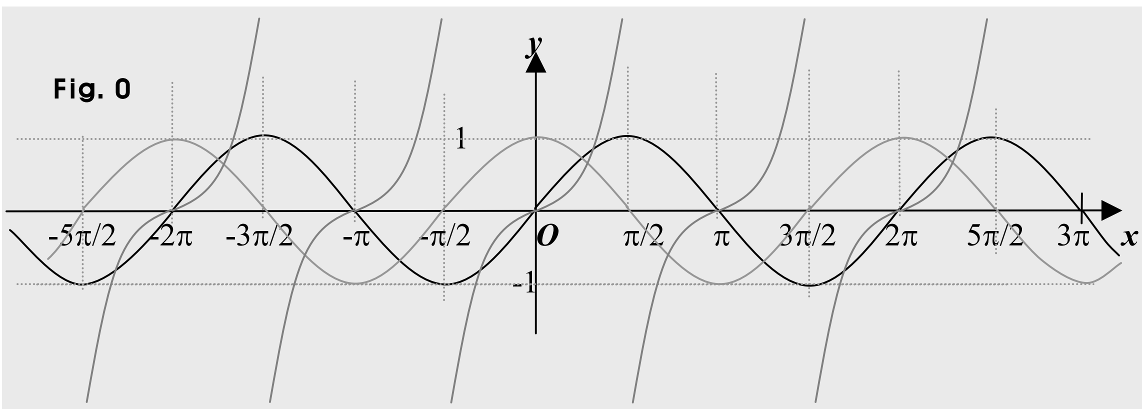
Unlike the sine and cosine functions above, the range of the tangent function ***h*** can be a set of all real numbers, each of which is a trig ratio, of course.

3. Trigonometric Functions

Note that this section covers the overview.

$$y = f(x) = \sin x \qquad y = g(x) = \cos x \qquad y = h(x) = \tan x$$

Fig. 0



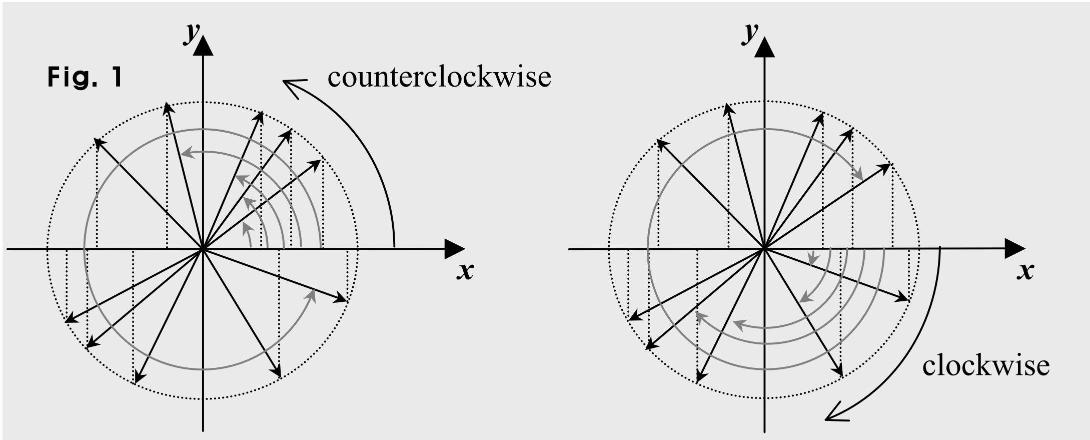
So what is a trigonometric function?

In coordinate trigonometry, we can come up with a set of tools, called trigonometric functions, often just called ***trig functions***. And a trig function is a function of an angle. Thus, it takes an angle as an input.

So for instance, in a trig function $\boldsymbol{y = f(x) = \sin x}$, the input variable $\boldsymbol{x}$ takes an angle, which can be made by the vector turning about the origin in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane.
And working with a trig function, we use a real number as an angle, and can use all real numbers as angles. Thus, the angles can be positive, 0, and negative.

So let's now first, get back to the vector turning in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane, and go over how angles get made and can be real numbers. Initially, the vector is at rest sitting on the $\boldsymbol{x}$-axis on the right of the origin. So the angle begins with 0 i.e., 0^{o}.

Fig. 1



Then, either counterclockwise or clockwise, in ***one way***, the vector keeps turning about the origin in the ***x-y*** plane, and keeps making angles. So for instance, if the vector has made one and a sixth of a complete turn counterclockwise, the angle made is

$\left(360+\frac{360}{6}\right)^{\text{o}} = 420^{\text{o}}$, which is $2\pi + \frac{2\pi}{6} = \frac{7\pi}{3}$.

Each and every moment, it makes an angle, because an amount of turning is an angle. Initially, the vector is at rest sitting on the ***x***-axis on the right of the origin. So the angle made is 0 (in radians) i.e., 0^{o}.

And we can have the vector make as many complete turns as necessary. If the vector starts and keeps turning counterclockwise, and $\boldsymbol{\theta}$ is the angle made, $\boldsymbol{\theta}$ is positive each and every moment. So we get $\mathbf{0^o \le \boldsymbol{\theta} \le 360^o \cdot \boldsymbol{n}}$. Or we get $\mathbf{0 \le \boldsymbol{\theta} \le 2\boldsymbol{n}\pi}$ (in radians).

Or if the vector starts and keeps turning clockwise, the angle $\boldsymbol{\theta}$ is negative each and every moment. So we get $\mathbf{-360^o \cdot \boldsymbol{n} \le \boldsymbol{\theta} \le 0^o}$. Or we get $\mathbf{-2\boldsymbol{n}\pi \le \boldsymbol{\theta} \le 0}$.

So putting together all the angles that can be made, we can put them the way as follows.

$\mathbf{-360^o \cdot \boldsymbol{n} \le \boldsymbol{\theta} \le 360^o \cdot \boldsymbol{n}}$ in degrees or $\mathbf{-2\boldsymbol{n}\pi \le \boldsymbol{\theta} \le 2\boldsymbol{n}\pi}$ in radians,

where π is 3.141592…, which is a real number, and $\boldsymbol{n}$ is an integer ≥ 0, so $\boldsymbol{n} = 0, 1, 2, \ldots$

Thus, we can get $-\infty < \boldsymbol{\theta} < \infty$, where ∞ indicates infinity.

An angle in radians is a ratio of an arc to its radius.

A ratio is a real number, so an angle in radians is a real number as -5, $-\pi$, 0, 1.5π, 12, …

Thus, $\boldsymbol{\theta}$ can be all real numbers, so every input is a real number as $-\frac{\pi}{3}, \sqrt{3}, \frac{\pi}{4}, 7, \ldots$

What then about the outputs?

Each output is a trig ratio, and is a number. It's because a value of **sin *x*** is a trig ratio, which is a ratio, which is a real number. And just saying 'real' in math expressions, we mean a real number or all real numbers.
So each value of ***x*** is real, since it's an angle in radians, and the value of ***f*(*x*)** is real, since it's a trig ratio.
Thus, for instance, assuming ***f*** is a sine function, its expression is **sin *x***, and the domain is a set of all real numbers meaning a set of all angles in radians, we can put the trig function ***f*** the way as follows.

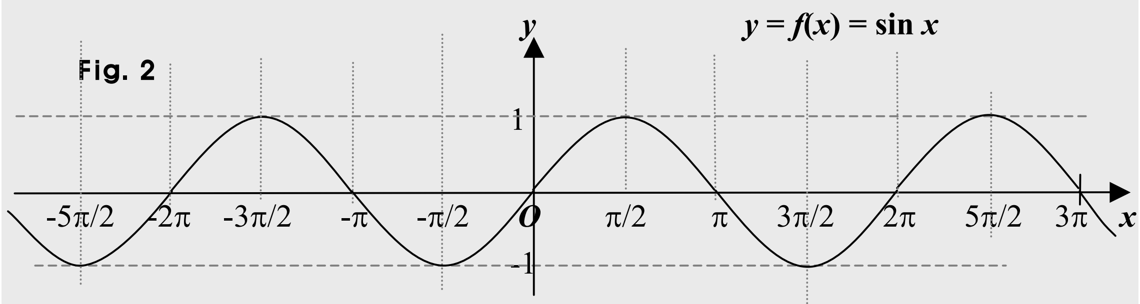
$$y = f(x) = \sin x \text{ for } x \text{ real.}$$

Or it's simply put this way, too: $y = f(x) = \sin x$.

So if no domain is specified in a function such as $y = k(x) = 2x$, the domain is assumed to be a set of all real numbers or the largest possible set. What then about the range of ***f***?

The range can be a set of all numbers from –1 to 1, and each of the numbers is a trig ratio, which is an output, which is the value of ***f*(*x*)**, which is the value of ***y***.
So the range of the sine function ***f*** can be put this way, too: $-1 \le y \le 1$, or $|y| \le 1$.

Fig. 2



And the sine function ***f*** above can be called the prototype, which is thus, in the most basic form. And assuming ***F*** is a sine function, too, and using a general form, we can put the function ***F*** the way as follows.

$$y = F(x) = A\cdot\sin\{w(x + a)\} + b \text{ for } x \text{ real, where } A, w, a, \text{ and } b \text{ are constants.}$$

And we can just put it this way, too: $y = F(x) = A\cdot\sin w(x + a) + b$ for x real.

Or more simply this way, too: $y = F(x) = A \cdot \sin w(x + a) + b$.

Note that $w(x + a)$ represents an angle, and A and b are constants.
And the function F is defined for x real, so the domain is a set of all real numbers.
What then about the range?

The range is a set of all numbers from $-|A| + b$ to $|A| + b$, and is the set of all outputs.

So the range of F can be put this way, too: $-|A| + b \le y \le |A| + b$.

And we call $|A|$ the *amplitude*, $|w|$ is the *frequency*, $\dfrac{2\pi}{|w|}$ is the *period*, and a is the *phase*.

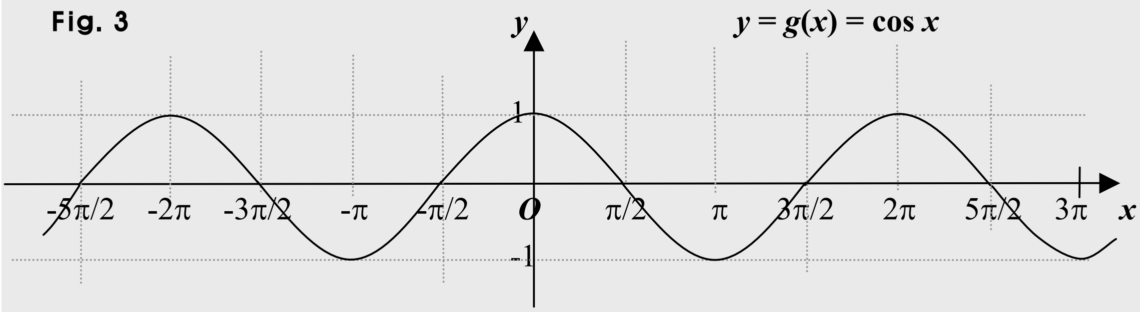
And the same is true for cosine functions, too.
So for instance, assuming g is a cosine function, its expression is $\cos x$, and the domain is a set of all real numbers, we can put the function g the way as follows.

$$y = g(x) = \cos x \text{ for } x \text{ real.}$$

Or simply this way, too: $y = g(x) = \cos x$.   What then about the range?

The range can be a set of all real numbers from -1 to 1, and each of the numbers is a trig ratio, which is an output, which is the value of $g(x)$, which is the value of y.
So the range of the cosine function g can be put this way, too: $-1 \le y \le 1$, or $|y| \le 1$.

Fig. 3



And the cosine function g above can be called the prototype, which is thus, in the most basic form. And assuming G is a cosine function, too, and using a general form, we can put the function G the way below.

$$y = G(x) = A \cdot \cos\{w(x + a)\} + b \text{ for } x \text{ real, where } A, w, a, \text{ and } b \text{ are constants.}$$

And of course, we can just put it this way, too: $y = G(x) = A \cdot \cos w(x + a) + b$ for x real.

Or more simply this way, too: $y = G(x) = A \cdot \cos w(x + a) + b$.

Note also that $w(x + a)$ represents an angle, and A and b are constants.

And the function G is defined for x real, so the domain is the set of all real numbers. What then about the range?

The range is a set of all numbers from $-|A| + b$ to $|A| + b$, and is the set of all outputs.
So the range of G can be put this way, too: $-|A| + b \le y \le |A| + b$.
Then, we call $|A|$ the amplitude, $|w|$ is the frequency, $\dfrac{2\pi}{|w|}$ is the period, and a is the phase.
And we will get to the details on A, w, a, and b in one of the sections where we discuss trig functions.
What about tangent functions?

For instance, assuming h is a tangent function, and its expression is $\tan x$, we can put the trig function h the way as follows.

$$y = h(x) = \tan x.$$

What then about the domain?

In the tangent function h, x takes a real number as an angle, too. Note however, that x cannot take all angles, so the domain of h is not a set of all real numbers.

The tangent function $\tan x$ cannot be defined if the vector is on the y-axis. It's because the adjacent is 0 when the vector is on the y-axis. So it can't be defined if the angle x is as follows.
$90^\circ = 1 \cdot 90^\circ$, $270^\circ = 3 \cdot 90^\circ$, $450^\circ = 5 \cdot 90^\circ$, $630^\circ = 7 \cdot 90^\circ$, and so forth. And the same is true, also, if the angle is -90°, $-270^\circ = -3 \cdot 90^\circ$, $-450^\circ = -5 \cdot 90^\circ$, …. And we know $90^\circ = \dfrac{\pi}{2}$.

So in short, $\tan x$ cannot be defined for $x = \dfrac{n\pi}{2}$, where n is an ***odd*** integer.

What then is the domain of the tangent function $y = h(x) = \tan x$?

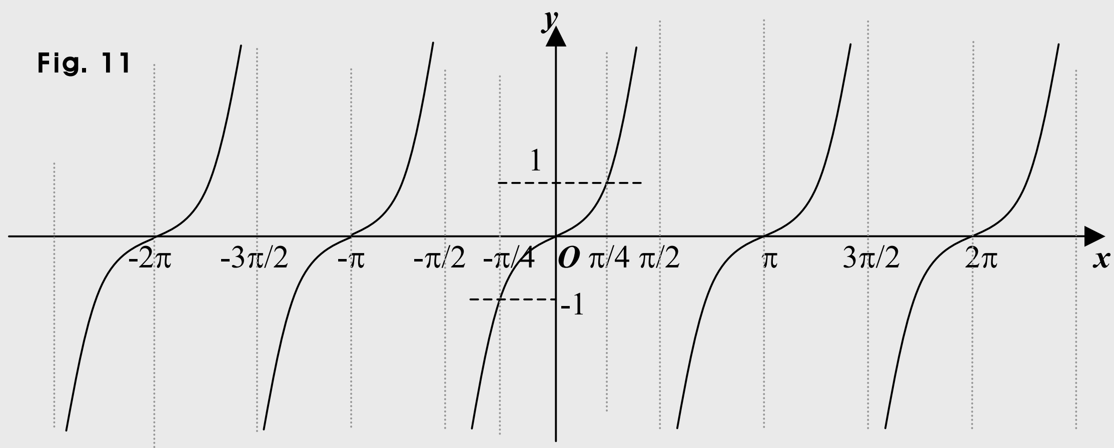
The domain of $\boldsymbol{h}$ is a set of all angles less $\frac{n\pi}{2}$ where $\boldsymbol{n}$ is an odd integer.

That is, a set of all real numbers other than $\frac{n\pi}{2}$ where $\boldsymbol{n}$ is an integer odd.

So setting $\boldsymbol{h(x)} = \mathbf{tan}\ \boldsymbol{x}$, we mean $\boldsymbol{h(x)} = \mathbf{tan}\ \boldsymbol{x}$ for $x \neq \frac{n\pi}{2}$ where $\boldsymbol{n}$ is an odd integer.

Note that if the domain is not specified as in the case of $\boldsymbol{y} = \boldsymbol{h(x)} = \mathbf{tan}\ \boldsymbol{x}$, we just assume that the domain is the set of all possible values that can be the inputs. In other words, the domain is the larges set of values that the input variable can take.
What then about the range?

Fig. 11



Unlike the sine and the cosine functions above, the range of the tangent function **tan** $\boldsymbol{x}$ is a set of all real numbers, each of which is a trig ratio, of course.

And the tangent function $\boldsymbol{h}$ above can be called the prototype, which is thus, in the most basic form. And assuming $\boldsymbol{H}$ is a tangent function, too, and using a general form, we can put it the way below.

$y = H(x) = A\cdot\tan\{w(x + a)\} + b$, where A, w, a, and b are constant.

And of course, we can just put it this way, too: $y = H(x) = A\cdot\tan w(x + a) + b$.

Note also that $\boldsymbol{w(x + a)}$ represents an angle, and $\boldsymbol{A}$ and $\boldsymbol{b}$ are constant.

What then about the domain?

We know the fact that **tan** $\boldsymbol{\theta}$ cannot be defined for $\boldsymbol{\theta} = \frac{\boldsymbol{n\pi}}{2}$ where $\boldsymbol{n}$ is an odd integer.

Also, we know $\boldsymbol{\theta}$ is the angle in **tan** $\boldsymbol{\theta}$, and in **tan** $\boldsymbol{w(x+a)}$, $\boldsymbol{w(x+a)}$ is the angle.

So if $\boldsymbol{w(x+a)} = \frac{\boldsymbol{n\pi}}{2}$ where $\boldsymbol{n}$ is an odd integer, the function $\boldsymbol{H}$ cannot be defined.

In other words, if $\boldsymbol{w(x+a)} \neq \frac{\boldsymbol{n\pi}}{2}$ where $\boldsymbol{n}$ is an odd integer, the function $\boldsymbol{H}$ is defined.

So finding the domain of $\boldsymbol{H}$, we can get it the way as follows.

$\boldsymbol{w(x+a) = \frac{n\pi}{2} \Rightarrow x+a = \frac{n\pi}{2w} \Rightarrow x = \frac{n\pi}{2w} - a}$. So if $\boldsymbol{x \neq \frac{n\pi}{2w} - a}$, the function $\boldsymbol{H}$ is defined.

Thus, the domain of $\boldsymbol{H}$ is a set of all real numbers less $\boldsymbol{\frac{n\pi}{2w} - a}$, where $\boldsymbol{n}$ is an odd integer, that is, the function $\boldsymbol{H}$ is not defined for $\boldsymbol{x = \frac{n\pi}{2w} - a}$, where $\boldsymbol{n}$ is odd, but is defined for any other real number. What then about the range?

The range is a set of all real numbers. And as in the case of the sine and the cosine functions above, $|\boldsymbol{w}|$ is called the frequency, and $\boldsymbol{a}$ is called the phase, too. However, the period is not $\boldsymbol{\frac{2\pi}{|w|}}$ but $\boldsymbol{\frac{\pi}{|w|}}$. And we will get to see how it is the case in a separate section. What about the amplitude?

Tangent functions have no amplitude. So $\boldsymbol{A}$ is not the amplitude, but can be an amplifier. If $|\boldsymbol{A}|$ is bigger, the magnitude of the output is bigger for the same input.

And in the next section, we will begin with the sine function prototype, where the amplitude is 1, the phase is 0, and the period is 2π (i.e., 360°), so the frequency is 1.

And we have another set of three trig functions, which are the reciprocals of the ones introduced above.
One is a cosecant function, and its expression is **csc** $\boldsymbol{x}$, which is the reciprocal of **sin** $\boldsymbol{x}$, so the cosecant function can be just called the reciprocal of the sine function.
Another is a secant function, and its expression is **sec** $\boldsymbol{x}$, which is the reciprocal of **cos** $\boldsymbol{x}$, so the secant function can be just called the reciprocal of the cosine function.

And the other is a cotangent function, and its expression is **cot** x, which is the reciprocal of **tan** x, so the cotangent function can be called the reciprocal of the tangent function.

And we have their inverses, too. The inverse of a function is called an inverse function, and is often just called the inverse. What is an inverse function though?

If a function K is one-to-one, we can get the inverse.

Then, the domain of the function K is the range of the inverse, and the range of the function K is the domain of the inverse.

So for instance, if of the function K, the domain is X, and the range is Y, Y is the domain of the inverse, and X is the range.
And the same is true for the inverse of a trig function, too.

We know that the domain of a trig function such as $f(x) = \sin x$ or $g(x) = \cos x$ is a set of angles, and the range is a set of trig ratios. (Both sets are number sets though, since we just use numbers as angles, and ratios are numbers.)

So the domain of the inverse of a trig function such as f or g above is a set of trig ratios, and the range is a set of angles. Of course, the trig function that can have the inverse has to be one-to-one.

And we can define the inverses of the three basic trig functions the way as follows:

$y = f(x) = \sin x \Leftrightarrow x = f^{-1}(y) = \sin^{-1} y$, $\quad y = g(x) = \cos x \Leftrightarrow x = g^{-1}(y) = \cos^{-1} y$, and

$y = h(x) = \tan x \Leftrightarrow x = h^{-1}(y) = \tan^{-1} y$, where x is an angle, and y is a trig ratio.

And by convention, specifying a function, we usually use x as the input variable, and use y as the output variable. And the same is true for inverse functions, too.
And if not ambiguous, we can use a function name over and over.

So to begin with, a sine function $y = f(x) = \sin x$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ is one-to-one.

Thus, assuming S is the inverse of the sine function f, we get $y = S(x) = \sin^{-1} x$ for $|x| \le 1$, where x is a trig ratio, and y is an angle. And the range of S is $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

Next, a cosine function $\boldsymbol{y = g(x) = \cos x}$ for $\mathbf{0 \le x \le \pi}$ is one-to-one.

So assuming $\boldsymbol{C}$ is the inverse of the function $\boldsymbol{g}$, we get $\boldsymbol{y = C(x) = \cos^{-1} x}$ for $\boldsymbol{|x| \le 1}$, where $\boldsymbol{x}$ is a trig ratio, and $\boldsymbol{y}$ is an angle. And the range of $\boldsymbol{C}$ is $\mathbf{0 \le y \le \pi}$.

And next, a tangent function $\boldsymbol{y = h(x) = \tan x}$ for $-\frac{\pi}{2} < \boldsymbol{x} < \frac{\pi}{2}$ is one-to-one.

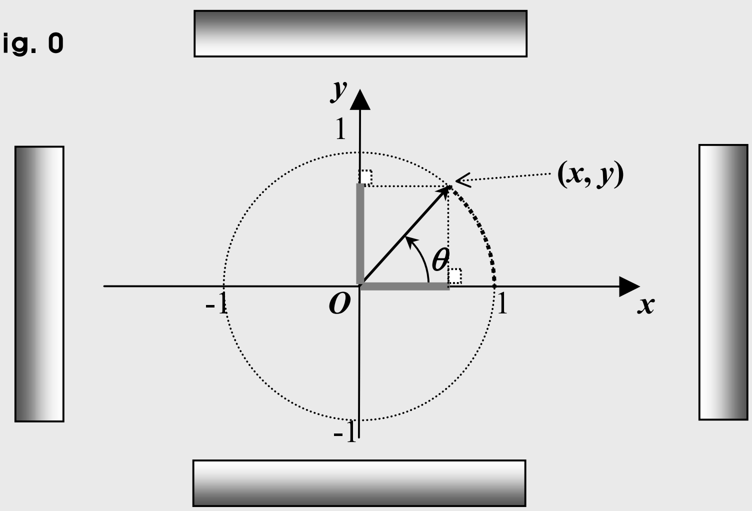
So assuming $\boldsymbol{T}$ is the inverse of the tangent function $\boldsymbol{h}$, we get $\boldsymbol{y = T(x) = \tan^{-1} x}$ for $\boldsymbol{x}$ real, where $\boldsymbol{x}$ is a trig ratio, and $\boldsymbol{y}$ is an angle. And the range of $\boldsymbol{T}$ is $-\frac{\pi}{2} < \boldsymbol{y} < \frac{\pi}{2}$.

If not sure of an inverse function itself, refer to **BASIC FUNCTIONS**.

And of course, we will cover the details on each of all the trig functions above in a separate section.

4. Sine Functions

Fig. 0



Let's first, quickly go over the basics of trigonometric functions, and then, move on to the details on sine functions. So what is a trigonometric function?

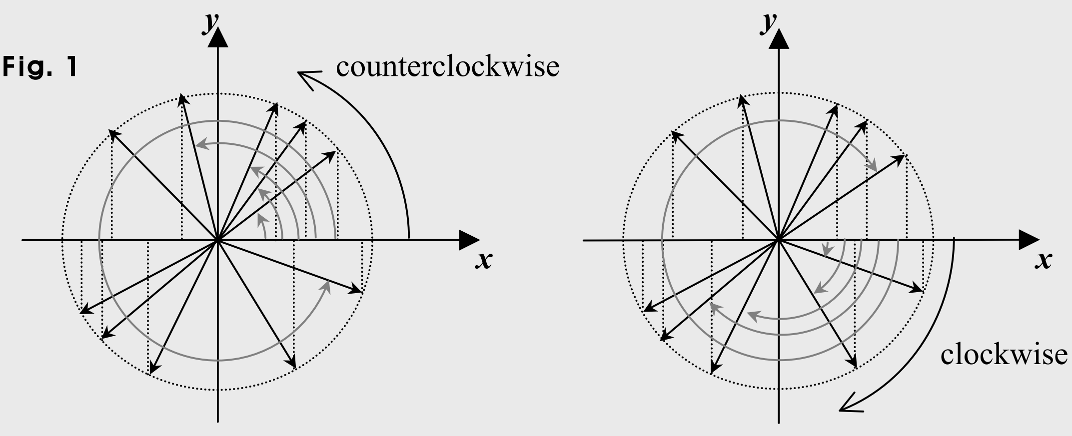
It is often called a trig function and is a function of which each output is a trig ratio called the sine, the cosine, or the tangent. For instance, putting a sine function in the ***x*-*y*** coordinate system, we can put it the way as follows.

$y = f(x) = \sin x$, where x is a real number indicating an angle.
In short, $y = f(x) = \sin x$ for x real.

Thus, the sine function $f(x)$ above is a function of an angle, so takes an angle as an input, and produces a trig ratio called the sine as an output. Specifically, in $y = f(x) = \sin x$, the input variable x takes an angle, the expression $\sin x$ produces a trig ratio, an output called the sine, and the output variable y gets the sine.
Also, the same is true for a cosine function such as $y = g(x) = \cos x$, and for a tangent function such as $y = h(x) = \tan x$.

And the angle is made by the vector turning about the origin in the ***x*-*y*** plane.

Fig. 1



Either counterclockwise or clockwise, in ***one way***, the vector keeps turning about the origin in the ***x*-*y*** plane, and keeps making angles. Each and every moment, it makes an angle, because an amount of turning is an angle. Initially, the vector is at rest sitting on the ***x***-axis on the right of the origin. Then, the angle made is 0, i.e., 0^{o}.

And we can have the vector make as much turn as we need or as many complete turns as necessary. If the vector starts and keeps turning counterclockwise, the angle made is positive every moment. Or if the vector starts and keeps turning clockwise, the angle made is negative every moment.
So assuming $\boldsymbol{\theta}$ is the angle made and putting together all the angles can be made, we get

$$-360^{o}\cdot n \leq \theta \leq 360^{o}\cdot n \text{ in degrees or } -2n\pi \leq \theta \leq 2n\pi \text{ in radians,}$$

where π is 3.141592…, which is real, and n is an integer ≥ 0, so $n = 0, 1, 2, \ldots$ So we can get $-\infty < \theta < \infty$, where ∞ indicates infinity.
Thus, θ can be all real numbers, and every input is a real number.

Suppose now, in the ***x*-*y*** plane where a vector of length 1 is turning ***counterclockwise*** about the origin, we place two lamps the way in Fig. 2 below so that the shadow of the vector is made on the ***x***-axis. And let's call the shadow the projection, and call the value of the projection simply the projection, too.

Also, if the projecion is on the ***left*** of the origin on the ***x***-axis, or is ***below*** the origin on the ***y***-axis, the projection is ***negative***.

Fig. 2

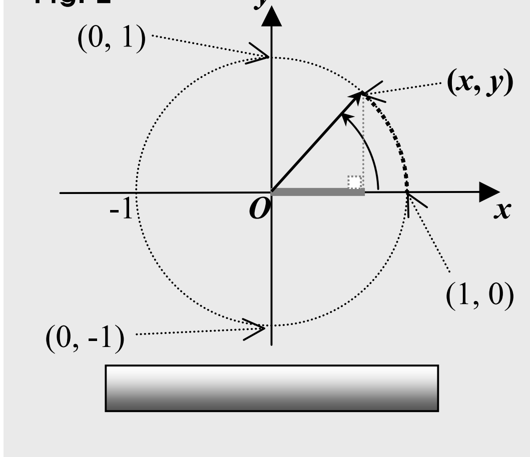
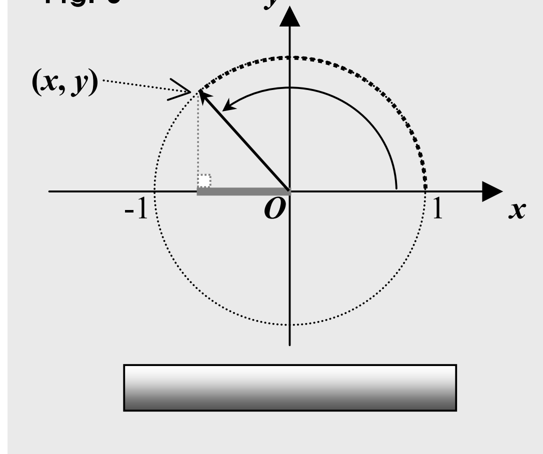


Fig. 3



So first, if the vector is above the ***x***-axis while turning, we can see a linear motion where as in Fig. 2, the projection on the ***x***-axis shrinks changing its value from 1 to 0 as the vector approaches the ***y***-axis, and then as in Fig. 3, expands changing its value from 0 to –1 as the vector approaches the ***x***-axis.

Next, if the vector is below the ***x***-axis while turning, we can see another linear motion where as in Fig. 4, the projection on the ***x***-axis shrinks changing its value from –1 to 0 as the vector approaches the ***y***-axis, and then as in Fig. 5, expands changing its value from 0 to 1 as the vector approaches the ***x***-axis.

Fig. 4

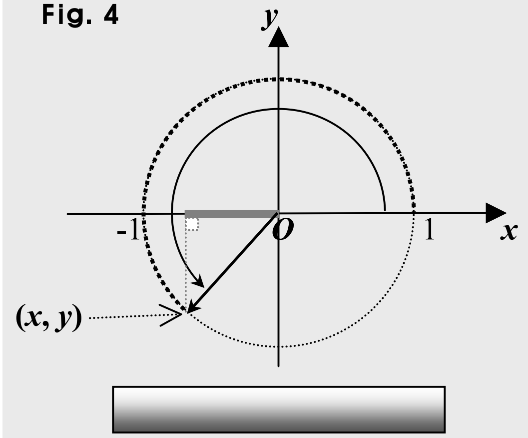
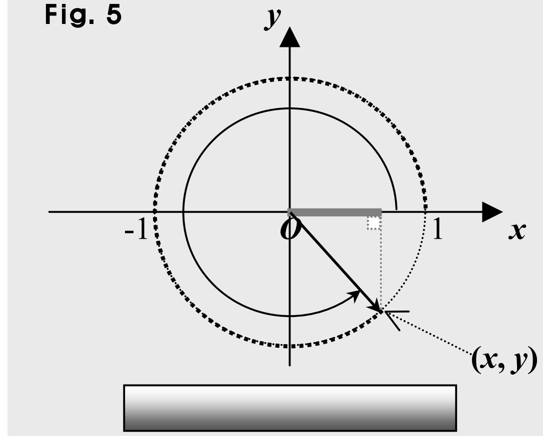
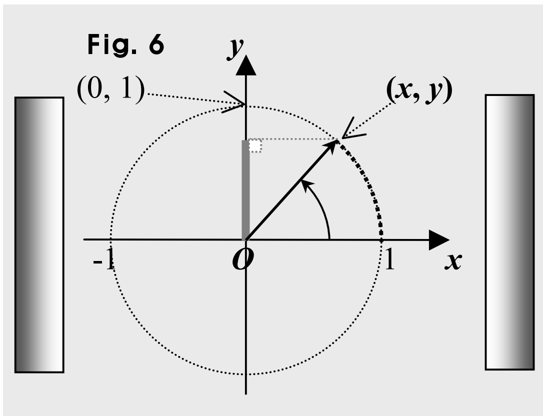
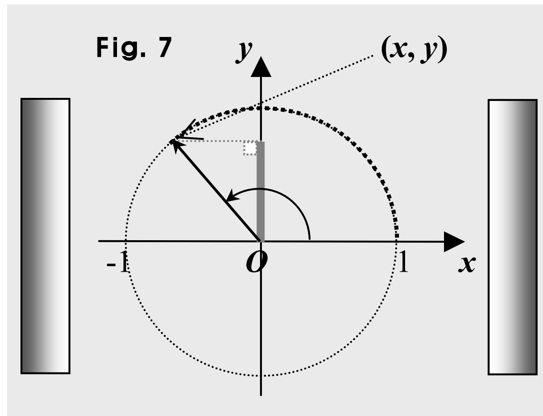


Fig. 5



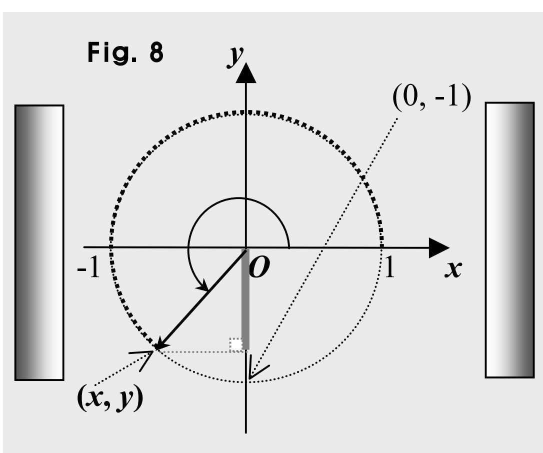
Suppose next, we place two lamps the way in Fig. 6 below so that the projection is made on the ***y***-axis.

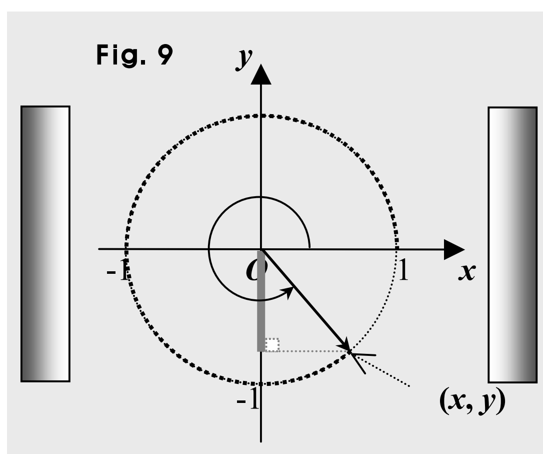




Then first, if the vector is above the ***x***-axis while turning, we can see a linear motion where as in Fig. 6, the projection on the ***y***-axis expands changing its value from 0 to 1 as the vector approaches the ***y***-axis, and then as in Fig. 7, shrinks changing its value from 1 to 0 as the vector approaches the ***x***-axis.

And next, if the vector is below the ***x***-axis while turning, we can see another linear motion where as in Fig. 8, the projection on the ***y***-axis expands changing its value from 0 to –1 as the vector approaches the ***y***-axis, and then as in Fig. 9, shrinks changing its value from –1 to 0 as the vector approaches the ***x***-axis.



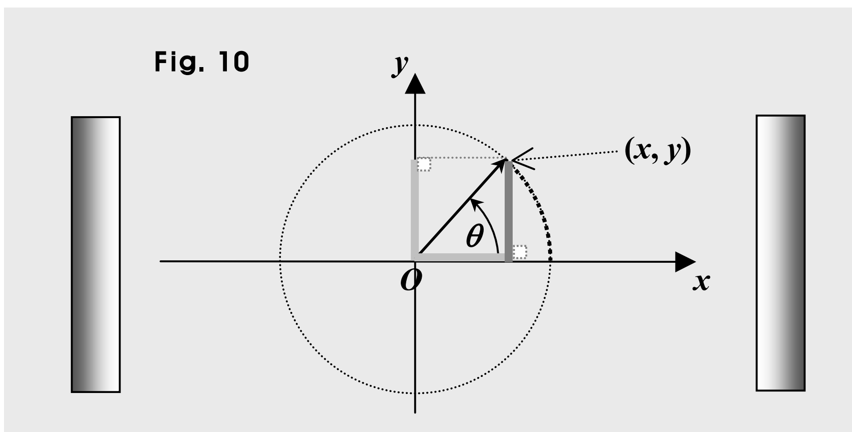


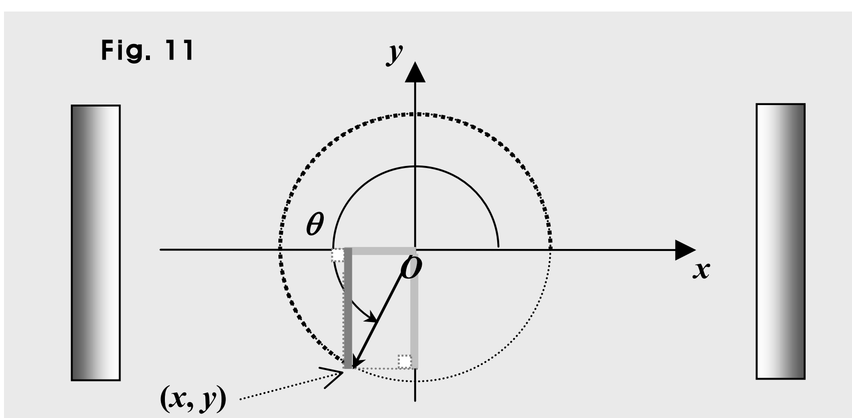
Also, we can see that such linear motions keep repeating as the vector keeps turning. So now, what are we going to do with those projections?

We are going to come up with trig functions and their graphs.

At every moment, assuming in Fig. 10 and Fig. 11 below, the |vector| is hypotenuse, we get a right triangle where the adjacent is the projection on the ***x***-axis, and the opposite is the projection on the ***y***-axis.

Notice that in Fig. 10 and Fig. 11, the line segment in dark gray is the same in size and length as the projection in light gray on the ***y***-axis. Thus, we can use the projection on the ***y***-axis as the opposite.





And depending on the angle θ made by the vector, the projections get determined. In short, the angle made determines the projections. So what makes such a right triangle is the |vector| and the angle θ made by the vector.
And we have the statement as follows.

In coordinate trigonometry,
the $\boldsymbol{x}$-coordinate at the terminal point is the adjacent,
the $\boldsymbol{y}$-coordinate is the opposite,
the |vector| is the hypotenuse,
and the vector turned makes the angle.

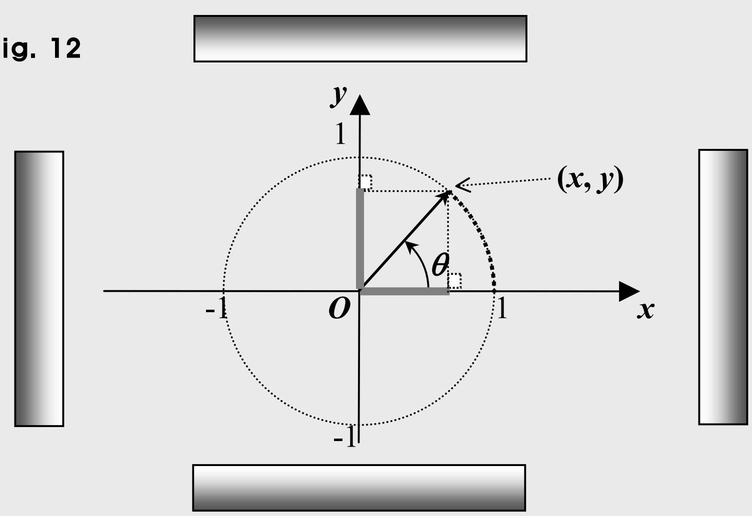
Therefore, of the terminal point, the $\boldsymbol{x}$-coordinate is the projection on the $\boldsymbol{x}$-axis, and the $\boldsymbol{y}$-coordinate is the projection on the $\boldsymbol{y}$-axis.

And as in Fig. 12, if $(\boldsymbol{x}, \boldsymbol{y})$ is the terminal point, and the |vector| is 1, then since the hypotenuse is 1, we can say that the sine is just $\boldsymbol{y}$ for any angle $\boldsymbol{\theta}$, that is, $\mathbf{sin}\ \boldsymbol{\theta} = \boldsymbol{y}$, since $\mathbf{sin}\ \boldsymbol{\theta} = \frac{\boldsymbol{y}}{1}$, because the sine is $\frac{\textbf{\textit{opposite}}}{\textbf{\textit{hypotenuse}}}$, the opposite is $\boldsymbol{y}$, and the hypotenuse is 1,

and can say that the cosine is just $\boldsymbol{x}$ for any angle $\boldsymbol{\theta}$, that is, $\mathbf{cos}\ \boldsymbol{\theta} = \boldsymbol{x}$, since $\mathbf{cos}\ \boldsymbol{\theta} = \frac{\boldsymbol{x}}{1}$,

because the cosine is $\frac{\textbf{\textit{adjacent}}}{\textbf{\textit{hypotenuse}}}$, the adjacent is $\boldsymbol{x}$, and the hypotenuse is 1.

Fig. 12



Thus, if the |vector| is 1, then for any angle, the projection on the $\boldsymbol{y}$-axis is the value of the sine itself, and the projection on the $\boldsymbol{x}$-axis is the value of the cosine itself, so in short,

the projection on the $\boldsymbol{y}$-axis is $\mathbf{sin}\ \boldsymbol{\theta}$, and the projection on the $\boldsymbol{x}$-axis is $\mathbf{cos}\ \boldsymbol{\theta}$.

What then about the tangent?

The tangent is $\dfrac{y}{x}$, since the tangent is the opposite over the adjacent

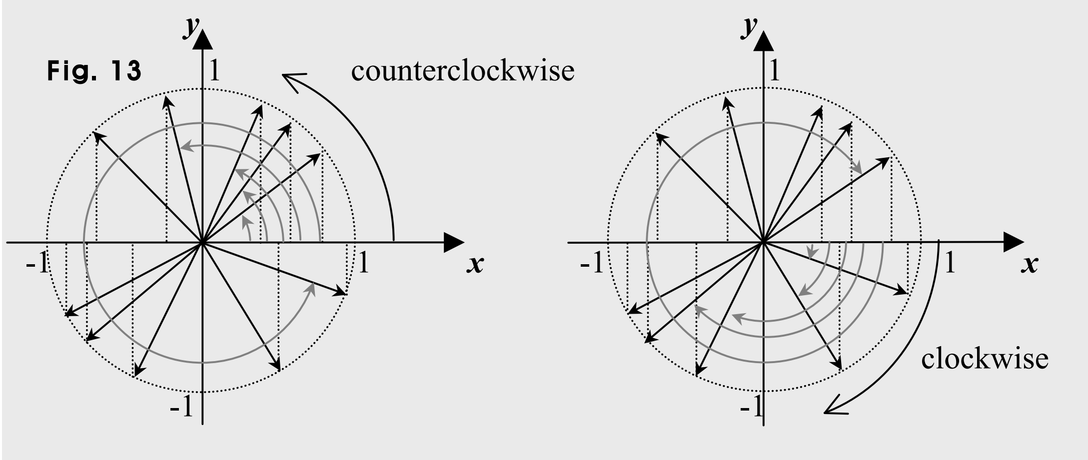
So we can say that the tangent is $\dfrac{\text{the sine}}{\text{the cosine}}$, since $\mathbf{cos}\ \boldsymbol{\theta} = \boldsymbol{x}$, and $\mathbf{sin}\ \boldsymbol{\theta} = \boldsymbol{y}$.

In short, $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.

Now, in coordinate trigonometry, using those projections, along with the angle $\boldsymbol{\theta}$ stated above, we can come up with the three basic trig functions, along with their graphs.

Of each of the basic three, an input is an angle, which is made by the vector turning in the ***x*-*y*** plane.

Fig. 13



So in the sine function $\boldsymbol{y = f(x) = \sin x}$, the input variable $\boldsymbol{x}$ gets an angle at a time.

Usually though, we just use real numbers as 0 or π as angles, since angles are in radian.

So assuming $\boldsymbol{f}$ is a sine function, its expression is $\mathbf{sin}\ \boldsymbol{x}$, and the domain is a set of all angles, we say that the domain is a set of all real numbers,
and can put the sine function $\boldsymbol{f}$ the way as follows. $\boldsymbol{y = f(x) = \sin x}$ for $\boldsymbol{x}$ real,
or simply this way, too: $\boldsymbol{y = f(x) = \sin x}$.

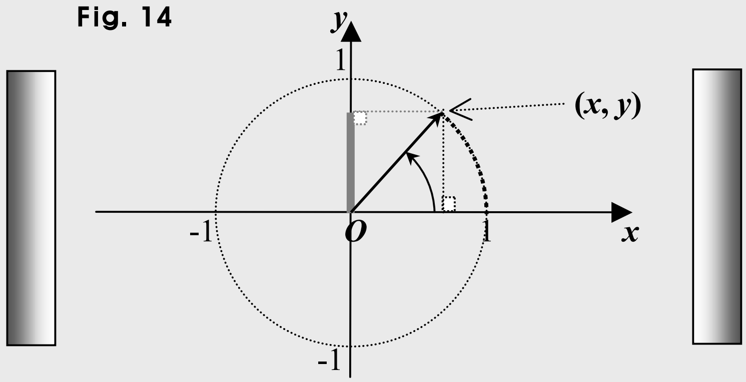
What then about outputs?

They are trig ratios called the sines, because in the sine function $y = f(x) = \sin x$, the x gets an angle, so each value of $\sin x$, that is, each value of $f(x)$ is a trig ratio called the sine, the ratio of the opposite to the hypotenuse, which is a real number.
What then about the range?

The range is a set of all real numbers from –1 to 1, and each of the numbers is a trig ratio, which is an output, which is the value of $f(x)$, which is the value of y.
So the range of the sine function f can be put this way, too: $-1 \le y \le 1$, or $|y| \le 1$.

Also, the same is true for the cosine function $y = g(x) = \cos x$.

Fig. 14



If the |vector| = 1. the sine is y at (x, y), and the cosine is x at (x, y).
That is, if the |vector| = 1, $\sin \theta$ = the projection on the y-axis, and $\cos \theta$ = the projection on the x-axis.
What then are the maximum and the minimum of the x-value and the y-value at (x, y)?

The maximum is 1, and the minimum is –1.
If the |vector| is 1, the projection on the y-axis covers all the numbers from –1 to 1, because the radius of the circle the terminal point traces is 1. And the same is true for the projection on the x-axis, also.

For any angle θ, assuming the |vector| is 1, that is, the hypotenuse is 1, and the terminal point is (x, y), we get $\sin \theta = y$, which is of course, the projection on the y-axis, and we get $\cos \theta = x$, which is of course, the projection on the x-axis.
Now, as the angle θ changes from 0 to 2π, what values does $\sin \theta$ get?

Fig. 15

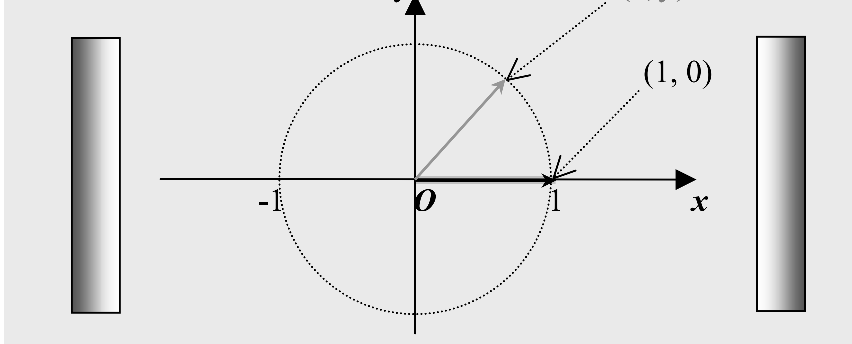


Fig. 16

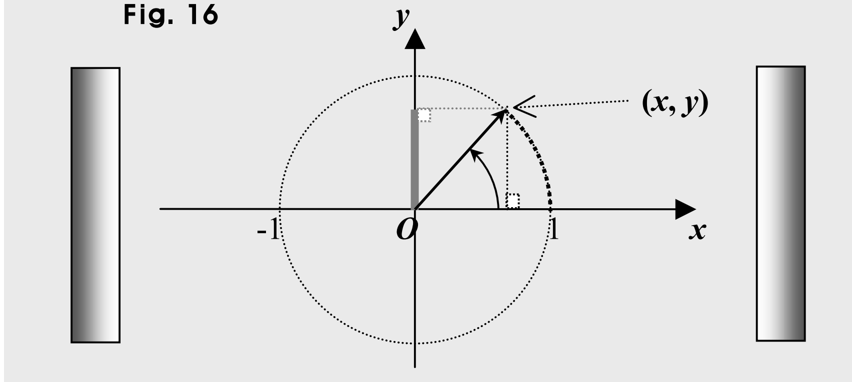
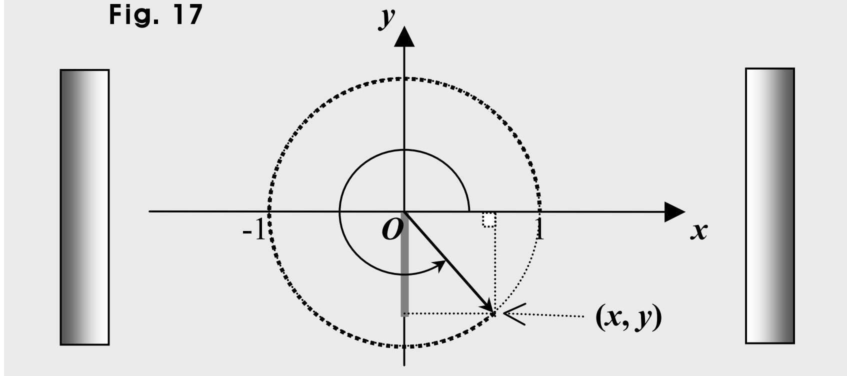


Fig. 17



If the |vector| is 1, the projection on the ***y***-axis covers all the numbers from –1 to 1, so the value of **sin** $\boldsymbol{\theta}$ covers all the numbers from –1 to 1.

So if $\mathbf{0 \leq \theta \leq 2\pi}$, we get $\mathbf{-1 \leq \sin \theta \leq 1}$, that is, $\mathbf{|\sin \theta| \leq 1}$.

If $\mathbf{2\pi \leq \theta \leq 4\pi}$, we get $\mathbf{-1 \leq \sin \theta \leq 1}$, too, that is, $\mathbf{|\sin \theta| \leq 1}$.

Thus, if $\mathbf{0 \leq \theta \leq 2n\pi}$, where $\boldsymbol{n}$ is an integer $\geq \mathbf{1}$, we get $\mathbf{|\sin \theta| \leq 1}$.
That is, no matter how many complete turns the vector may make, we get $\mathbf{|\sin \theta| \leq 1}$.

Suppose now again, the vector of length 1 is at rest sitting on the $\boldsymbol{x}$-axis on the right of the origin. Suppose this time, the vector starts turning clockwise.

Then, we can see the projection on the $\boldsymbol{y}$-axis starts growing downward from 0.

Fig. 18

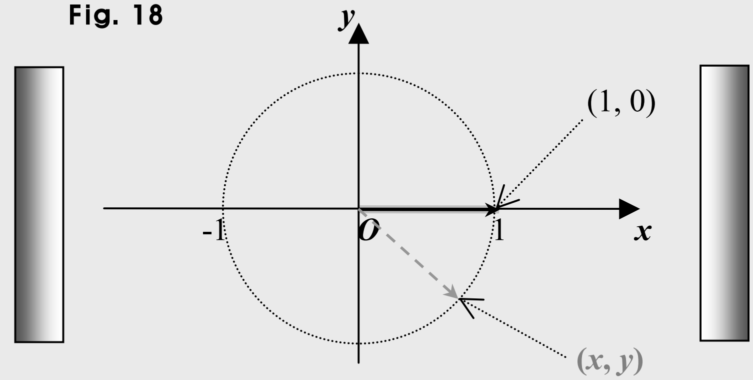
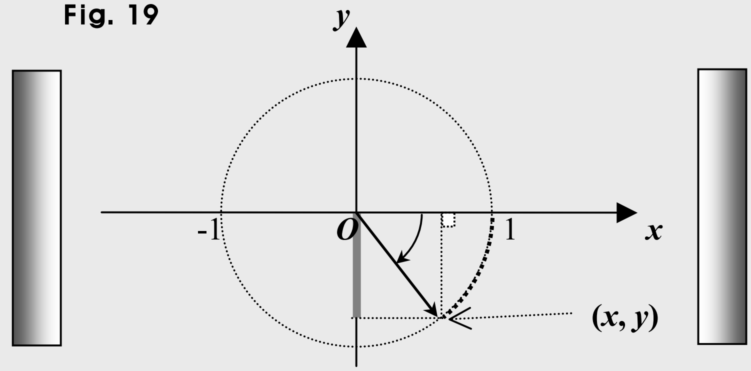
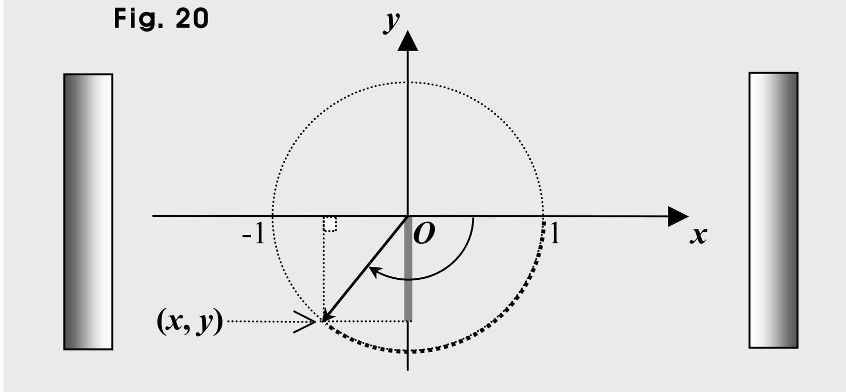


Fig. 19



So first, if the vector is below the $\boldsymbol{x}$-axis while turning, we can see a linear motion where as in Fig. 19, the projection on the $\boldsymbol{y}$-axis expands changing its value from 0 to –1 as the vector approaches the $\boldsymbol{y}$-axis, and then as in Fig. 20, shrinks changing its value from –1 to 0 as the vector approaches the $\boldsymbol{x}$-axis.

Fig. 20



And next, if the vector is above the ***x***-axis while turning, we can see another linear motion where as in Fig. 21, the projection on the ***y***-axis expands changing its value from 0 to 1 as the vector approaches the ***y***-axis, and then as in Fig. 22, shrinks changing its value from 1 to 0 as the vector approaches the ***x***-axis.

Fig. 21

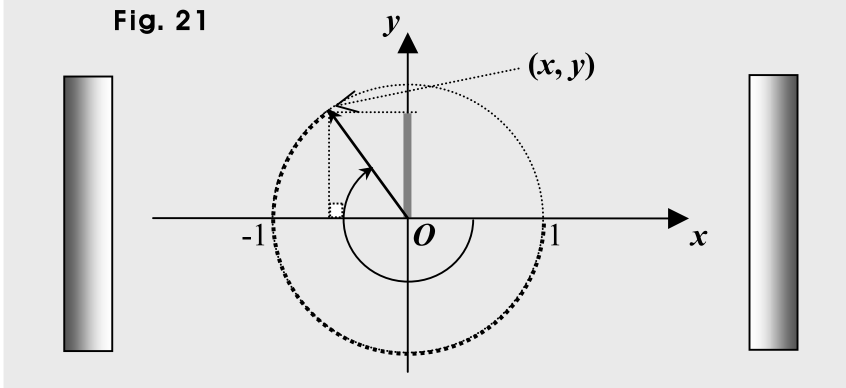
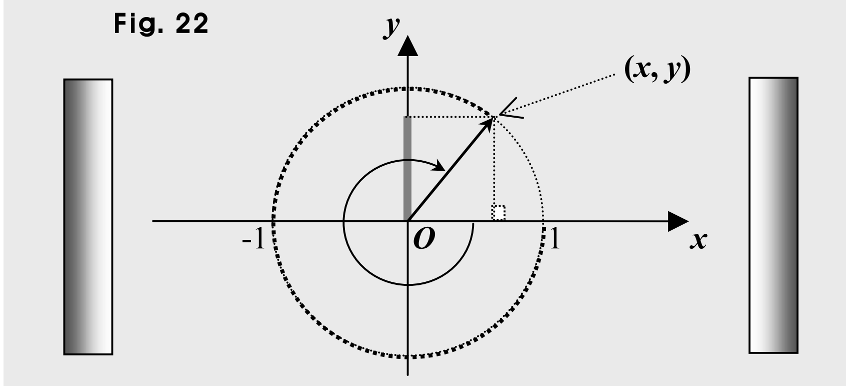


Fig. 22



Then, we can see that the maximum of the $\boldsymbol{y}$-value at $\mathbf{(x, y)}$ is 1, and the minimum is –1, since the terminal point makes a circle of radius 1 centered at the origin if the vector makes a complete turn.
So now, as the angle θ changes from 0 to -2π, what values does $\mathbf{sin}\ \theta$ get?

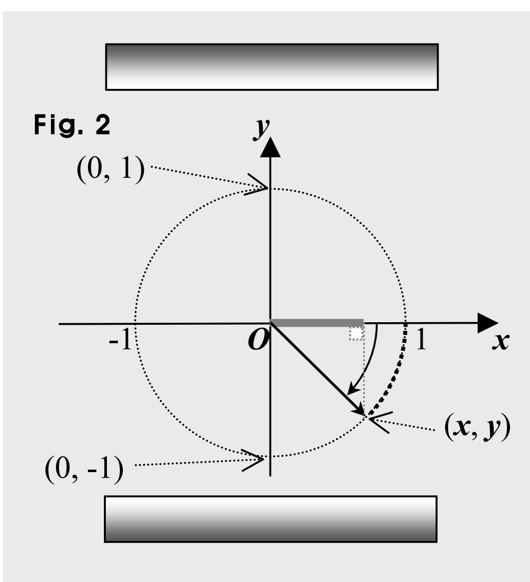
If the |vector| is 1, the projection on the $\boldsymbol{y}$-axis covers all the numbers from –1 to 1, so the value of $\mathbf{sin}\ \theta$ covers all the numbers from –1 to 1.

So if $\mathbf{-2\pi \le \theta \le 0}$, we get $\mathbf{-1 \le sin\ \theta \le 1}$, that is, $\mathbf{|sin\ \theta| \le 1}$.
If $\mathbf{-4\pi \le \theta \le -2\pi}$, we get $\mathbf{-1 \le sin\ \theta \le 1}$, too, that is, $\mathbf{|sin\ \theta| \le 1}$.

Thus, if $\mathbf{2n\pi \le \theta \le 0}$, where $\boldsymbol{n}$ is an integer $\mathbf{\le -1}$, we get $\mathbf{|sin\ \theta| \le 1}$.

So now, putting threads together, we can say that if the |vector| is 1, and $\mathbf{(x, y)}$ is the terminal point, we get $\mathbf{sin\ \theta = y}$ for any angle θ, and that for $\mathbf{-2n\pi \le \theta \le 2n\pi}$, where $\boldsymbol{n}$ is an integer $\mathbf{\ge 1}$, we get $\mathbf{|sin\ \theta| \le 1}$.

And the same is true for $\mathbf{cos}\ \theta$, too.
So if $\mathbf{-2n\pi \le \theta \le 2n\pi}$, where $\boldsymbol{n}$ is an integer $\mathbf{\ge 1}$, we get $\mathbf{|cos\ \theta| \le 1}$.



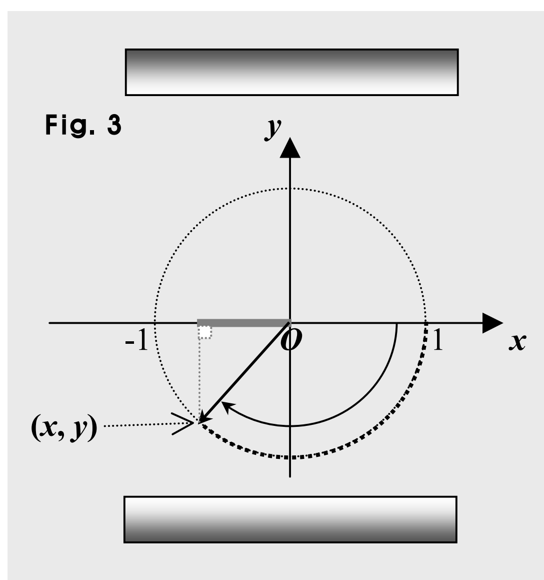


Fig. 4

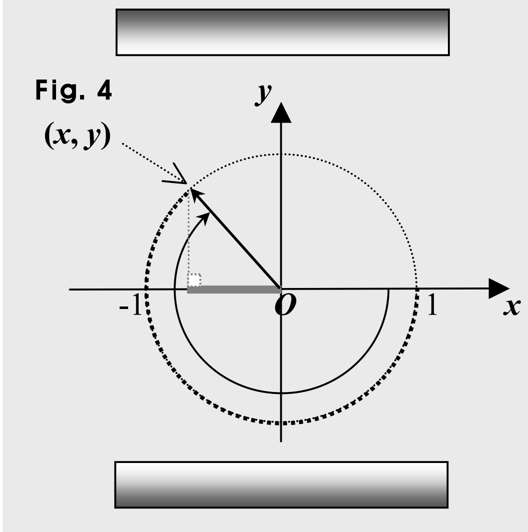
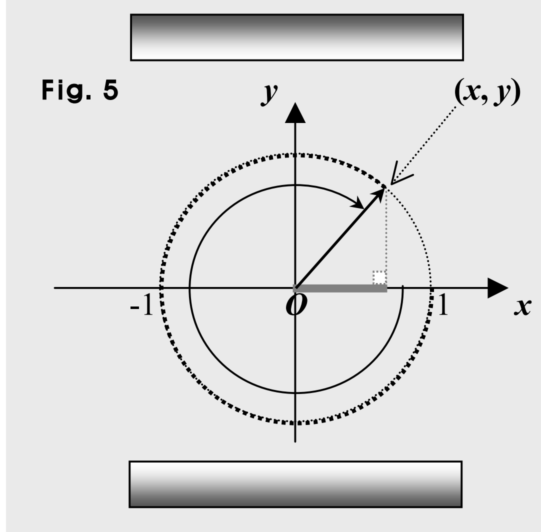


Fig. 5

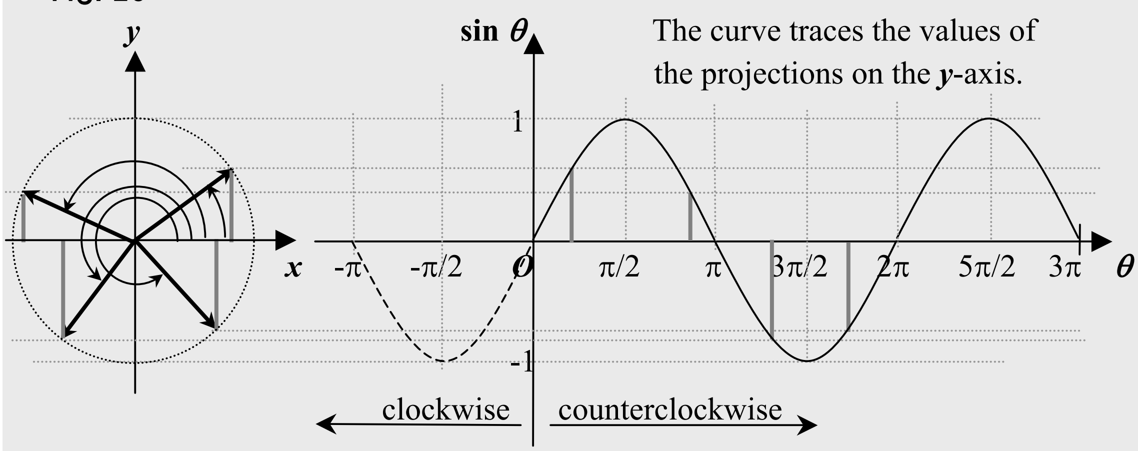


So using the facts above, we can form a function of the angle θ. How?

As the angle θ changes, the sine trig ratio $\sin\theta$ changes. So we can take as an input, each angle that θ gets, and take as an output, each trig ratio that $\sin\theta$ produces. Assuming thus, the function is f, the domain is a set of all angles, and s is the output variable, we can set $s = f(\theta) = \sin\theta$. And we call the function f a sine function. What then about the curve of the sine function $s = f(\theta) = \sin\theta$?

Assuming the |vector| is 1, and the terminal point is (x, y), we get $\sin\theta = y$, which is the projection on the y-axis. So we can get the curve of $\sin\theta$ the way as follows.

Fig. 23



And of course, we can put in the x-y system, the function $s = f(\theta) = \sin\,\theta$.   How?

Replacing s with y, and θ with x, we get $y = f(x) = \sin x$, which is now in the x-y system. And we can use numbers as angles.

Thus, if the domain is a set of all angles, we say that the domain is a set of all real numbers, and we can just set $y = f(x) = \sin x$ for x real.

What if we just set $y = f(x) = \sin x$?   That is, what if we set it with no domain?

In the function $f(x)$, the domain is assumed to be a set all real numbers, since $f(x)$ can be defined for all real numbers. If no domain is specified in the function definition, we just assume the largest possible domain.

And we can put in the x-y plane, the curve of the sine function $y = f(x) = \sin x$ the way as follows.

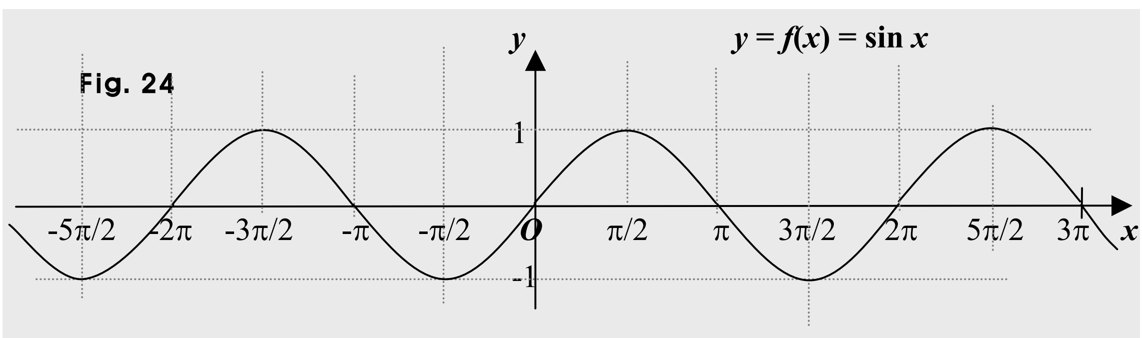


Fig. 24

The curve in Fig. 24 above is called the sine curve, and is no other than the curve in Fig. 23 above. And we know that the output can be every value from -1 to 1.

So the range is $-1 \leq y \leq 1$, that is, $|y| \leq 1$.

And of course, if the domain is not a set of all real numbers, the range can be other than the one above. Suppose for instance, $y = g(x) = \sin x$ for $0 \leq x \leq \pi$.

Then, as we can see in the graph above, the range is $0 \leq y \leq 1$.

And the sine function $\boldsymbol{y = f(x) = \sin x}$ can be called the prototype, and is thus, in the most basic form. Assuming $\boldsymbol{F}$ is a sine function, too, where the domain is a set of all real numbers, and using a general form, we can put it the way as follows.

$\boldsymbol{y = F(x) = A \cdot \sin \{w(x + a)\} + b}$ for $\boldsymbol{x}$ real, where $\boldsymbol{A}$, $\boldsymbol{w}$, $\boldsymbol{a}$, and $\boldsymbol{b}$ are constant.

Note that $\boldsymbol{w(x + a)}$ represents an angle.

And we can just put $\boldsymbol{F(x)}$ this way, too: $\boldsymbol{y = F(x) = A \cdot \sin w(x + a) + b}$ for $\boldsymbol{x}$ real.

Or more simply, this way, too: $\boldsymbol{y = F(x) = A \cdot \sin w(x + a) + b}$, since in this case, we can assume that $\boldsymbol{A}$, $\boldsymbol{w}$, $\boldsymbol{a}$, and $\boldsymbol{b}$ are constant.

Then, the range is a set of all numbers from $\boldsymbol{-|A| + b}$ to $\boldsymbol{|A| + b}$.

We know $\boldsymbol{y}$ is the output variable that takes every output, and the range is the set of all the outputs.

So the range of $\boldsymbol{F}$ can be put this way, too: $\boldsymbol{-|A| + b \le y \le |A| + b}$.

That is, the curve of $\boldsymbol{F}$ is bounded by the interval where $\boldsymbol{-|A| + b \le y \le |A| + b}$.

Then, $\boldsymbol{2|A|}$ is the width of the curve (or wave), and $\boldsymbol{|A|}$ is called the amplitude.

So the amplitude indicates half the width of the curve (or wave).

What then is the amplitude of the sine function $\boldsymbol{y = f(x) = \sin x}$?

Since the curve of $\boldsymbol{f}$ is bounded by the interval where $\boldsymbol{-1 \le y \le 1}$, the amplitude is 1.

And $\boldsymbol{|w|}$ is called the frequency, $\dfrac{2\pi}{|w|}$ is the period, and $\boldsymbol{a}$ is called the phase.

So in the function $\boldsymbol{y = f(x) = \sin x}$, the phase is 0, and the frequency is 1, so the period is 2π (i.e., 360°). How though?

The prototype in sine functions can be put this way: $\boldsymbol{y = f(x) = \sin x}$.

And using a general form, we can put a sine function called $\boldsymbol{F}$ the way as follows.

$\boldsymbol{y = F(x) = A \cdot \sin \{w(x + a)\} + b}$ for $\boldsymbol{x}$ real, where $\boldsymbol{A}$, $\boldsymbol{w}$, $\boldsymbol{a}$, and $\boldsymbol{b}$ are constant.

Then, $\boldsymbol{a}$ is called the phase, $|\boldsymbol{w}|$ is called the frequency, and $\dfrac{2\pi}{|\boldsymbol{w}|}$ is the period.

And we can put the prototype into the general form the way as follows.

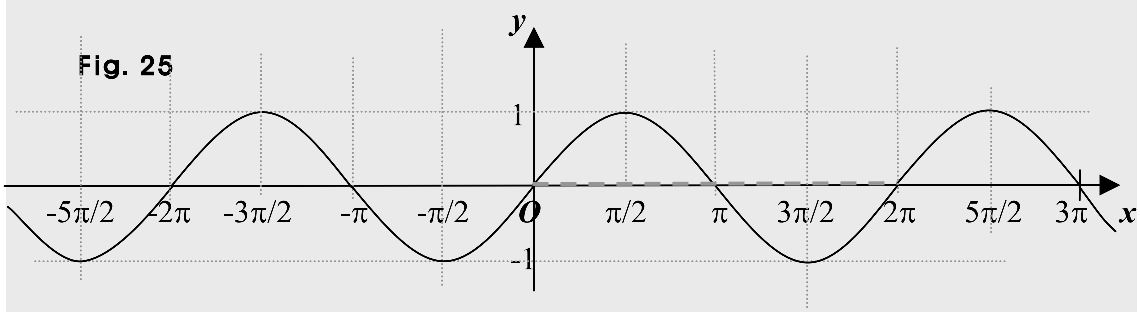
$$\boldsymbol{y = f(x) = 1 \cdot \sin 1(x + 0) + 0}.$$

Thus, the phase $\boldsymbol{a}$ is 0, and the frequency $|\boldsymbol{w}|$ is 1, so the period $\dfrac{2\pi}{|\boldsymbol{w}|}$ is 2π.

What do we mean by the period though?

Putting in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane, the curve of the sine function $\boldsymbol{f}$ above, we can get this:

Fig. 25



Then, we can see that a part of the cuve repleats itself.

And the part is in fact, the smallest part repeating itself.   What then is the part?

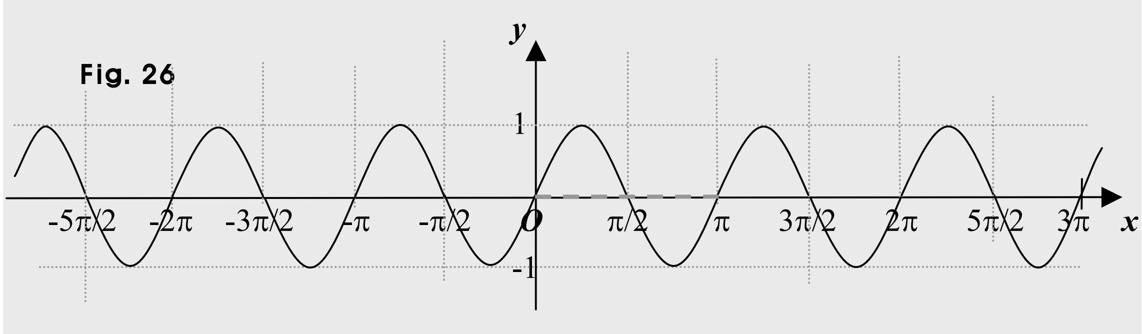
It is the part from 0 to 2π. So the part repeats itself every 2π interval. And we call such an interval a period. So the interval 2π is the period in the sine function $\boldsymbol{y = f(x) = \sin x}$.

Thus, we call a sine function a *periodic* function. And of course, the same is true, too, for cosine functions, tangent functions, and other trig functions such as secant functions. The sine function ***f*** above is the prototype, and thus, is in the most basic form.

So the period **2π** can be called the *basic period* in sine functions. In other words, the interval **2π** can be called the *basic interval* in sine functions.

And for instance, in the general form, setting ***w*** to 2, ***a*** and ***b*** to 0 each, and ***A*** to 1, we get a new function where $\boldsymbol{y = g(x) = \sin 2x}$ for ***x*** real. Note that $\boldsymbol{2x}$ represents an angle. Then, putting in the ***x-y*** plane, the curve of the sine function ***g*** above, we get this:

Fig. 26



Then again, we can see that a part of the cuve repleats itself.
And the part is of course, the smallest part repeating itself. What then, is the part?

It is the part from 0 to π. So the part repeats itself every π interval.
And we call such an interval a period. So the interval π is the period in the function ***g***.
How many times does the smallest part stated above appear in the basic interval 2π?

It appears twice in the basic interval 2π, which is the basic period. Then, we say the frequency is 2 in the function ***g***. What then do we mean by the frequency?

It is the number of times the smallest part repeating itself appears in the basic inverval 2π, which is the basic period. In short, the frequency is the number of times the smallest part appears in 2π. More precisely though, we want to put the frequency the way below.

In case of sine functions, the frequency is the number of times the samllest part repeating itself appears in the basic period (or the basic interval), which is 2π.
And the same is true for cosine functions, too. So the basic period (interval) in cosine functions is 2π, too. The basic period however, in tangent functions is not 2π but π.
Thus in general, we can put the frequency the way as follows.

The frequency is, of the curve of a trig function, the number of times the smallest part repeating itself appears in its basic period or its basic interval.
And in the general form $\boldsymbol{A}\cdot\textbf{sin}\ \boldsymbol{w}(\boldsymbol{x}+\boldsymbol{a})+\boldsymbol{b}$, $|\boldsymbol{w}|$ is the frequency.
Then, given a sine function in the general form, how can we get its period?

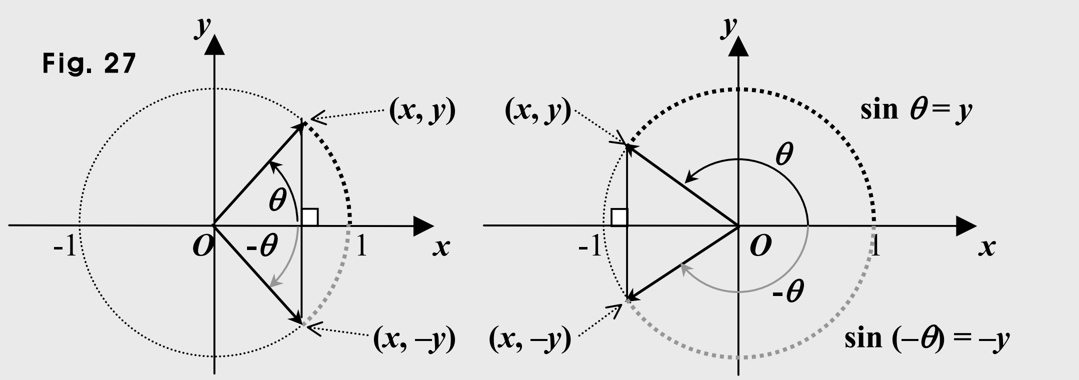
We know that $|\boldsymbol{w}|$ is the frequency, that is, the number of times the smallest part repeating itself appears in the basic interval (basic period) 2π, and that the interval that fits the smallest part is the period.
So using the frequency $|\boldsymbol{w}|$ and the basic interval 2π, how can we express the period?

We can put the period this way: $\dfrac{2\pi}{|\,w\,|}$. Why not just $\boldsymbol{w}$ but $|\boldsymbol{w}|$ though?

We can have a sine function where $\boldsymbol{y}=\boldsymbol{h}(\boldsymbol{x})=\textbf{sin}\ (-\boldsymbol{x})$ for $\boldsymbol{x}$ real. Then, the frequency in the sine function $\boldsymbol{h}$ is $|-1|$, simply because a frequency is positive.
And we can put $\boldsymbol{h}$ this way, too: $\boldsymbol{y}=\boldsymbol{h}(\boldsymbol{x})=-\textbf{sin}\ \boldsymbol{x}$ for $\boldsymbol{x}$ real, because $\textbf{sin}\ (-\boldsymbol{x})=-\textbf{sin}\ \boldsymbol{x}$.
How?

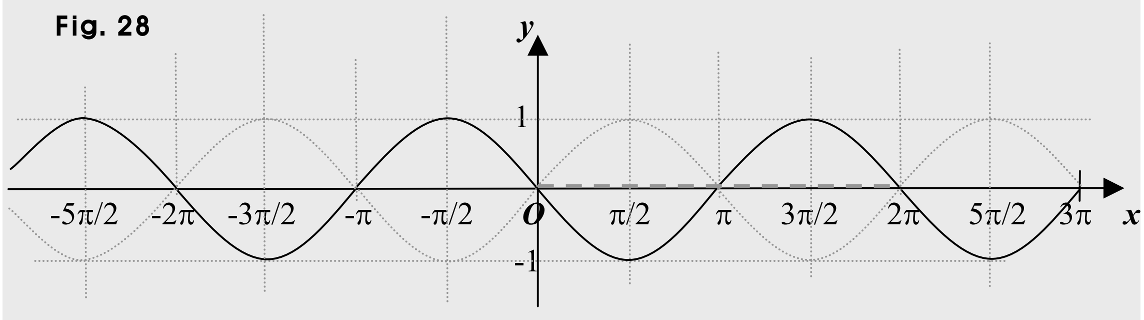
If the vector (hypotenuse) is of length 1, we can get this:

Fig. 27



So we get **sin (–*θ*) = –sin *θ***. And putting in the ***x-y*** plane, the curve of ***h***, we get this:

Fig. 28



And we can see that the period of ***h*** is 2π, too.
And next, what do we mean by the phase?

It can be called a horizontal shift, too. How?

Suppose first $\boldsymbol{y = F(x) = A\cdot \sin w(x + a) + b}$ for $\boldsymbol{x}$ real, where $\boldsymbol{A}$, $\boldsymbol{w}$, $\boldsymbol{a}$, and $\boldsymbol{b}$ are constant.
Then, $\boldsymbol{a}$ is called the phase, $|\boldsymbol{w}|$ is called the frequency, and $\dfrac{2\pi}{|\boldsymbol{w}|}$ is the period.

And next, assuming $\boldsymbol{y = Q(x) = A\cdot \sin wx + b}$ for $\boldsymbol{x}$ real, and shifting the curve of the function $\boldsymbol{Q(x)}$ by $\boldsymbol{-a}$ in the direction of the $\boldsymbol{x}$-axis, we get the curve of the function $\boldsymbol{F}$ specified above.

So the two curves themselves of ***F*** and ***Q*** are the same, and moving the curve of ***Q*** in the amount of ***–a*** along the ***x***-axis, we get the curve of ***F***.

In other words, the curve of ***Q*** gets moved to the left in the amount of ***a***.

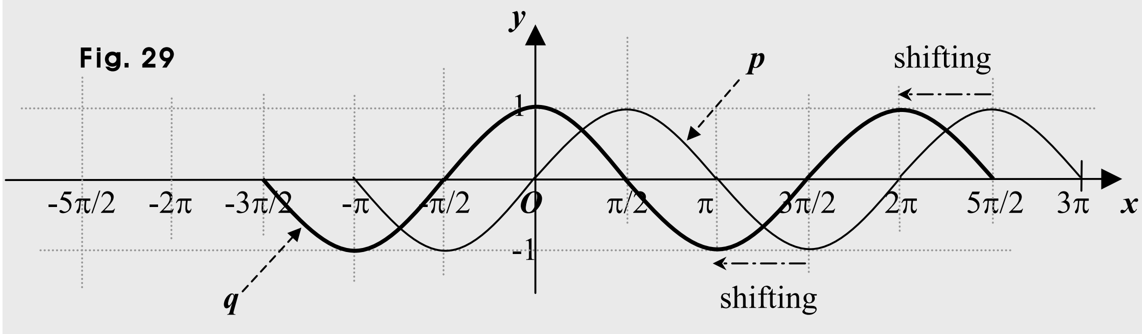
So if the phase ***a*** itself is positive, the curve gets shifted to the left, and if *negative*, the curve moves to the *right*.

Assuming for instance, shifting the curve of $\boldsymbol{p(x) = \sin x}$ for $\boldsymbol{-\pi \le x \le 3\pi}$ by $-\dfrac{\pi}{2}$ in the direction of the ***x***-axis, that is, to the left, we get the curve of a function below.

$$\boldsymbol{q(x) = \sin\left(x + \frac{\pi}{2}\right)} \text{ for } -\frac{3\pi}{2} \le \boldsymbol{x} \le \frac{5\pi}{2}.$$

So putting in a graph the two curves, that is, the curves of ***p*** and ***q***, we get this:

Fig. 29



Thus, the curve of ***p*** gets moved to the left in the amount of $\frac{\pi}{2}$. And the new curve is the curve of ***q***. So we can call the phase a *horizontal* shift, too.
What then about the constant ***b*** in $y = Q(x) = A\cdot\sin wx + b$?

It can be called a *vertical* shift.
So the constant ***b*** makes a curve move vertically and in the amount of ***b***.
In other words, it makes a curve move in the amount of ***b*** in the direction of the ***y***-axis.

So for instance, assuming $y = V(x) = A\cdot\sin w(x + a)$ for ***x*** real, and shifting the curve of the function $V(x)$ by ***b*** in the direction of the ***y***-axis, we can get the curve of the function

$$y = F(x) = A\cdot\sin w(x + a) + b \text{ for } x \text{ real.}$$

So the two curves themselves of ***F*** and ***V*** are the same, and moving the curve of ***V*** in the amount of ***b*** along the ***y***-axis, we get the curve of ***F***.

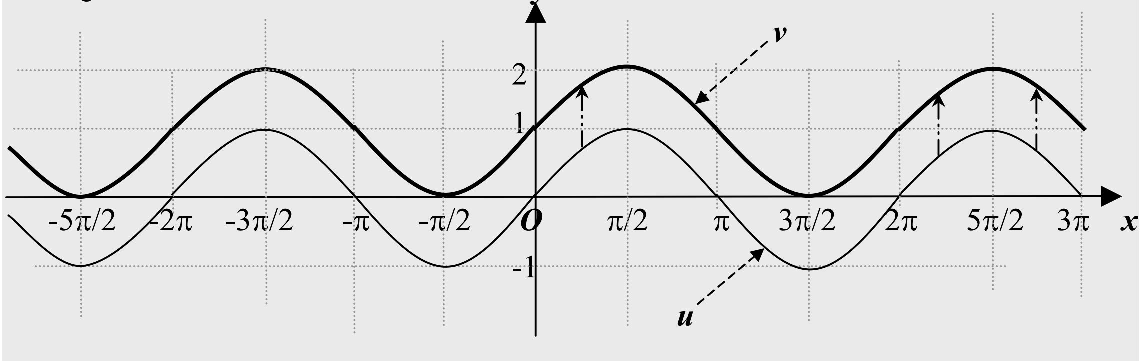
And if ***b*** is positive, the curve is shifted upward, and if negative, it moves downward.

Assuming for instance, shifting by 1 in the direction of the ***y***-axis, that is, upward, the curve of $u(x) = \sin x$ for ***x*** real, we get the curve of a function as follows.

$$v(x) = \sin x + 1 \text{ for } x \text{ real.}$$

Putting therefore, the two curves in a graph, we get the graph as follows.

Fig. 30



So the curve of $\boldsymbol{u}$ gets moved upward in the amount of 1.
And the new curve is the curve of $\boldsymbol{v}$. So we can call the $\boldsymbol{b}$ the amount of a vertical shift.
In short, $\boldsymbol{b}$ is a vertical shift.
And formally, we call such shifting a *transformation*, too. More specifically, it is called a *parallel transformation*, because it forces a curve to *translate* in a *parallel* manner.
If not sure of such a transformation, you may want to refer to the book **Function Transformations**.

And let's now put in a graph, for instance, the curve of a sine function as follows.

$$\boldsymbol{y = S(x) = -2\sin(-2x + \pi) + 1} \text{ for } \boldsymbol{x} \text{ real}$$

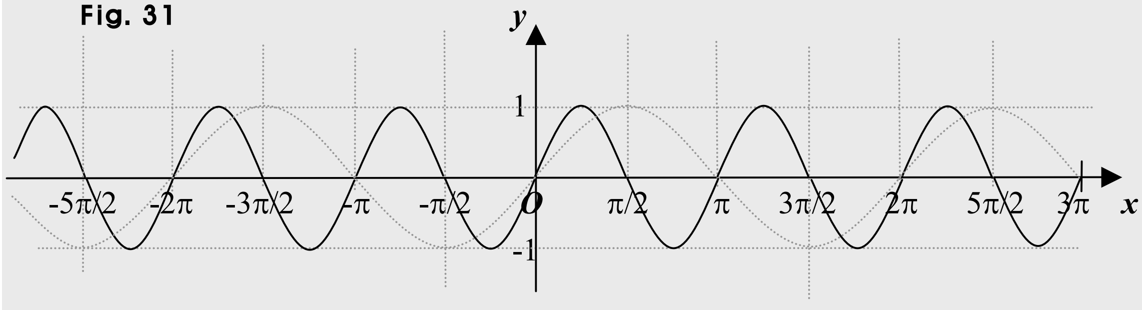
To begin with, we have $\mathbf{sin}\ (-x) = -\mathbf{sin}\ x$.
So we can get $\mathbf{sin}\ (-2x + \pi) = \mathbf{sin}\ \{-(2x - \pi)\} = -\mathbf{sin}\ (2x - \pi)$.
Thus, we get $-2\ \mathbf{sin}\ (-2x + \pi) + 1 = 2\ \mathbf{sin}\ (2x - \pi) + 1$.

And putting it in the general form, we get $\mathbf{2\ sin\ 2}(x - \frac{\pi}{2}) + 1$.

Then, we can now see that the amplitude is 2, the frequency is 2, the phase is $-\frac{\pi}{2}$, that is, the horizontal shift is $\frac{\pi}{2}$ to the left, and the vertical shift is 1 (upward).
And next, putting it in a graph, we may want to begin with the prototype, $\mathbf{sin}\ x$.
Then, setting first, the frequency to 2, we get $\mathbf{sin}\ 2x$. Note that $2x$ represents an angle.
We know that the period is the basic period (interval) divided by the frequency, and that the basic interval (period) for the sine is 2π. So we can see that the period is $\frac{2\pi}{2} = \pi$.

Thus, we can put the curve of **sin 2*x*** the way as follows.

Fig. 31

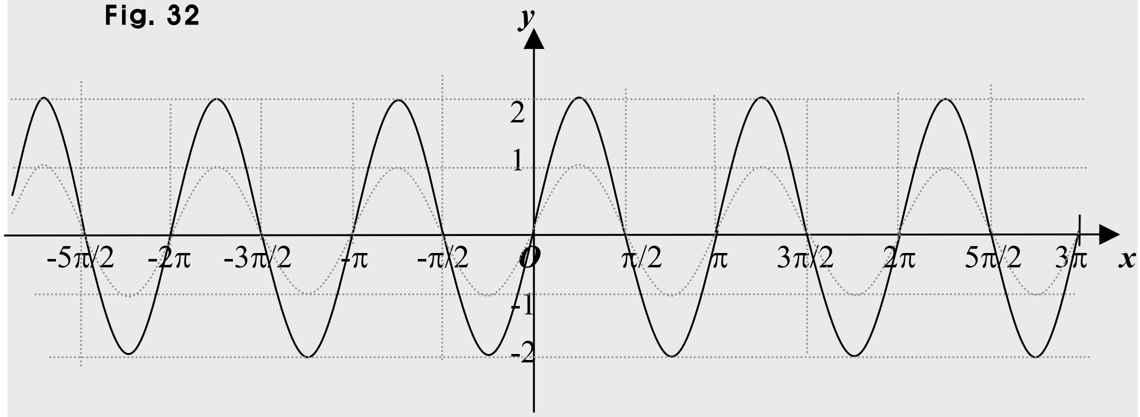


Next, setting the amplitude to 2, we get **2 sin 2*x***.

And the amplitude indicates half the width of the curve. So the half width is 2.

Thus, we can put in a graph, the curve of **2 sin 2*x*** the way as follows.

Fig. 32



Next, setting the phase to $-\frac{\pi}{2}$, we get **2 sin 2**$(x-\frac{\pi}{2})$.

We know shifting the curve with **sin *x*** by $-\frac{\pi}{2}$ in the direction of the ***x***-axis, we get the curve of **sin** $(x+\frac{\pi}{2})$.

So shifting the curve of **2 sin 2*x*** by not $-\frac{\pi}{2}$ but $\frac{\pi}{2}$ in the direction of the ***x***-axis, we get the curve of **2 sin 2**$(x-\frac{\pi}{2})$.

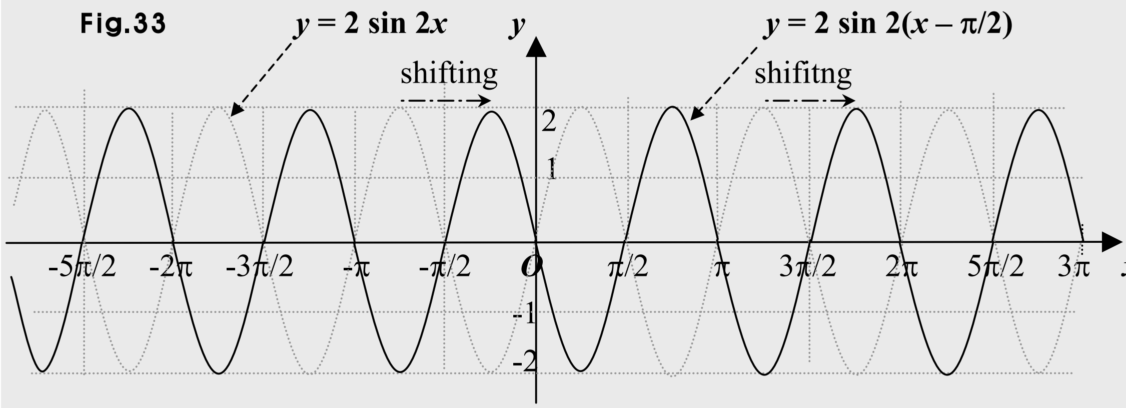
That is, moving the curve of **2sin 2*x*** in the amount of $\frac{\pi}{2}$ to the right, we get the curve of **2 sin 2(*x*** $-\frac{\pi}{2}$**)**. So we do a horizontal shifting to the right in the amount of $\frac{\pi}{2}$.

Note however, if the domain is a set of all real numbers, the curve of $\mathbf{2\sin 2}(\boldsymbol{x}-\frac{\pi}{2})$ is in fact, no other than the curve of $\mathbf{2\sin 2}(\boldsymbol{x}+\frac{\pi}{2})$.

If however, the phase is not a multiple of $\pm\frac{\pi}{2}$, the two curves are different.

Thus, we can put in a graph, the curve of $\mathbf{2\sin 2}(\boldsymbol{x}-\frac{\pi}{2})$ the way as follows.

Fig.33

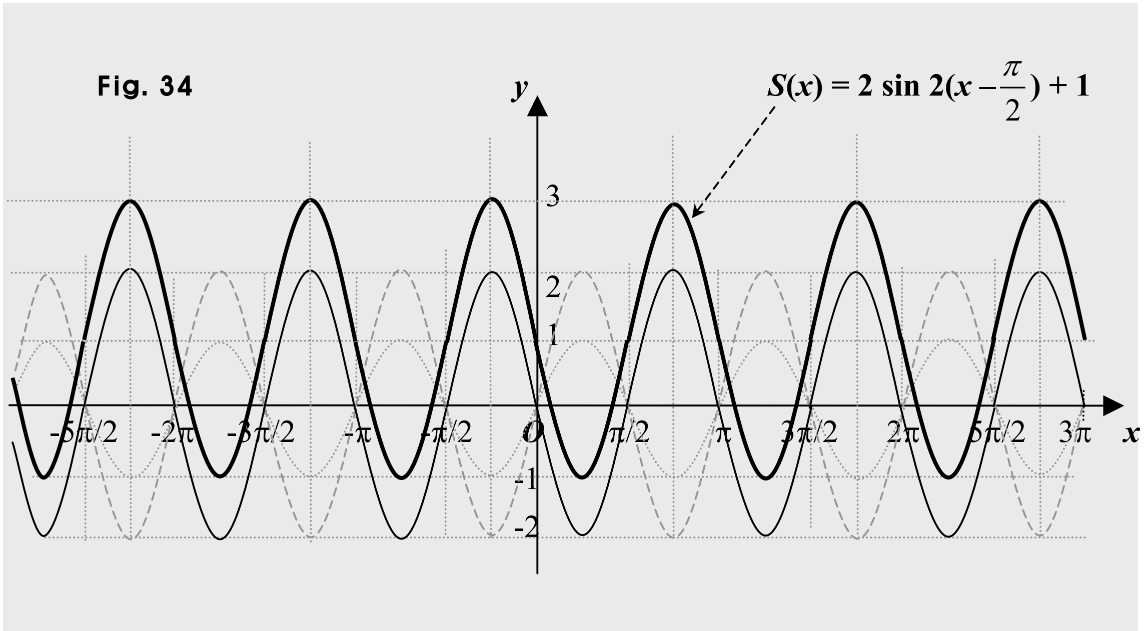


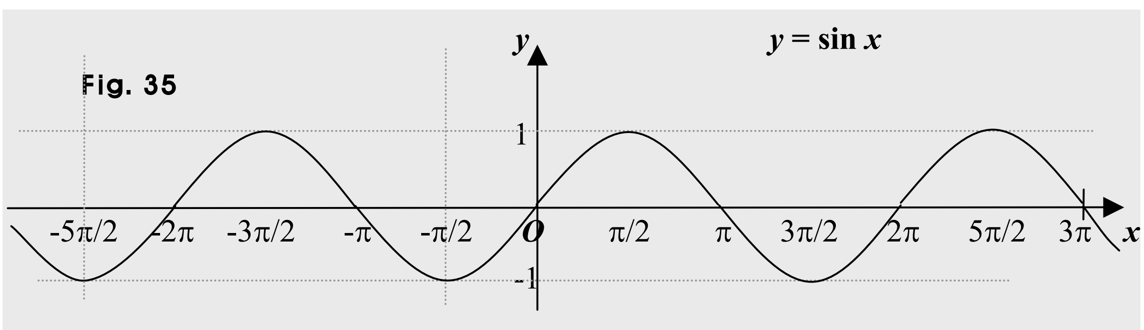
And next, setting the vertical shift to 1, we get $\mathbf{2\sin 2}(\boldsymbol{x}-\frac{\pi}{2}) + \mathbf{1}$.

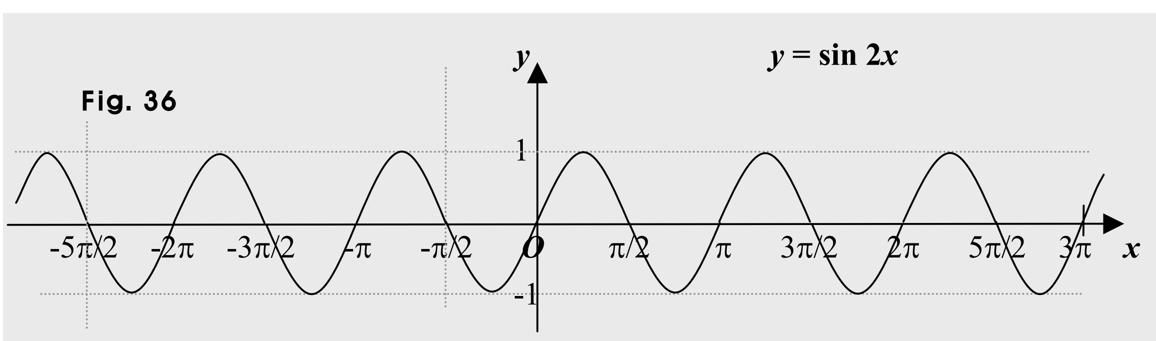
Then, we move upward the curve of $\mathbf{2\sin 2}(\boldsymbol{x}-\frac{\pi}{2})$ in the amount of 1 to get the curve of $\mathbf{2\sin 2}(\boldsymbol{x}-\frac{\pi}{2}) + \mathbf{1}$, which is the curve of the function ***S*** given.

Note that $\boldsymbol{S}(\boldsymbol{x}) = \mathbf{2\sin 2}(\boldsymbol{x}-\frac{\pi}{2}) + \mathbf{1} = \mathbf{-2\sin}(\mathbf{-2}\boldsymbol{x} + \boldsymbol{\pi}) + \mathbf{1}$.

Thus, we can put in a graph, the curve of the function ***S*** the way as follows.







Note that $\boldsymbol{y = S(x) = -2 \sin (-2x + \pi) + 1 = 2 \sin 2(x - \frac{\pi}{2}) + 1}$.

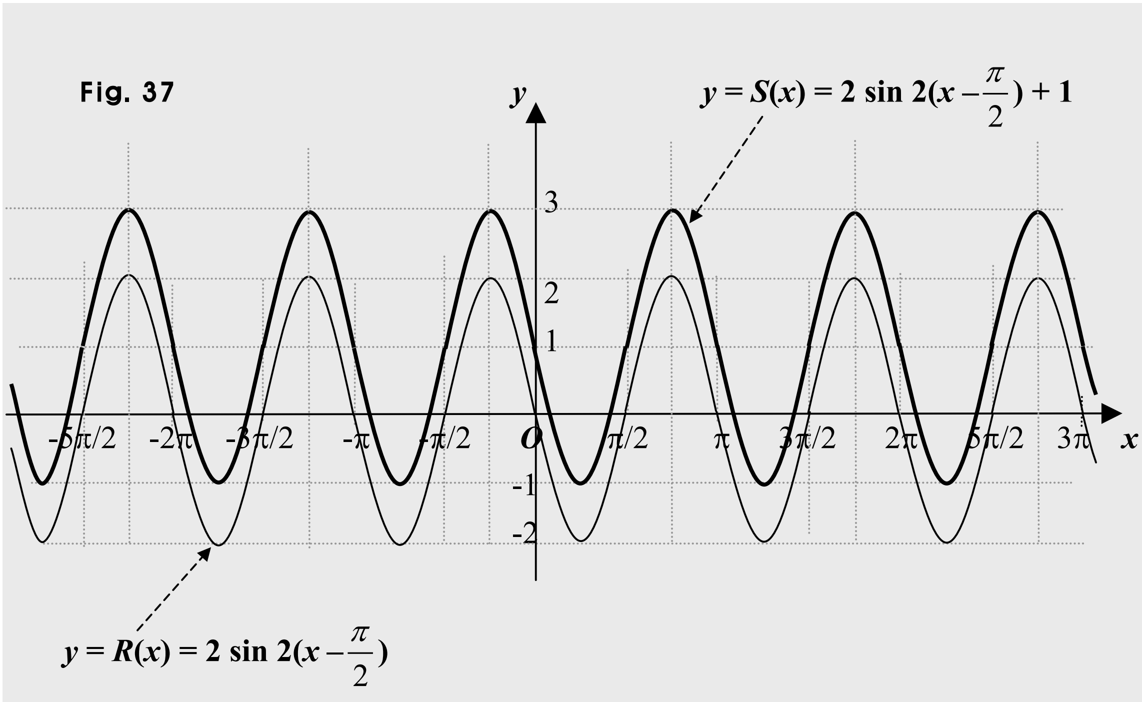


Fig. 37

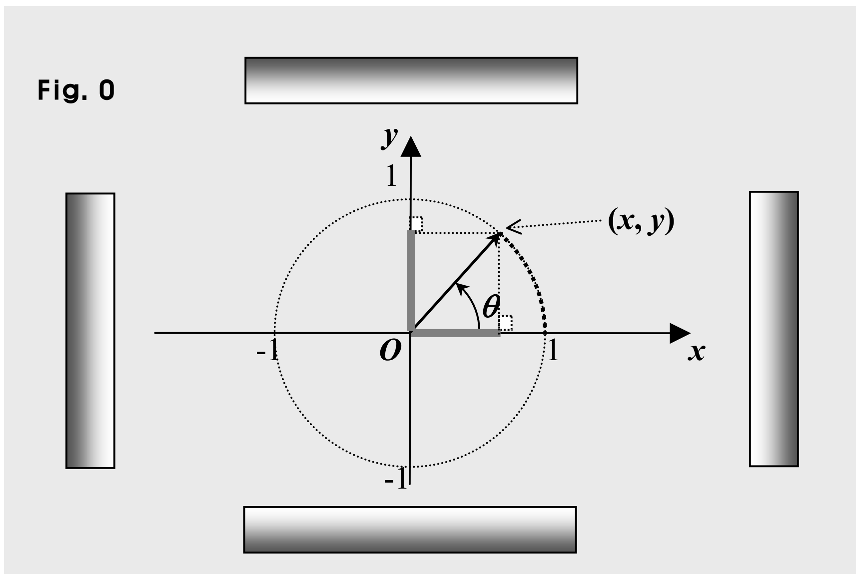
Note that not $\boldsymbol{x}$ but $\mathbf{2}(\boldsymbol{x} - \frac{\pi}{2})$ represnets an angle for the function ***S*** or ***R***.

What then about $\boldsymbol{x}$?

It's the input variable, and can take all real numbers in the funciton ***S*** or ***R***, so the domain of ***S*** or ***R*** is a set of all real numbers. The numbers though, in the domain do not need to be angles. So $\boldsymbol{x}$ can represent an angle, too, but if it does, the angle is not for the function ***S*** or ***R***, but for something else such as an amount of turning of a planet abuot its axis of rotation.

We can use the input variable $\boldsymbol{x}$ for many purposes, so it can reptesent time, weight, distance, area, volume, and thus, anything that can change its value or amount.

5. Cosine Functions



Let's first again, quickly go over the basics of trigonometric functions, and then, move on to the details on cosine functions. So what is a trigonometric function?

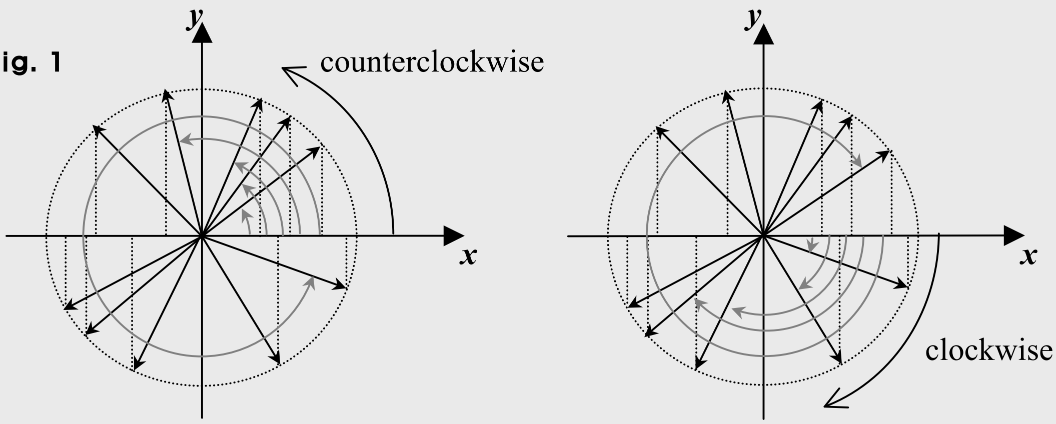
It is often called a trig function and is a function of which each output is a trig ratio called the sine, the cosine, or the tangent. For instance, putting a cosine function in the ***x-y*** coordinate system, we can put it the way as follows.

$\boldsymbol{y = f(x) = \cos x}$, where $\boldsymbol{x}$ is a real number indicating an angle.
In short, $\boldsymbol{y = f(x) = \cos x}$, where $\boldsymbol{x}$ is real.

Therefore, the cosine function $\boldsymbol{f(x)}$ above is a function of an angle, so takes an angle as an input, and produces a trig ratio called the cosine as an output.
Specifically, in $\boldsymbol{y = f(x) = \cos x}$, the input variable $\boldsymbol{x}$ takes an angle, the expression $\boldsymbol{\cos x}$ produces an output called the cosine, and the output variable $\boldsymbol{y}$ gets the cosine.
Also, the same is true for a sine function such as $\boldsymbol{y = g(x) = \sin x}$, and for a tangent function such as $\boldsymbol{y = h(x) = \tan x}$.

And the angle is made by the vector turning about the origin in the ***x-y*** plane.

Fig. 1



Either counterclockwise or clockwise, in ***one way***, the vector keeps turning about the origin in the ***x-y*** plane, and keeps making angles. Each and every moment, it makes an angle, because an amount of turning is an angle. Initially, the vector is at rest sitting on the ***x***-axis on the right of the origin. Then, the angle made is 0 in radians, i.e., 0^{o}.

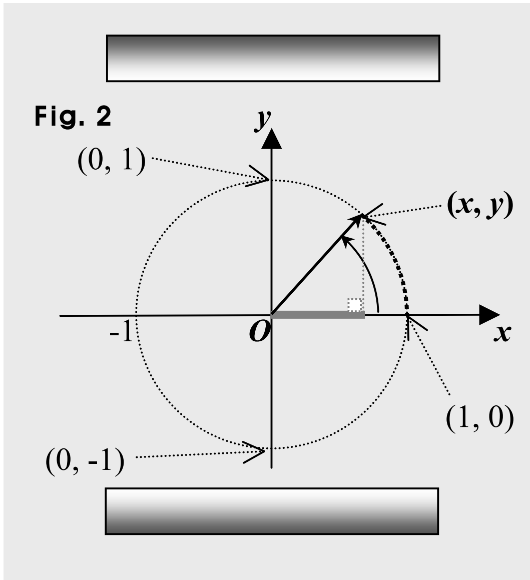
And we can have the vector make as much turn as we need or as many complete turns as necessary. If the vector starts and keeps turning counterclockwise, the angle made is positive every moment. Or if the vector starts and keeps turning clockwise, the angle made is negative every moment.
So assuming θ is the angle made and putting together all the angles can be made, we get

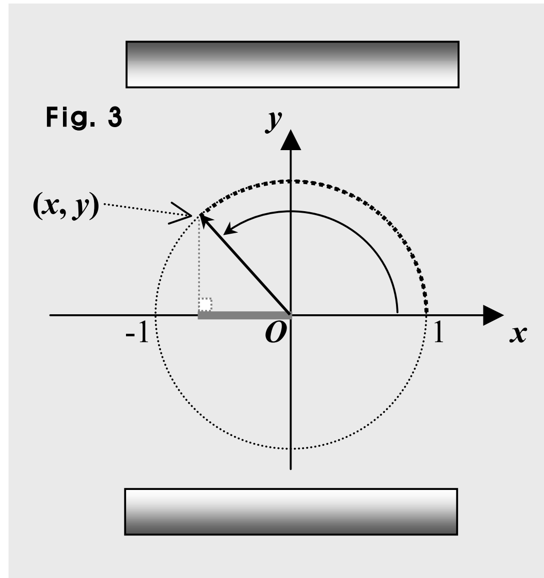
$$-360^{o}\cdot n \le \theta \le 360^{o}\cdot n \text{ in degrees or } -2n\pi \le \theta \le 2n\pi \text{ in radians,}$$

where π is 3.141592…, which is real, and n is an integer ≥ 0, so $n = 0, 1, 2, \ldots$ So we can get $-\infty < \theta < \infty$, where ∞ indicates infinity.
Thus, θ can be all real numbers, and every input is a real number.

Suppose now again, in the ***x-y*** plane where a vector of length 1 is turning ***counterclockwise*** about the origin, we place two lamps the way in Fig. 2 below so that the shadow of the vector is made on the ***x***-axis. And let's call the shadow the projection, and call the value of the projection just the projection, too.

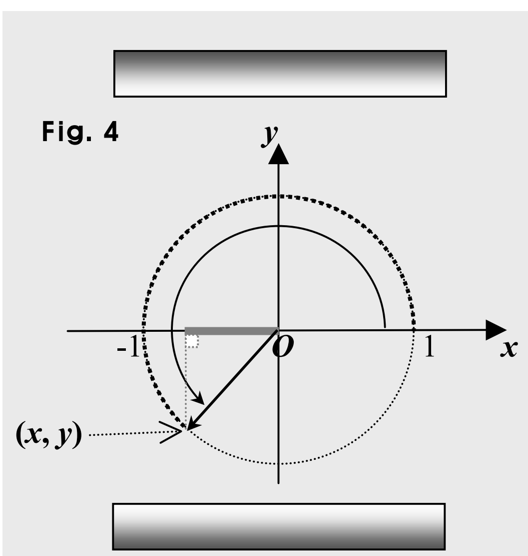
Also, if the projecion is on the ***left*** of the origin on the ***x***-axis, or is ***below*** the origin on the ***y***-axis, the projection is ***negative***.

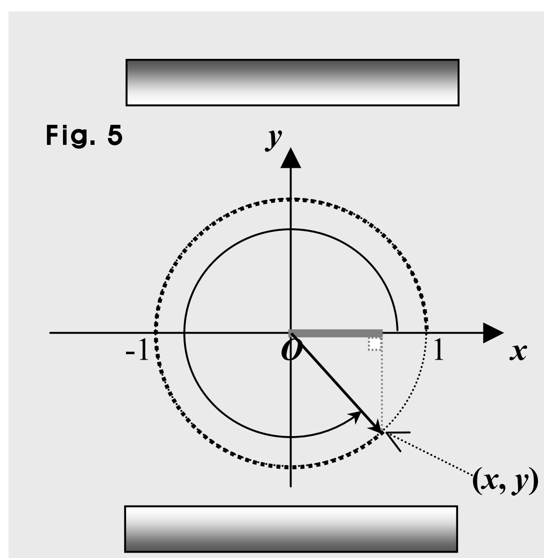




So first, if the vector is above the ***x***-axis while turning, we can see a linear motion where as in Fig. 2, the projection on the ***x***-axis shrinks changing its value from 1 to 0 as the vector approaches the ***y***-axis, and then as in Fig. 3, expands changing its value from 0 to –1 as the vector approaches the ***x***-axis.

Next, if the vector is below the ***x***-axis while turning, we can see another linear motion where as in Fig. 4, the projection on the ***x***-axis shrinks changing its value from –1 to 0 as the vector approaches the ***y***-axis, and then as in Fig. 5, expands changing its value from 0 to 1 as the vector approaches the ***x***-axis.



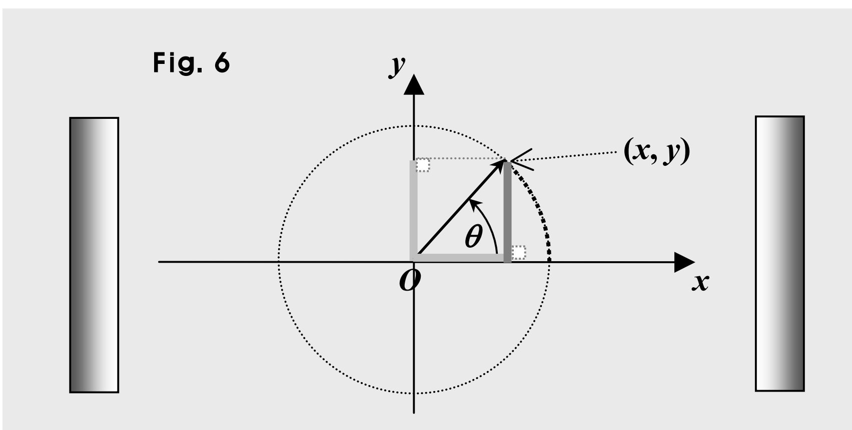


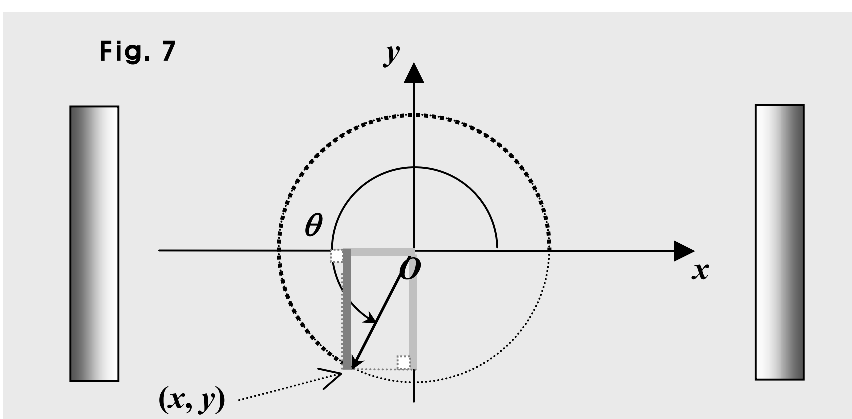
Also, we can see that such linear motions keep repeating as the vector keeps turning. So now, what are we going to do with those projections?

We are going to come up with trig functions and their graphs.

At every moment, assuming in Fig. 6 and Fig. 7 below, the |vector| is hypotenuse, we get a right triangle where the adjacent is the projection on the ***x***-axis, and the opposite is the projection on the ***y***-axis.

Notice that in Fig. 6 and Fig. 7, the line segment in dark gray is the same in size and length as the projection in light gray on the ***y***-axis. Thus, we can use the projection on the ***y***-axis as the opposite.





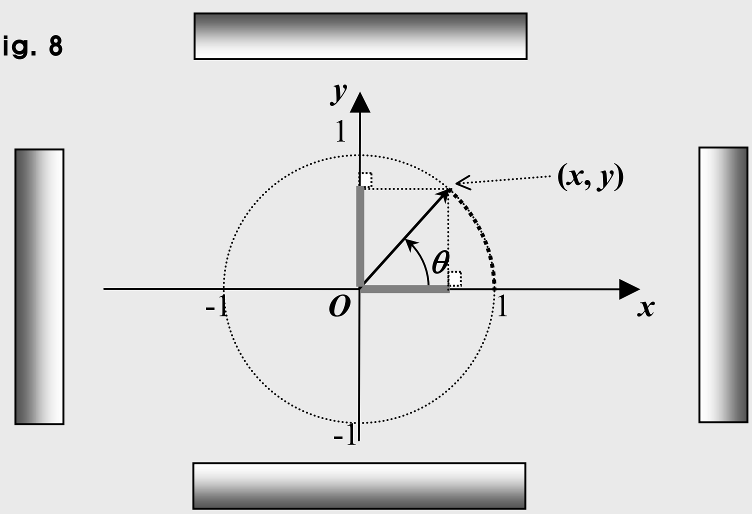
And depending on the angle θ made by the vector, the projections get determined. In short, the angle made determines the projections. So what makes such a right triangle is the |vector| and the angle θ made by the vector.
And we have the statement as follows.

In coordinate trigonometry,
the x-coordinate at the terminal point is the adjacent,
the y-coordinate is the opposite,
the |vector| is the hypotenuse,
and the vector turned makes the angle.

So we can say that of the terminal point, the x-coordinate is the projection on the x-axis, and the y-coordinate is the projection on the y-axis.

And if (x, y) is the terminal point, and the |vector| is 1, then since the hypotenuse is 1, the adjacent is x, and the opposite is y, we can say that the cosine is just x for any angle θ, i.e., **cos** $\theta = x$, and that the sine is just y for any angle θ, that is, **sin** $\theta = y$.

Fig. 8



Thus, if the |vector| is 1, for any angle, the projection on the x-axis is the value of the cosine itself, and the projection on the y-axis is the value of the sine itself, so in short,

the projection on the x-axis is **cos** θ, and the projection on the y-axis is **sin** θ.

What then about the tangent?

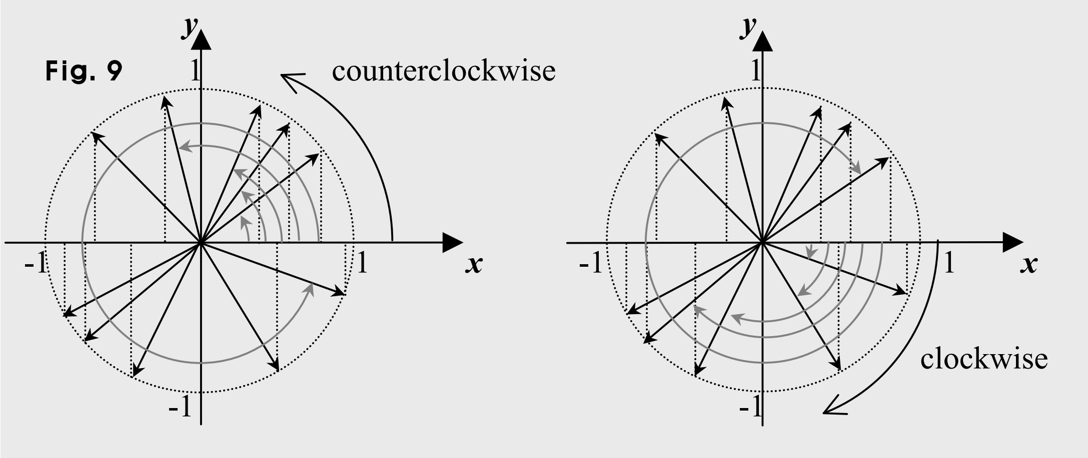
The tangent is $\frac{y}{x}$, since the tangent is the opposite over the adjacent. So we can say that the tangent is $\frac{\text{the sine}}{\text{the cosine}}$, since $\mathbf{cos}\ \boldsymbol{\theta} = \boldsymbol{x}$, and $\mathbf{sin}\ \boldsymbol{\theta} = \boldsymbol{y}$.

Thus, in short, $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.

Now, in coordinate trigonometry, using those projections, along with the angle $\boldsymbol{\theta}$ stated above, we can come up with the three basic trig functions, along with their graphs.

Of each of the basic three functions, each input is an angle, which is made by the vector turning in the ***x*-*y*** plane.

Fig. 9



So in the cosine function ***y*** = ***f***(***x***) = **cos *x***, the input variable ***x*** gets an angle at a time.

Usually though, we just use real numbers as 0 or π as angles, since angles are in radian.

So assuming ***f*** is a cosine function, its expression is **cos *x***, and the domain is a set of all angles, we say that the domain is a set of all real numbers,

and can put the cosine function ***f*** the way as follows. ***y*** = ***f***(***x***) = **cos *x*** for ***x*** real,
or simply this way, too: ***y*** = ***f***(***x***) = **cos *x***.

What then about outputs?

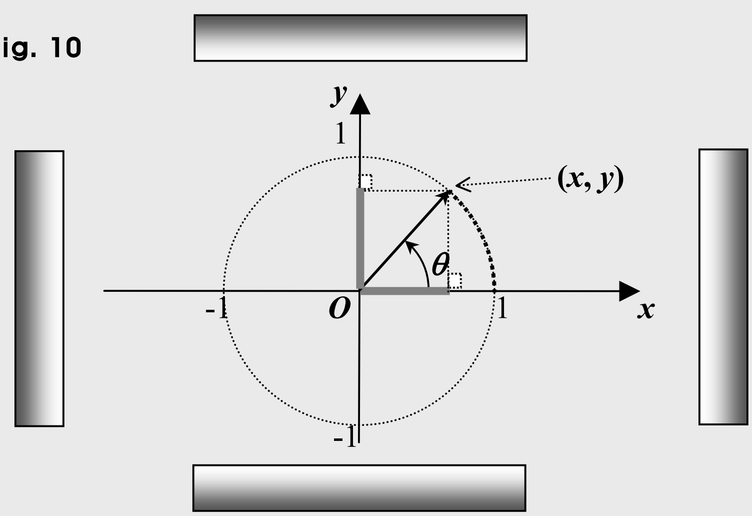
They are trig ratios called the cosines,
because in the cosine function $\boldsymbol{y = f(x) = \cos x}$, the $\boldsymbol{x}$ gets an angle,
so each value of $\boldsymbol{\cos x}$, that is, each value of $\boldsymbol{f(x)}$ is a trig ratio called the cosine,
the ratio of the adjacent to the hypotenuse, which is a real number.

What then about the range?

The range is a set of all real numbers from –1 to 1, and each of the numbers is a trig ratio,
which is an output, which is the value of $\boldsymbol{f(x)}$, which is the value of $\boldsymbol{y}$.
So the range of the cosine function $\boldsymbol{f}$ can be put this way, too: $\boldsymbol{-1 \le y \le 1}$, or $\boldsymbol{|y| \le 1}$.

Also, the same is true for a sine function such as $\boldsymbol{y = g(x) = \sin x}$.

Fig. 10



If the |vector| = 1. the cosine is $\boldsymbol{x}$ at $\boldsymbol{(x, y)}$, and the sine is $\boldsymbol{y}$ at $\boldsymbol{(x, y)}$.
That is, if the |vector| = 1, $\boldsymbol{\cos\theta}$ = the projection on the $\boldsymbol{x}$-axis, and $\boldsymbol{\sin\theta}$ = the projection on the $\boldsymbol{y}$-axis.

What then are the maximum and the minimum of the $\boldsymbol{x}$-value and the $\boldsymbol{y}$-value at $\boldsymbol{(x, y)}$?

The maximum is 1, and the minimum is –1.
If the |vector| is 1, the projection on the x-axis covers all the numbers from –1 to 1, because the radius of the circle the terminal point traces is 1. And the same is true for the projection on the y-axis, too.

For any angle θ, assuming the |vector| is 1, that is, the hypotenuse is 1, and the terminal point is (x, y), we get **cos** $\theta = x$, which is of course, the projection on the x-axis, and we get **sin** $\theta = y$, which is of course, the projection on the y-axis.
Now, as the angle θ changes from 0 to 2π, what values does **cos** θ get?

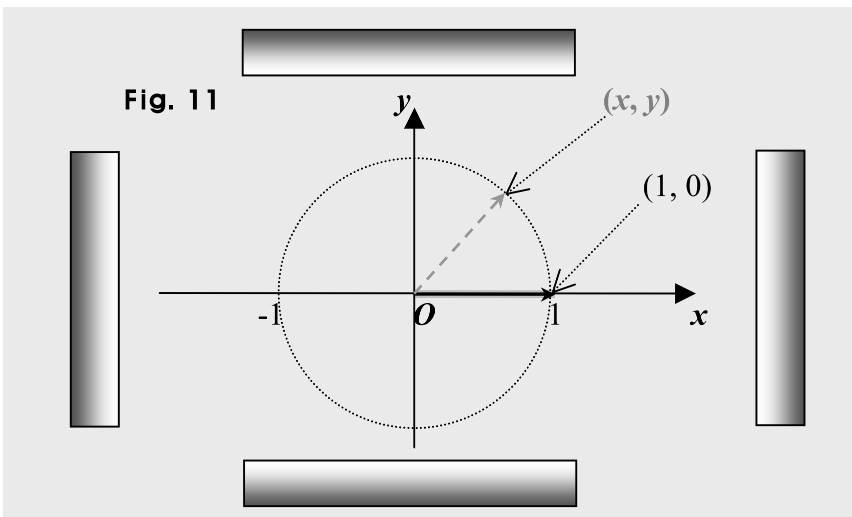


Fig. 11

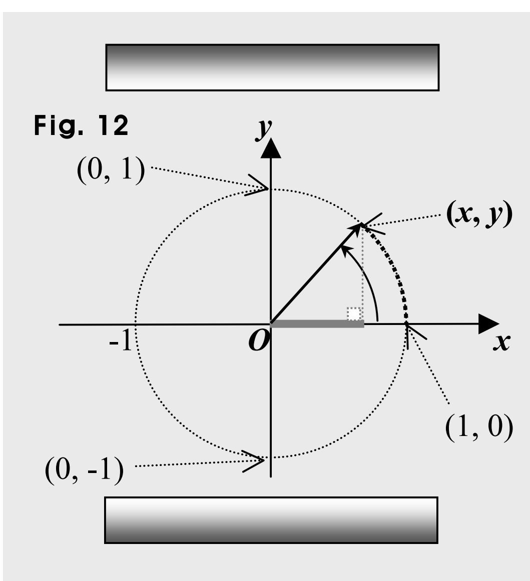


Fig. 12

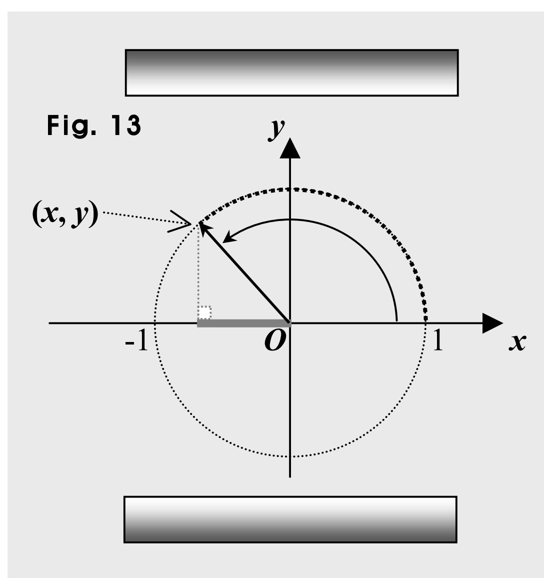
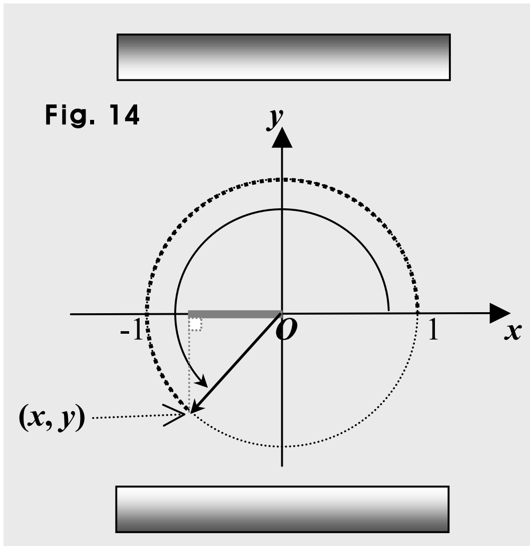
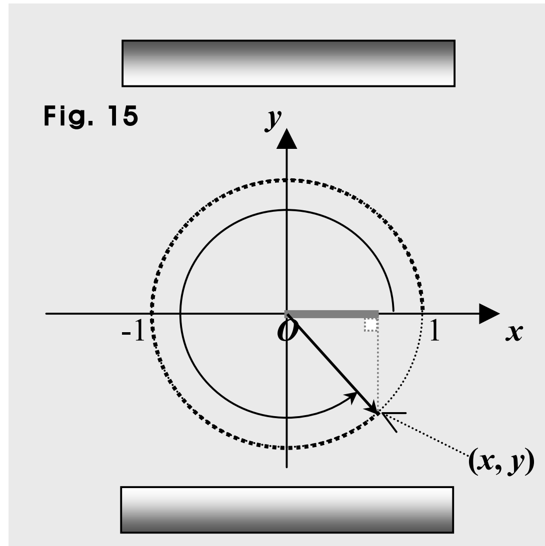


Fig. 13





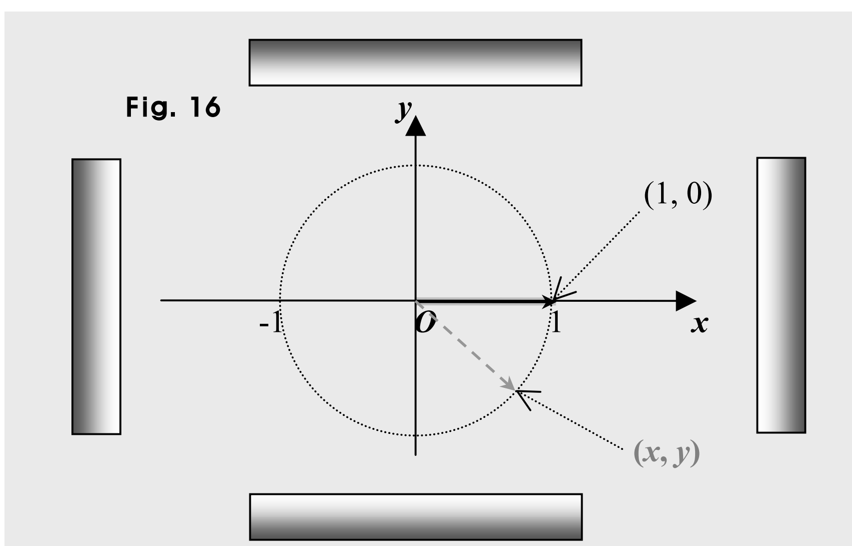
If the |vector| is 1, the projection on the x-axis covers all the numbers from –1 to 1, so the value of **cos** θ covers all the numbers from –1 to 1.

So if $0 \le \theta \le 2\pi$, we get $-1 \le \cos\theta \le 1$, that is, $|\cos\theta| \le 1$.
If $2\pi \le \theta \le 4\pi$, we get $-1 \le \cos\theta \le 1$, too, that is, $|\cos\theta| \le 1$.

Thus, if $0 \le \theta \le 2n\pi$, where n is an integer ≥ 1, we get $|\cos\theta| \le 1$.

That is, no matter how many complete turns the vector may make, we get $|\cos\theta| \le 1$.

Suppose now again, the vector of length 1 is at rest sitting on the x-axis on the right of the origin. Suppose this time, the vector starts turning clockwise.
Then, we can see the projection on the x-axis starts shrinking from 1.



So first, if the vector is below the ***x***-axis while turning, we can see a linear motion where as in Fig. 17, the projection on the ***x***-axis shrinks changing its value from 1 to 0 as the vector approaches the ***y***-axis, and then as in Fig. 18, expands changing its value from 0 to –1 as the vector approaches the ***x***-axis.

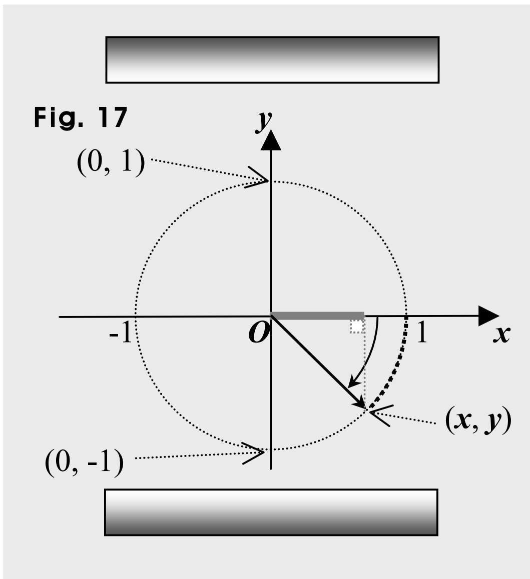


Fig. 17

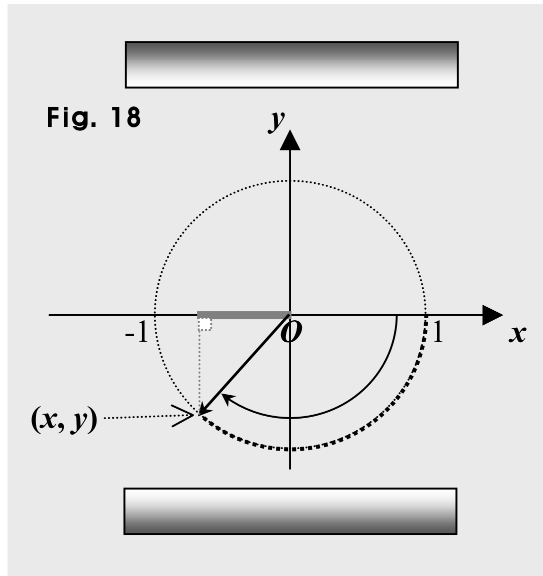


Fig. 18

And next, if the vector is above the ***x***-axis while turning, we can see another linear motion where as in Fig. 19, the projection on the ***y***-axis shrinks changing its value from –1 to 0 as the vector approaches the ***y***-axis, and then as in Fig. 20, expands changing its value from 0 to 1 as the vector approaches the ***x***-axis.

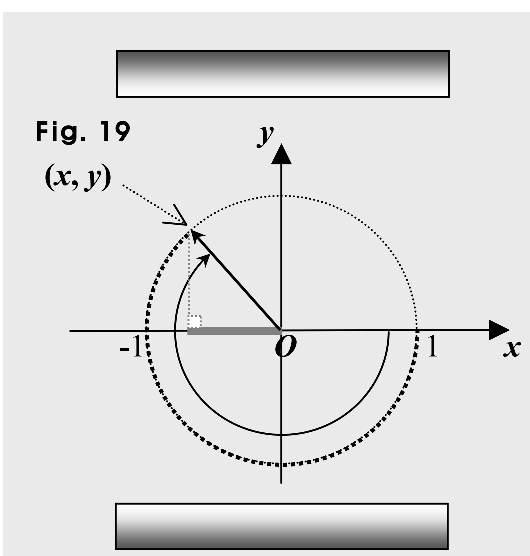


Fig. 19

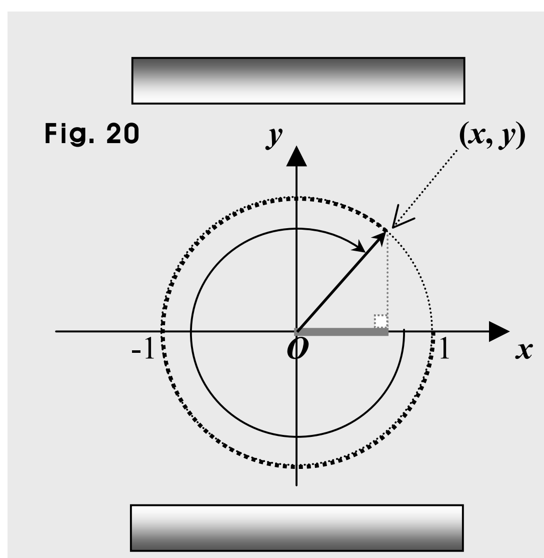


Fig. 20

Then, we can see that the maximum of the $\boldsymbol{x}$-value at $\boldsymbol{(x, y)}$ is 1, and the minimum is –1, since the terminal point makes a circle of radius 1 centered at the origin if the vector makes a complete turn.
So now, as the angle θ changes from 0 to -2π, what values does **cos** θ get?

If the |vector| is 1, the projection on the $\boldsymbol{y}$-axis covers all the numbers from –1 to 1, so the value of **cos** θ covers all the numbers from –1 to 1.

So if $\mathbf{-2\pi \le \theta \le 0}$, we get $\mathbf{-1 \le cos\ \theta \le 1}$, that is, $\mathbf{|cos\ \theta| \le 1}$.
If $\mathbf{-4\pi \le \theta \le -2\pi}$, we get $\mathbf{-1 \le cos\ \theta \le 1}$, too, that is, $\mathbf{|cos\ \theta| \le 1}$.

Thus, if $\mathbf{2n\pi \le \theta \le 0}$, where $\boldsymbol{n}$ is an integer $\mathbf{\le -1}$, we get $\mathbf{|cos\ \theta| \le 1}$.

So now, putting threads together, we can say that if the |vector| is 1, and $\boldsymbol{(x, y)}$ is the terminal point, we get $\mathbf{cos}\ \theta = \boldsymbol{x}$ for any angle θ, and that for $\mathbf{-2n\pi \le \theta \le 2n\pi}$, where $\boldsymbol{n}$ is an integer $\mathbf{\ge 1}$, we get $\mathbf{|cos\ \theta| \le 1}$.

And the same is true for **sin** θ, also.
So if $\mathbf{-2n\pi \le \theta \le 2n\pi}$, where $\boldsymbol{n}$ is an integer $\mathbf{\ge 1}$, we get $\mathbf{|sin\ \theta| \le 1}$.

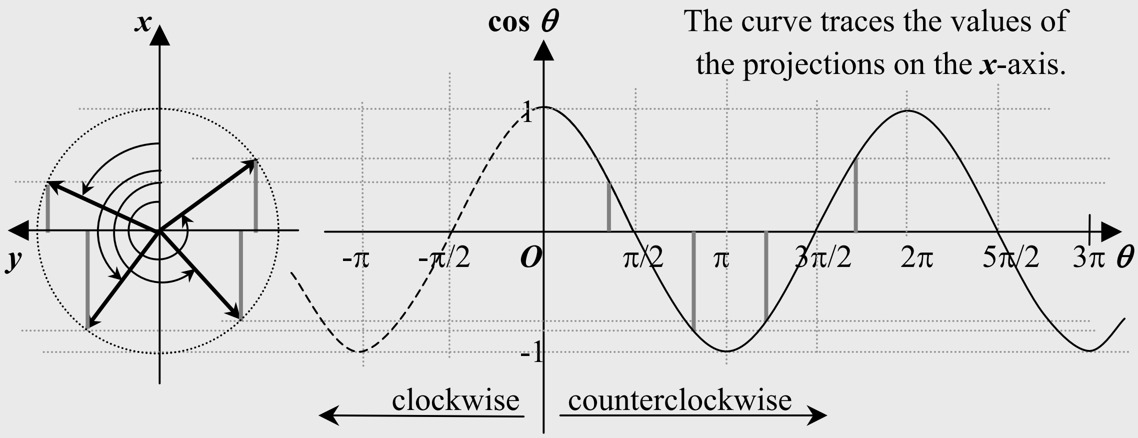
Thus, using the facts above, we can form a function of the angle θ. How?

As the angle θ changes, the cosine trig ratio **cos** θ changes. So we can take as an input each angle that θ gets, and take as an output each trig ratio that **cos** θ produces.

Assuming thus, the function is $\boldsymbol{f}$, the domain is a set of all angles, and $\boldsymbol{c}$ is the output variable, we can set $\boldsymbol{c = f(\theta)} = \mathbf{cos}\ \theta$. And we call the function $\boldsymbol{f}$ a cosine function.
What then about the curve of the cosine function $\boldsymbol{c = f(\theta)} = \mathbf{cos}\ \theta$?

Assuming the |vector| is 1, and the terminal point is $\boldsymbol{(x, y)}$, we get $\mathbf{cos}\ \theta = \boldsymbol{x}$, which is the projection on the $\boldsymbol{x}$-axis. In short, the projection on the $\boldsymbol{x}$-axis is **cos** θ.
So this time, rotating by $\frac{\pi}{2}$ counterclockwise about the origin, the $\boldsymbol{x}$-$\boldsymbol{y}$ plane where the vector is turning, we can get the curve of the cosine function $\boldsymbol{f(\theta)}$ the way as follows.

Fig. 21

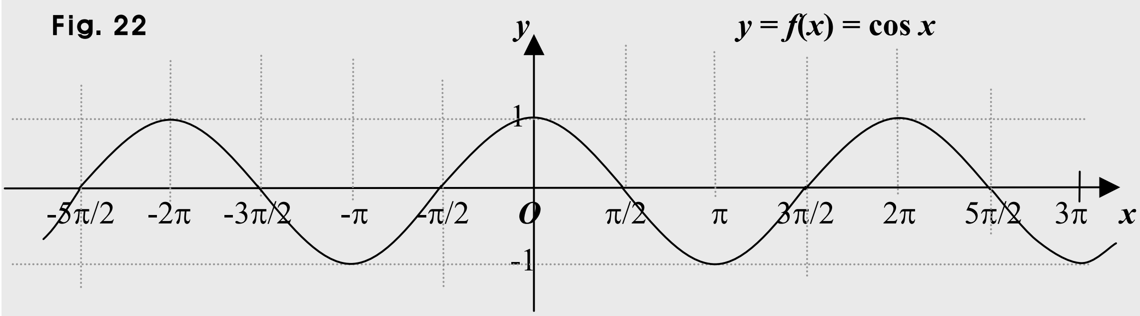


And of course, we can put in the ***x-y*** system the function $\boldsymbol{c = f(\theta) = \cos\theta}$. How?

Replacing ***c*** with ***y***, and ***θ*** with ***x***, we get $\boldsymbol{y = f(x) = \cos x}$, which is now in the ***x-y*** system. And we can use numbers as angles. Thus, if the domain is a set of all angles, we say that the domain is a set of all real numbers, and we can just set $\boldsymbol{y = f(x) = \cos x}$ for ***x*** real. What if we just set $\boldsymbol{y = f(x) = \cos x}$? That is, what if we set it without the domain?

The domain is assumed to be a set all real numbers, since $\boldsymbol{f(x)}$ can be defined for all real numbers. If no domain is specified in the function definition, we just assume the largest possible domain.
And we can put in the ***x-y*** plane, the curve of the cosine function $\boldsymbol{y = f(x) = \cos x}$ the way as follows.

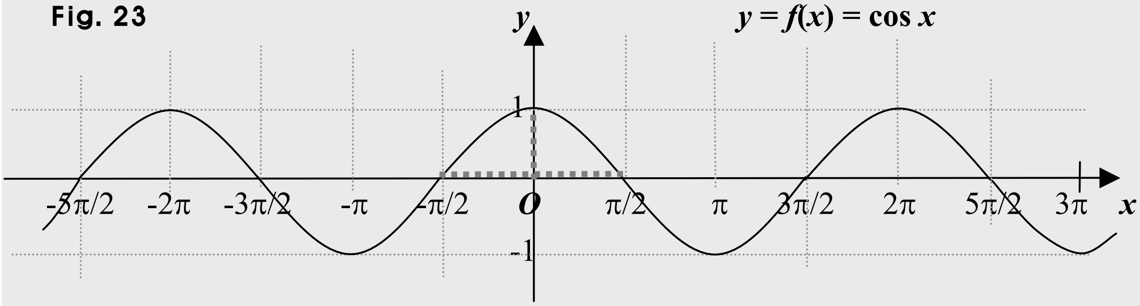
Fig. 22



The curve in Fig. 22 above is called the cosine curve, and is no other than the curve in Fig. 21 above. And we know that the output can be every value from –1 to 1.
So the range is $\boldsymbol{-1 \le y \le 1}$, that is, $\boldsymbol{|y| \le 1}$.

And of course, if the domain is not a set of all real numbers, the range can be other than the one above. Suppose for instance, $y = g(x) = \cos x$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

Fig. 23



Then, as we can see in the graph above, the range is $0 \le y \le 1$.

And the cosine function $y = f(x) = \cos x$ can be called the prototype, and is thus, in the most basic form. Assuming F is a cosine function, too, where the domain is a set of all real numbers, and using a general form, we can put it the way below.

> $y = F(x) = A \cdot \cos \{w(x + a)\} + b$ for x real, where A, w, a, and b are constant.

Note that $w(x + a)$ represents an angle.

And we can just put it this way, too: $y = F(x) = A \cdot \cos w(x + a) + b$ for x real.

Or more simply, this way, too: $y = F(x) = A \cdot \cos w(x + a) + b$, since in this case, we can assume that A, w, a, and b are constant.

Then, the range is a set of all numbers from $-|A| + b$ to $|A| + b$.

So the range of F can be put this way, too: $-|A| + b \le y \le |A| + b$.

That is, the curve of F is bounded by the interval where $-|A| + b \le y \le |A| + b$.

Then, $2|A|$ is the width of the curve (or wave), and $|A|$ is called the amplitude.
So the amplitude indicates half the width of the curve (or wave).
What then is the amplitude of the cosine function $y = f(x) = \cos x$?

Since the curve of f is bounded by the interval where $-1 \le y \le 1$, the amplitude is 1.

And $|w|$ is called the frequency, $\dfrac{2\pi}{|w|}$ is the period, and a is called the phase.

So in the function $y = f(x) = \cos x$, the phase is 0, and the frequency is 1, so the period is 2π (i.e., 360°). How though?

The prototype in cosine functions can be put this way: $y = f(x) = \cos x$.

And using a general form, we can put a cosine function called F the way as follows.

$y = F(x) = A{\cdot}\cos\{w(x + a)\} + b$ for x real, where A, w, a, and b are constant.

Then, a is called the phase, $|w|$ is called the frequency, and $\dfrac{2\pi}{|w|}$ is the period.

And we can put the prototype into the general form the way as follows.

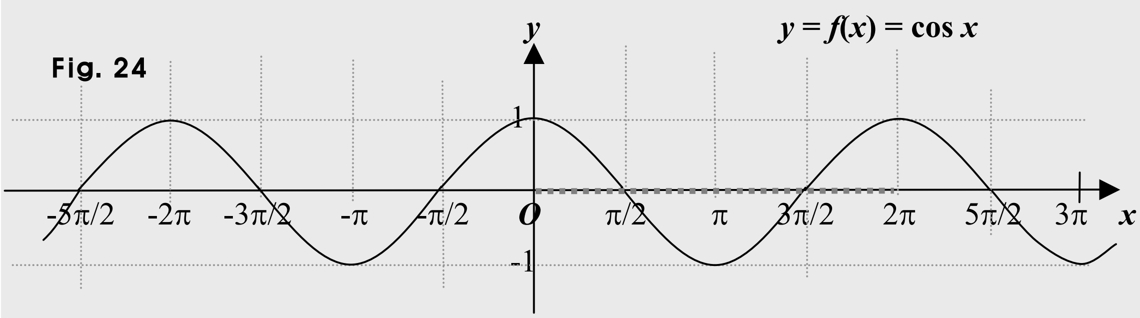
$$y = f(x) = 1{\cdot}\cos 1(x + 0) + 0$$

Thus, the phase a is 0, and the frequency $|w|$ is 1, so the period $\dfrac{2\pi}{|w|}$ is 2π.

What do we mean by the period though?

Putting in the x-y plane, the curve of the cosine function f above, we can get this:

Fig. 24



Then, we can see that a part of the cuve repleats itself.

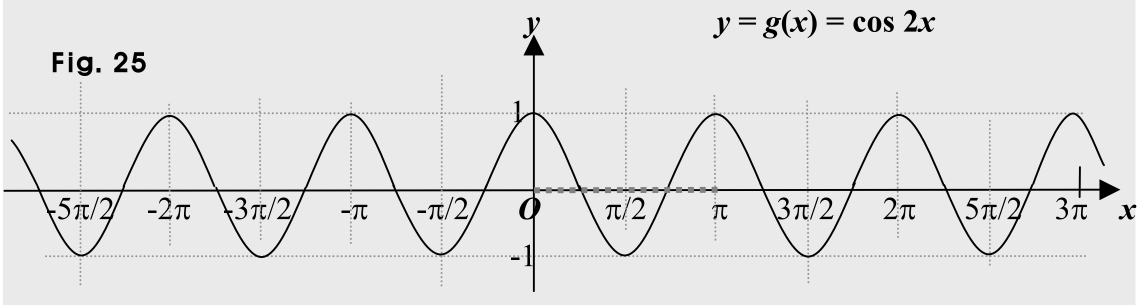
And the part is in fact, the smallest part repeating itself. What then is the part?

It is the part from 0 to 2π. So the part repeats itself every 2π interval. We call such an interval a period. So the interval 2π is the period in the cosine function $y = f(x) = \cos x$.

Thus, we call a cosine function a *periodic* function. And of course, the same is true, too, for sine functions, tangent functions, and other trig functions as cosecant functions.
The cosine function ***f*** above is the prototype, and thus, is in the most basic form.
So the period **2π** can be called the *basic period* in cosine functions. In other words, the interval **2π** can be called the *basic interval* in cosine functions.

And for instance, in the general form, setting ***w*** to 2, ***a*** and ***b*** to 0 each, and ***A*** to 1, we get a new function where: $y = g(x) = \cos 2x$ for x real.
Then, putting in the ***x-y*** plane, the curve of the cosine function ***g*** above, we get this:

Fig. 25



Then again, we can see that a part of the cuve repleats itself.
And the part is of course, the smallest part repeating itself. What then is the part?

It is the part from 0 to π. So the part repeats itself every π interval.
And we call such an interval a period. So the interval π is the period in the function ***g***.
How many times does the smallest part stated above appear in the basic interval 2π?

It appears twice in the basic interval 2π, which is the basic period. Then, we say the frequency is 2 in the function ***g***. What then do we mean by the frequency?

It is the number of times the smallest part repeating itself appears in the basic inverval 2π, which is the basic period. In short, the frequency is the number of times the smallest part appears in 2π. More precisely though, we want to put the frequency the way below.

In case of cosine functions, the frequency is the number of times the samllest part repeating itself appears in the basic period (or the basic interval), which is 2π.

And the same is true for sine functions, too. So the basic period (interval) in sine functions is 2π, too. The basic period however, in tangent functions is not 2π but π.
So in general, we can put the frequency the way as follows.

The frequency is, of the curve of a trig function, the number of times the smallest part repeating itself appears in its basic period or its basic interval.
And in the general form $\boldsymbol{A \cdot \cos w(x + a) + b}$, $\boldsymbol{|w|}$ is the frequency.
Then, given a cosine function in the general form, how can we get its period?

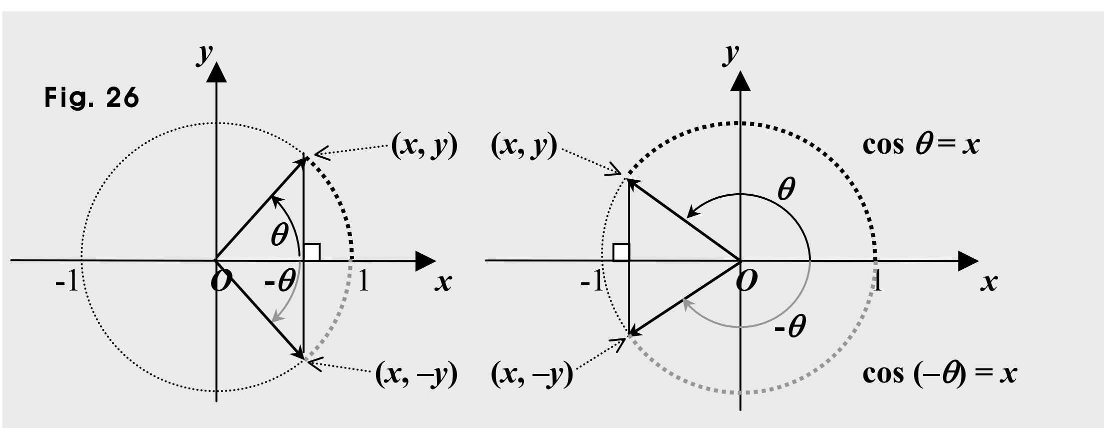
We know that $\boldsymbol{|w|}$ is the frequency, that is, the number of times the smallest part repeating itself appears in the basic interval (basic period) 2π, and that the interval that fits the smallest part is the period.
So using the frequency $\boldsymbol{|w|}$ and the basic interval 2π, how can we express the period?

We can put the period this way: $\dfrac{2\pi}{|w|}$. Why not just $\boldsymbol{w}$ but $\boldsymbol{|w|}$ though?

We can have a cosine function where $\boldsymbol{y = h(x) = \cos(-x)}$ for $\boldsymbol{x}$ real. Then, the frequency in the cosine function $\boldsymbol{h}$ is |-1|, simply because a frequency is positive.
And we can put $\boldsymbol{h}$ this way, too: $\boldsymbol{y = h(x) = \cos x}$ for $\boldsymbol{x}$ real. It's because $\boldsymbol{\cos(-x) = \cos x}$.
How?

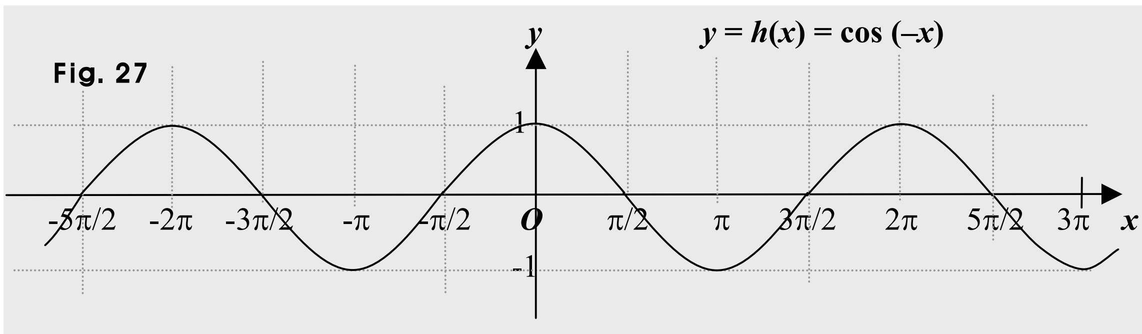
If the vector is of length 1, we can get this:

Fig. 26



So we get $\boldsymbol{\cos(-\theta) = \cos\theta}$.

And putting in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane, the curve of $\boldsymbol{h}$, we get this:



So we get the same curve as the one for $\boldsymbol{y = f(x) = \cos x}$, and the period of $\boldsymbol{h}$ is 2π, too.
And next, what do we mean by the phase?

It can be called a horizontal shift, too. How?

Suppose first $\boldsymbol{y = F(x) = A\cdot\cos w(x + a) + b}$ for $\boldsymbol{x}$ real, where $\boldsymbol{A}$, $\boldsymbol{w}$, $\boldsymbol{a}$, and $\boldsymbol{b}$ are constant.
Then, $\boldsymbol{a}$ is called the phase, $|\boldsymbol{w}|$ is called the frequency, and $\dfrac{2\pi}{|w|}$ is the period.

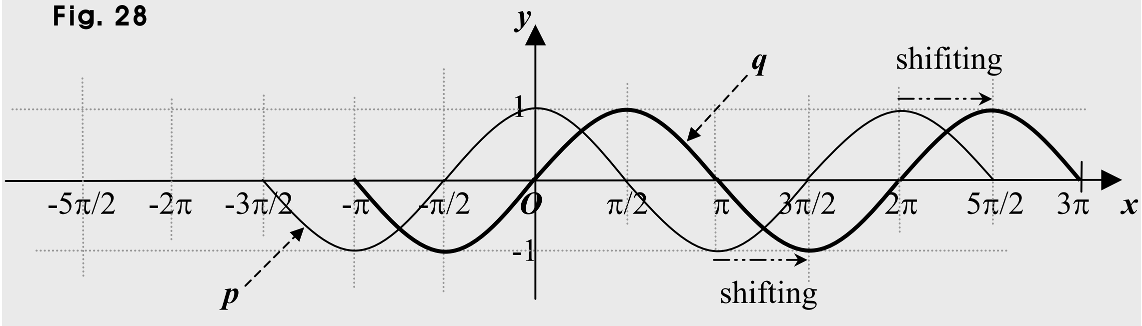
And next, assuming $\boldsymbol{y = Q(x) = A\cdot\cos wx + b}$ for $\boldsymbol{x}$ real, and shifting the curve of the function $\boldsymbol{Q(x)}$ by $\boldsymbol{-a}$ in the direction of the $\boldsymbol{x}$-axis, we get the curve of the function $\boldsymbol{F}$ specified above.

So the two curves themselves of $\boldsymbol{F}$ and $\boldsymbol{Q}$ are the same, and moving the curve of $\boldsymbol{Q}$ in the amount of $\boldsymbol{-a}$ along the $\boldsymbol{x}$-axis, we get the curve of $\boldsymbol{F}$.
In other words, the curve of $\boldsymbol{Q}$ gets moved to the left in the amount of $\boldsymbol{a}$.

So if the phase $\boldsymbol{a}$ itself is positive, the curve gets shifted to the left, and if *negative*, the curve moves to the *right*.
Assuming for instance, shifting the curve of $\boldsymbol{p(x) = \cos x}$ for $-\dfrac{3\pi}{2} \le \boldsymbol{x} \le \dfrac{5\pi}{2}$ by $\dfrac{\pi}{2}$ in the direction of the $\boldsymbol{x}$-axis, that is, to the right, we get the curve of a function as follows.
$\boldsymbol{q(x) = \cos (x - \frac{\pi}{2})}$ for $\boldsymbol{-\pi \le x \le 3\pi}$.

So putting in a graph the two curves, that is, the curves of $\boldsymbol{p}$ and $\boldsymbol{q}$, we get this:

Fig. 28



Thus, the curve of $\boldsymbol{p}$ gets moved to the right in the amount of $\dfrac{\pi}{2}$. And the new curve is the curve of $\boldsymbol{q}$. So we can call the phase a *horizontal* shift, too.
What then about the constant $\boldsymbol{b}$ in $\boldsymbol{y = Q(x) = A\cdot\cos wx + b}$?

It can be called a *vertical* shift.
So the constant $\boldsymbol{b}$ makes a curve move vertically and in the amount of $\boldsymbol{b}$.
In other words, it makes a curve move in the amount of $\boldsymbol{b}$ in the direction of the $\boldsymbol{y}$-axis.

So for instance, assuming $\boldsymbol{y = V(x) = A\cdot\cos w(x + a)}$ for $\boldsymbol{x}$ real, and shifting the curve of the function $\boldsymbol{V(x)}$ by $\boldsymbol{b}$ in the direction of the $\boldsymbol{y}$-axis, we can get the curve of the function

$$\boldsymbol{y = F(x) = A\cdot\cos w(x + a) + b} \text{ for } \boldsymbol{x} \text{ real.}$$

So the two curves themselves of $\boldsymbol{F}$ and $\boldsymbol{V}$ are the same, and moving the curve of $\boldsymbol{V}$ in the amount of $\boldsymbol{b}$ along the $\boldsymbol{y}$-axis, we get the curve of $\boldsymbol{F}$.

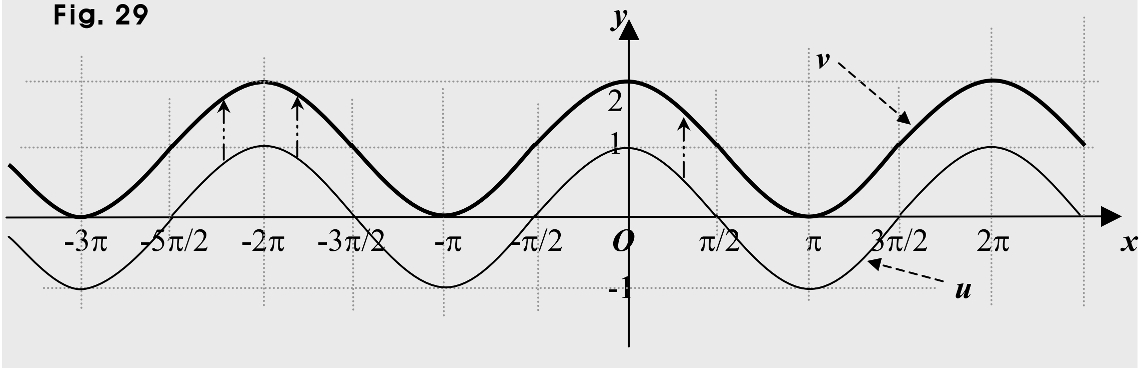
And if $\boldsymbol{b}$ is positive, the curve is shifted upward, and if negative, it moves downward.

Assuming for instance, shifting by 1 in the direction of the $\boldsymbol{y}$-axis, that is, upward, the curve of $\boldsymbol{u(x) = \cos x}$ for $\boldsymbol{x}$ real, we get the curve of a function as follows.

$$\boldsymbol{v(x) = \cos x + 1} \text{ for } \boldsymbol{x} \text{ real.}$$

Putting therefore, the two curves in a graph, we get the graph as follows.

Fig. 29



So the curve of ***u*** gets moved upward in the amount of 1.
And the new curve is the curve of ***v***. So we can call the ***b*** the amount of a vertical shift.
In short, ***b*** is a vertical shift.

And let's now put in a graph, for instance, the curve of a cosine function as follows.

$$\boldsymbol{y = C(x) = -2\cos(-2x + \pi) + 1} \text{ for } \boldsymbol{x} \text{ real}$$

To begin with, we have $\mathbf{cos}\,(-\theta) = \mathbf{cos}\ \theta$.
So we can get $\mathbf{cos}\,(-2x + \pi) = \mathbf{cos}\,\{-(2x - \pi)\} = \mathbf{cos}\,(2x - \pi)$.
Thus, we get $-2\,\mathbf{cos}\,(-2x + \pi) + 1 = -2\,\mathbf{cos}\,(2x - \pi) + 1$.

And putting it in the general form, we get $-2\,\mathbf{cos}\ 2(x - \frac{\pi}{2}) + 1$.

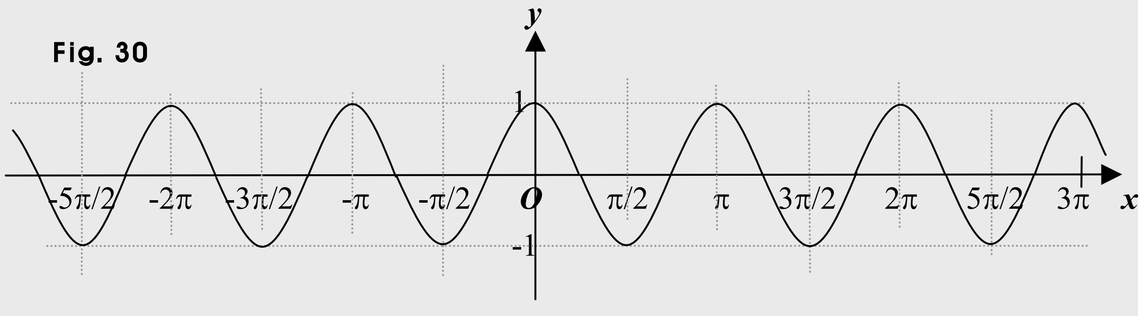
Then, we can now see that the amplitude is 2, the frequency is 2, the phase is $-\frac{\pi}{2}$, that is, the horizontal shift is $\frac{\pi}{2}$ to the right, and the vertical shift is 1 (upward).

And next, putting it in a graph, we may want to begin with the prototype, **cos *x***.
Then, setting first, the frequency to 2, we get **cos 2*x***.

We know that the period is the basic period (interval) divided by the frequency, and that the basic interval (period) for the cosine is 2π. So we can see that the period is $\frac{2\pi}{2} = \pi$.

Thus, we can put the curve of **cos 2*x*** the way as follows.

Fig. 30

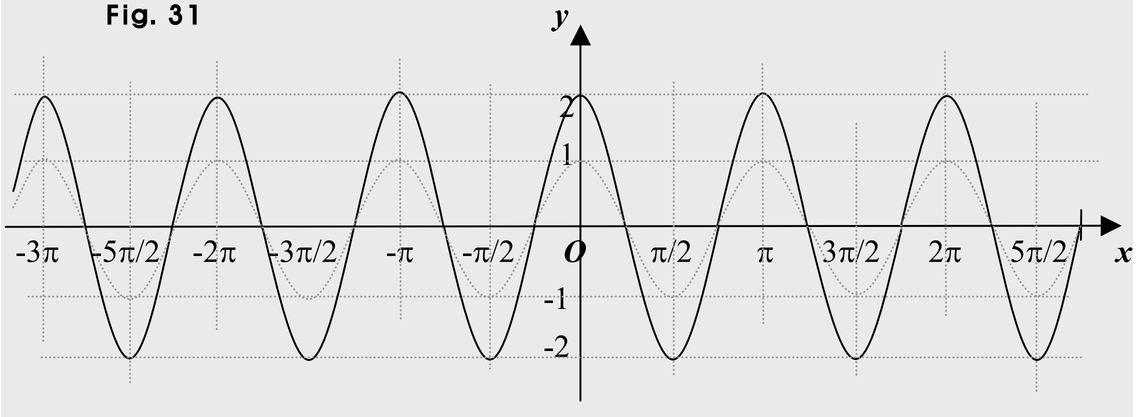


Next, setting the amplitude to 2, we get **2 cos 2*x***.

And the amplitude indicates half the width of the curve. So the half width is 2.

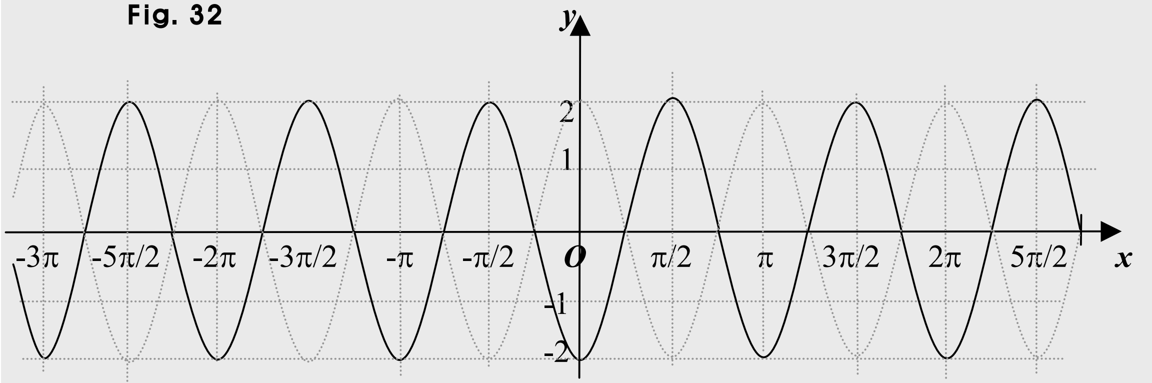
Thus, we can put in a graph, the curve of **2 cos 2*x*** the way as follows.

Fig. 31



Also, we want to put the negative sign in front. So we get **–2 cos 2*x***, and get this:

Fig. 32



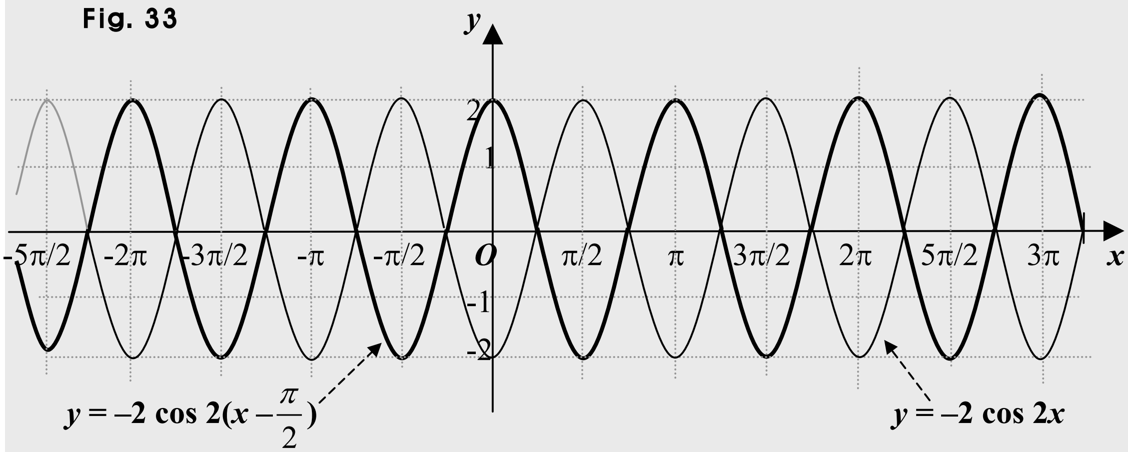
Next, setting the phase to $-\frac{\pi}{2}$, we get $\mathbf{-2\cos 2(x-\frac{\pi}{2})}$.

We know shifting the curve of $\mathbf{\cos x}$ by $-\frac{\pi}{2}$ in the direction of the $\boldsymbol{x}$-axis, we get the curve of $\mathbf{\cos(x+\frac{\pi}{2})}$.

So shifting the curve of $\mathbf{-2\cos 2x}$ by not $-\frac{\pi}{2}$ but $\frac{\pi}{2}$ in the direction of the $\boldsymbol{x}$-axis, we get the curve of $\mathbf{-2\cos 2(x-\frac{\pi}{2})}$. That is, moving the curve of $\mathbf{-2\cos 2x}$ in the amount of $\frac{\pi}{2}$ to the right, we get the curve of $\mathbf{-2\cos 2(x-\frac{\pi}{2})}$. So we do a horizontal shifting to the right in the amount of $\frac{\pi}{2}$.

Thus, we can put in a graph, the curve of $\mathbf{-2\cos 2(x-\frac{\pi}{2})}$ the way as follows.

Fig. 33



And next, setting the vertical shift to 1, we get $\mathbf{-2\cos 2(x-\frac{\pi}{2})+1}$.

Then, we move upward the curve of $\mathbf{-2\cos 2(x-\frac{\pi}{2})}$ in the amount of 1 to get the curve of $\mathbf{-2\cos 2(x-\frac{\pi}{2})+1}$, which is the curve of the function $\boldsymbol{C(x)}$ given.

Thus, we can put in a graph, the curve of the function $\boldsymbol{C}$ the way as follows.

{Note that $\boldsymbol{C(x) = -2\cos 2(x - \frac{\pi}{2}) + 1 = -2\cos(-2x + \pi) + 1}$.}

Fig. 34

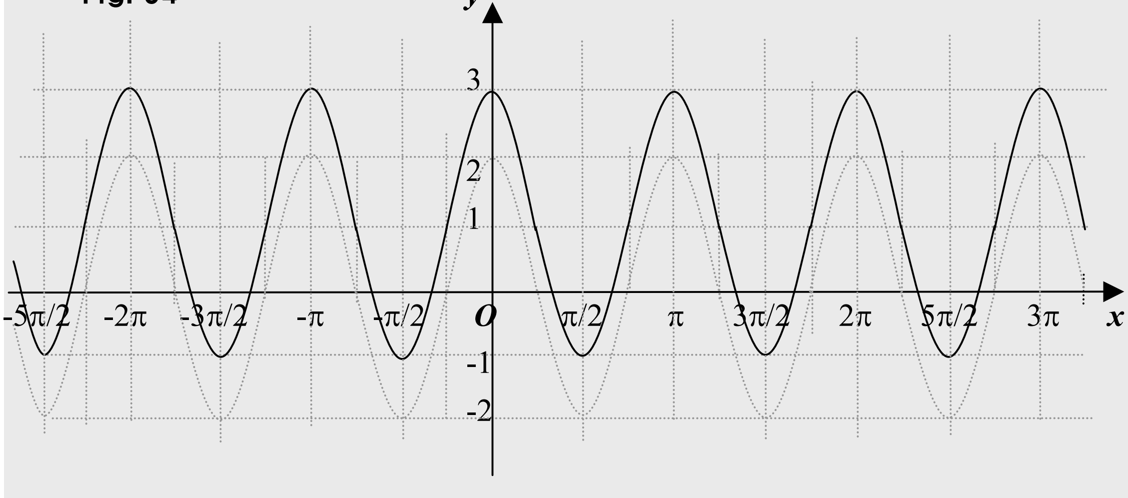


Fig. 35

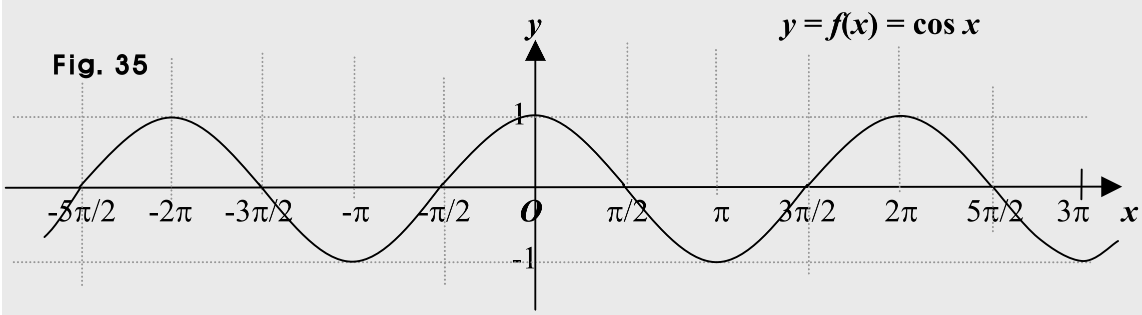


Fig. 36

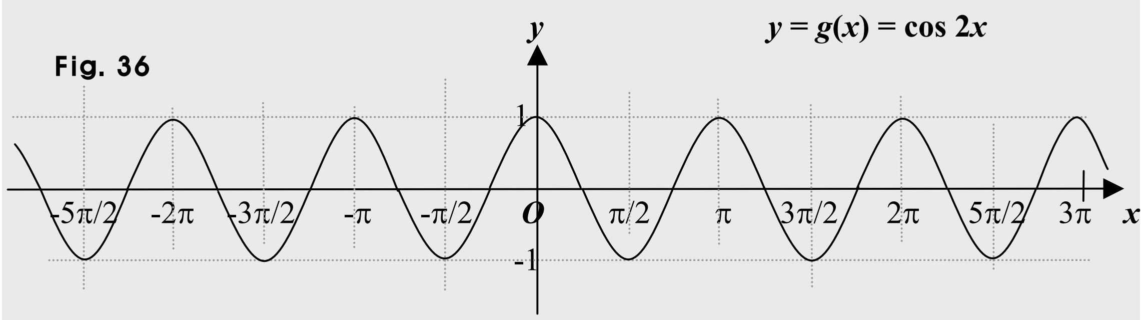


Fig. 37

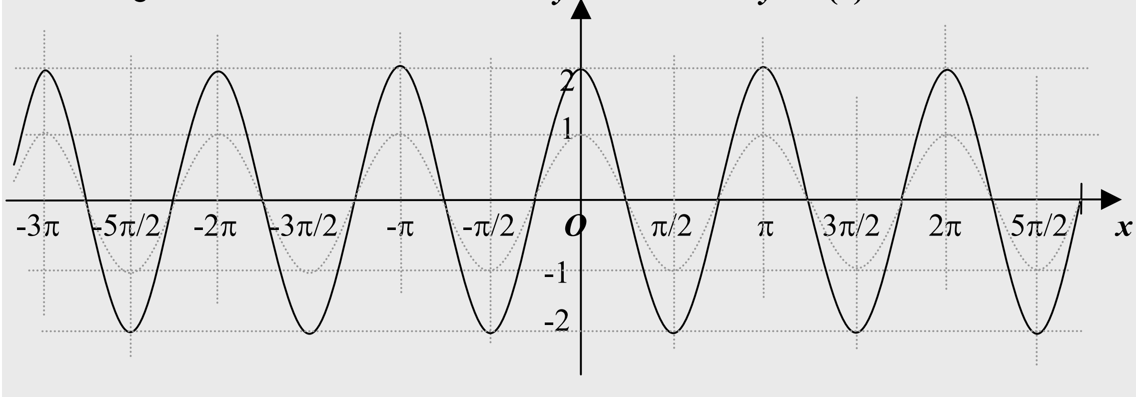


Fig. 38

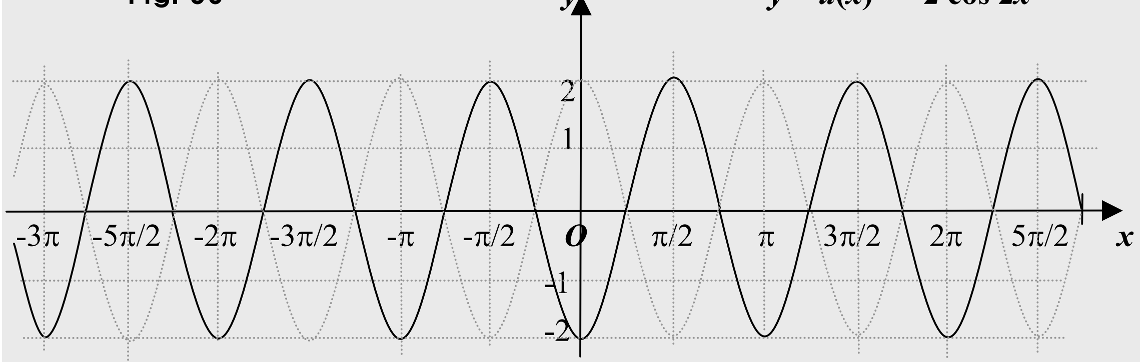
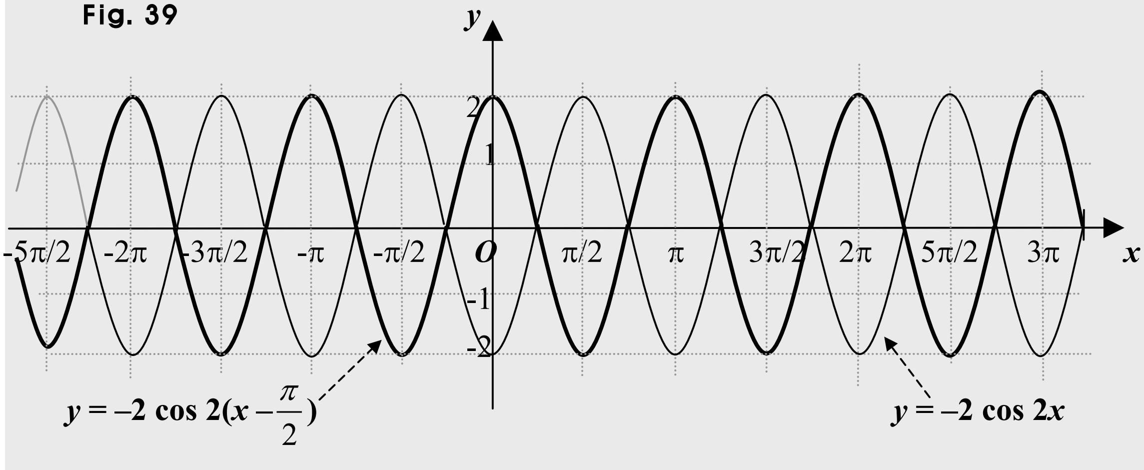


Fig. 39



Note that $\boldsymbol{C(x) = -2\cos 2(x - \frac{\pi}{2}) + 1 = -2\cos(-2x + \pi) + 1}$.

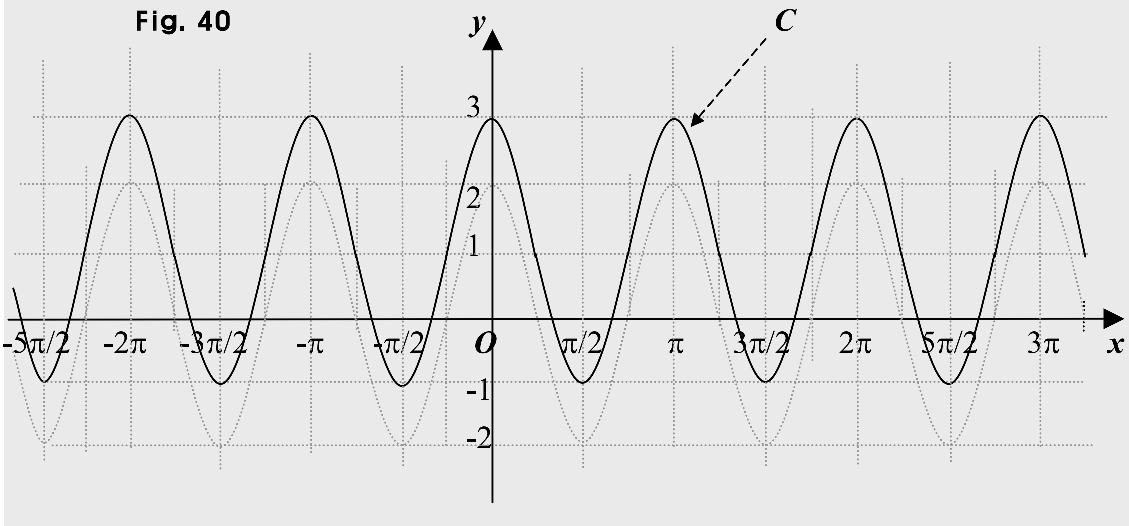
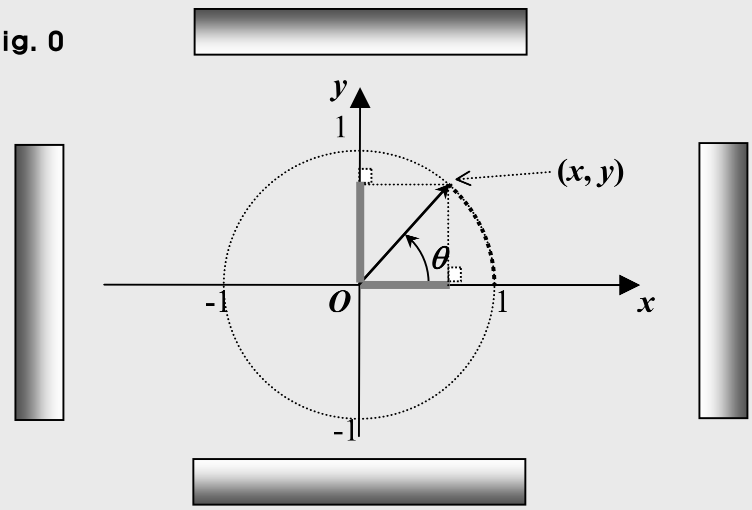


Fig. 40

6. Tangent Functions

Fig. 0



Let's first again, quickly go over the basics of trigonometric functions, and then, move on to the details on tangent functions. So what is a trigonometric function?

It is often called a trig function and is a function of which each output is a trig ratio called the sine, the cosine, or the tangent. For instance, putting a tangent function in the ***x-y*** coordinate system, we can put it the way as follows.

$$\boldsymbol{y = f(x) = \tan x},$$

where $\boldsymbol{x}$ is a real number indicating an angle, but $\boldsymbol{x} \neq \frac{\boldsymbol{n\pi}}{2}$ where $\boldsymbol{n}$ is an add integer.

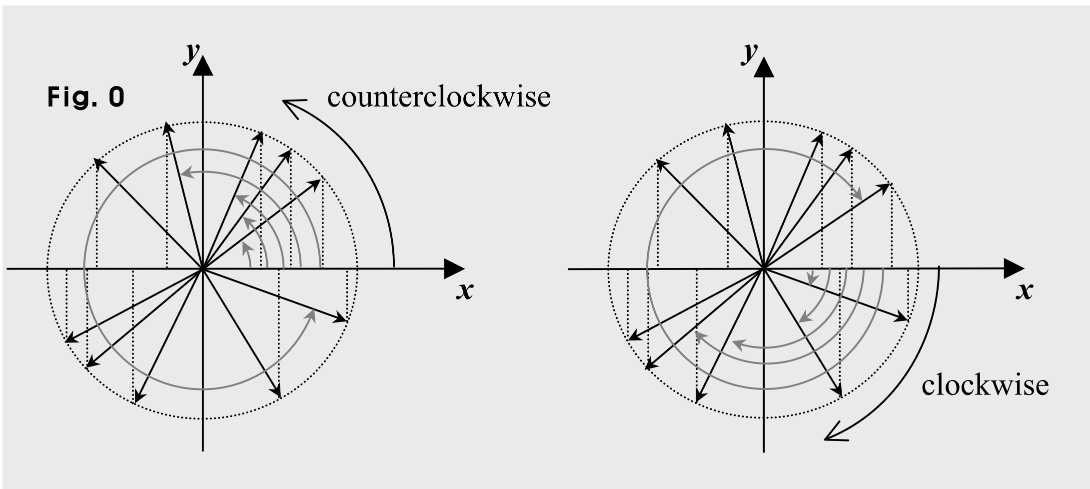
In short, $\boldsymbol{y = f(x) = \tan x}$, where $\boldsymbol{x}$ is real but not $\frac{\boldsymbol{n\pi}}{2}$ if $\boldsymbol{n}$ is odd.

Thus, the tangent function $\boldsymbol{f(x)}$ is a function of an angle, takes an angle as an input, but ***cannot be defined*** for $\boldsymbol{x} = \frac{\boldsymbol{n\pi}}{2}$ if $\boldsymbol{n}$ is odd,

and produces a trig ratio called the tangent as an output.

So in $y = f(x) = \tan x$, the input variable x takes an angle,
the expression $\tan x$ produces an output called the tangent at a time, but is
not defined for $x = \frac{n\pi}{2}$ if n is odd,
and the output variable y gets the tangent.

And the angle is made by the vector turning about the origin in the x-y plane.



Either counterclockwise or clockwise, in ***one way***,
the vector keeps turning about the origin in the x-y plane,
and keeps making angles.

Each and every moment, it makes an angle, because an amount of turning is an angle.

Initially, the vector is at rest sitting on the x-axis on the right of the origin.
Then, the angle made is 0 in radians i.e., 0^{o}.

And we can have the vector make as much turn as we need or as many complete turns as necessary.

If the vector starts and keeps turning counterclockwise, the angle made is positive every moment. Or if the vector starts and keeps turning clockwise, the angle made is negative every moment.

So assuming $\boldsymbol{\theta}$ is the angle made and putting together all the angles can be made, we get

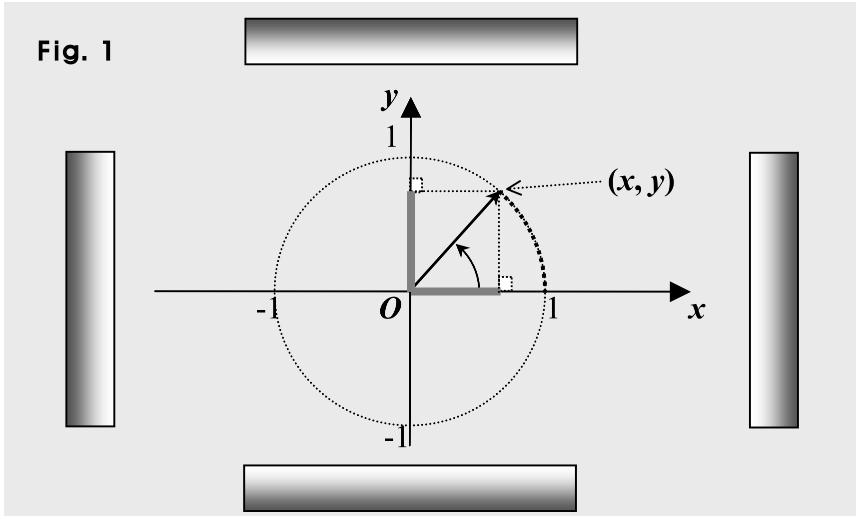
$$\mathbf{-360^o \cdot \boldsymbol{n} \leq \boldsymbol{\theta} \leq 360^o \cdot \boldsymbol{n}} \text{ in degrees or } \mathbf{-2\boldsymbol{n}\pi \leq \boldsymbol{\theta} \leq 2\boldsymbol{n}\pi} \text{ in radians,}$$

where π is 3.141592…, which is real, and $\boldsymbol{n}$ is an integer ≥ 0, so $\boldsymbol{n} = 0, 1, 2, \ldots$ Thus, we can get $-\infty < \boldsymbol{\theta} < \infty$, where ∞ indicates infinity.

So $\boldsymbol{\theta}$ can be all real numbers, every input is a real number, and yet, it is ***not*** the case we can use all angles for a tangent function as $\boldsymbol{y = f(x) = \tan x}$, because if $\boldsymbol{x} = \frac{\boldsymbol{n\pi}}{2}$ where $\boldsymbol{n}$ is odd, the adjacent is 0, so $\mathbf{tan}\ \boldsymbol{x}$ cannot be defined, since the tangent is $\frac{\text{the opposite}}{\text{the adjacent}}$ and no division by 0 is allowed. We'll see how it is 0 shortly.

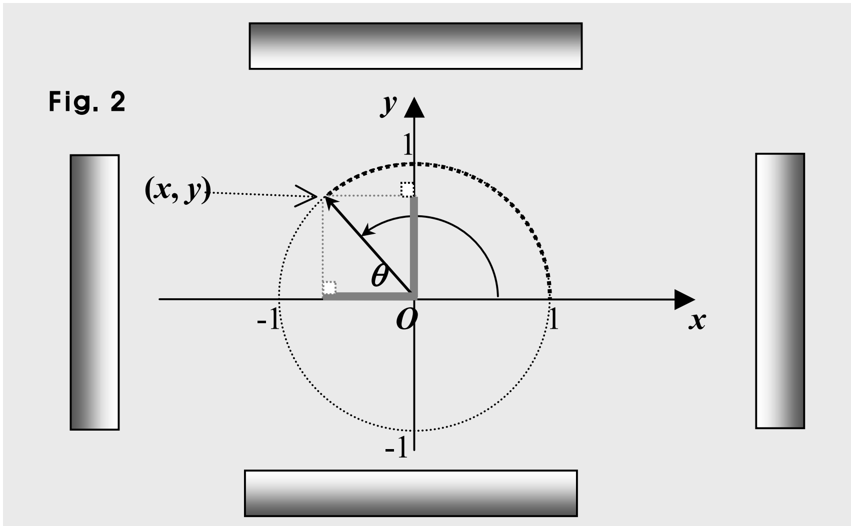
So suppose this time, that in the ***x-y*** plane where the vector is turning counterclockwise about the origin, we place four lamps the way in Fig. 1 so that the shadows of the vector are made on both axes, and that the |vector| is 1. And let's call the shadow the projection, and call the value of the projection the projection, too, for short.

Fig. 1



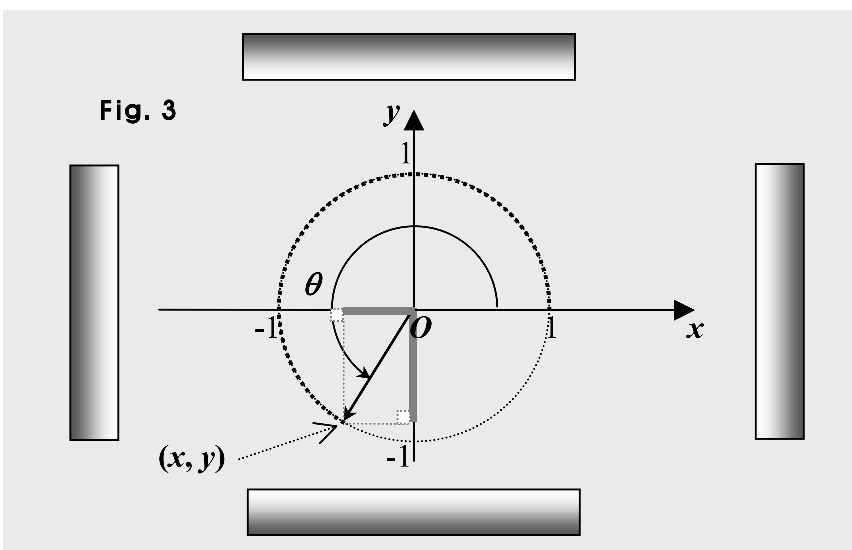
Then, as in Fig. 1 above, if the vector is above the $\boldsymbol{x}$-axis while turning, as the vector approaches the $\boldsymbol{y}$-axis, we can see linear motions where the projection on the $\boldsymbol{x}$-axis shrinks changing its value from 1 to 0, while the projection on the $\boldsymbol{y}$-axis expands changing its value from 0 to 1.

Fig. 2



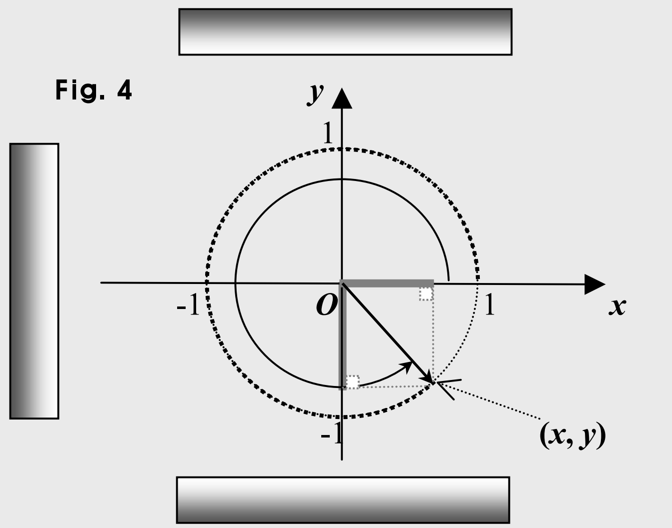
Also, if the projecion is on the ***left*** of the origin on the ***x***-axis, or is ***below*** the origin on the ***y***-axis, the projection is ***negative***.
Then, as in Fig. 2 above, if the vector is above the ***x***-axis while turning, as the vector approaches the ***x***-axis, we can see linear motions where the projection on the ***x***-axis expands changing its value from 0 to –1, while the projection on the ***y***-axis shrinks changing its value from 1 to 0.

Fig. 3



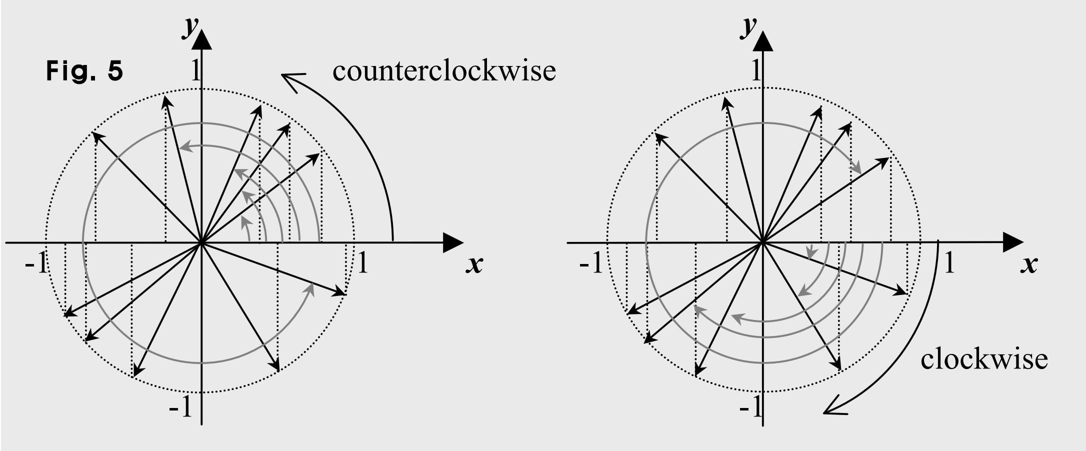
In Fig. 3 above, if the vector is below the ***x***-axis while turning, we can see linear motions where as the vector approaches the ***y***-axis, the projection on the ***x***-axis shrinks changing its value from –1 to 0, while the projection on the ***y***-axis expands changing its value from 0 to –1.

Fig. 4



And in Fig. 4 above, if the vector is below the ***x***-axis while turning, we can see linear motions where as the vector approaches the ***x***-axis, the projection on the ***x***-axis expands changing its value from 0 to 1, while the projection on the ***y***-axis shrinks changing its value from –1 to 0.

Fig. 5



Thus, as the vector keeps turning, we can see on one axis, a linear motion where the projection expands changing its value from 0 to 1 or from 0 to –1, while on the other axis, another linear motion where the projection shrinks changing its value from 1 to 0 or from –1 to 0. And such linear motions keep repeating as the vector keeps turning.

At every moment, we can use the |vector| as a hypotenuse, and use the projection on the ***x***-axis as the adjacent, and use the projection on the ***y***-axis as the opposite.

Fig. 6

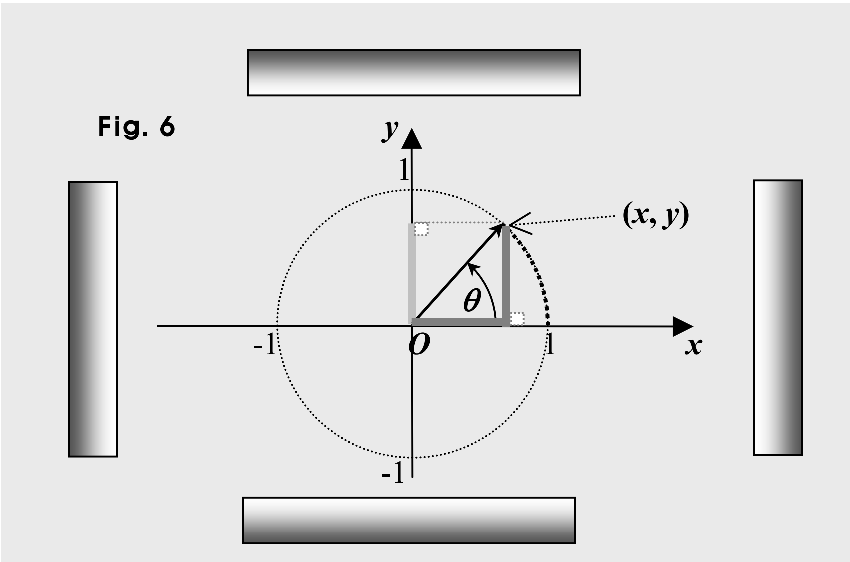
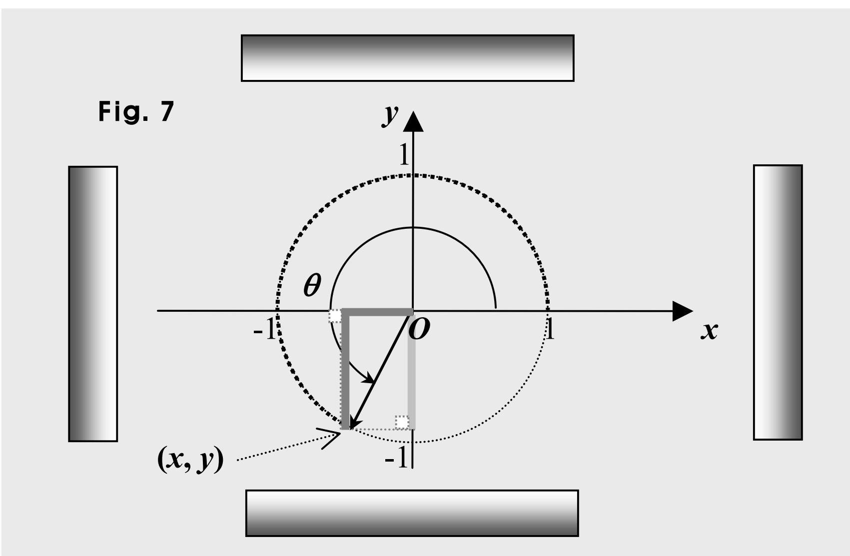


Fig. 7

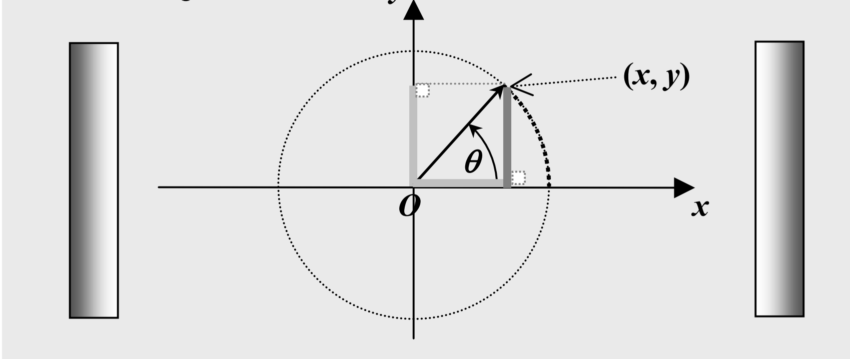


So now, what are we going to do with those projections?

We can come up with trig functions and their graphs. So this time, we are going to come up with a tangent function, and its graph.

At every moment, as in Fig. 8 below, we can see a right triangle where the hypotenuse is the |vector|, the opposite is the line segment in dark gray, which is the same as the projection in light gray on the ***y***-axis. So we can use the projection on the ***y***-axis as the opposite.

Fig. 8



And we have the statement as follows.

In coordinate trigonometry,
the ***x***-coordinate at the terminal point is the adjacent,
the ***y***-coordinate is the opposite,
the |vector| is the hypotenuse,
and the vector turned makes the angle.

Therefore, we can say that of the terminal point, the ***x***-coordinate is the projection on the ***x***-axis, and the ***y***-coordinate is the projection on the ***y***-axis.

In short,

x at (***x***, ***y***) is the projection on the ***y***-axis,

and ***y*** at (***x***, ***y***) is the projection on the ***y***-axis.

And the adjacent is ***x*** at (***x***, ***y***), and the opposite is ***y*** at (***x***, ***y***).

And depending on the angle θ made by the vector, the projections get determined. In short, the angle made determines the projections. So what makes such a right triangle is the |vector| and the angle θ made by the vector.

Fig. 9

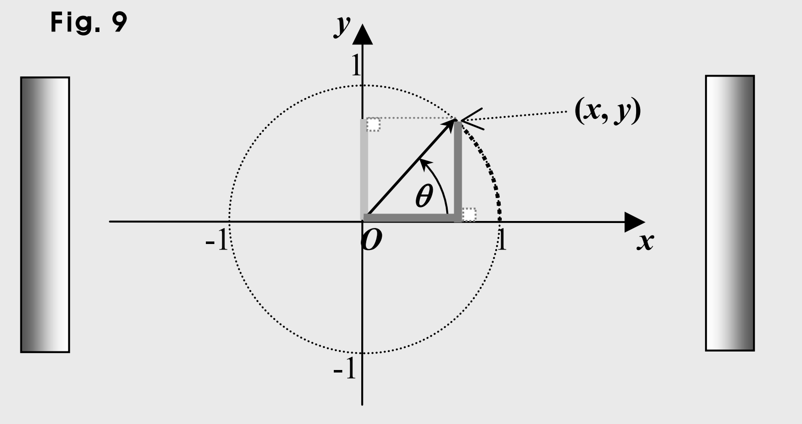
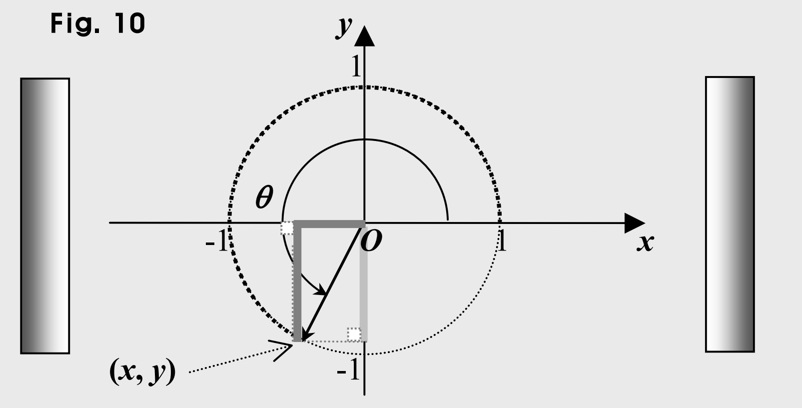


Fig. 10



And we use that angle θ to get trig ratios.

Note that if a projection is ***below*** the origin, or if a projection is on the ***left*** of the origin, its value is ***negative***. So in Fig.6, the two projections are negative.

Now, in coordinate trigonometry, using those projections, along with the angle θ stated above, we can come up with the basic trig functions, along with their graphs.

Of each of the basic three, each input is an angle, which is made by the vector turning in the ***x-y*** plane. So in the tangent function $\boldsymbol{y = f(x) = \tan x}$, the input variable $\boldsymbol{x}$ gets an angle at a time.

Usually though, we just use real numbers as 0 or π as angles, since angles are in radians. However, for **tan *x***, we cannot use the angles made by the vector if the vector is on the ***y***-axis, because the adjacent is 0, since the projection on the ***x***-axis is 0.

In this case, the angle $\boldsymbol{\theta} = \frac{\boldsymbol{n\pi}}{2}$ where ***n*** is odd, so ***n*** = ±1, ±3, ±5, …

In Fig. 11 and 12, the projections on the ***x***-axis are 0, since the vector is on the ***y***-axis.

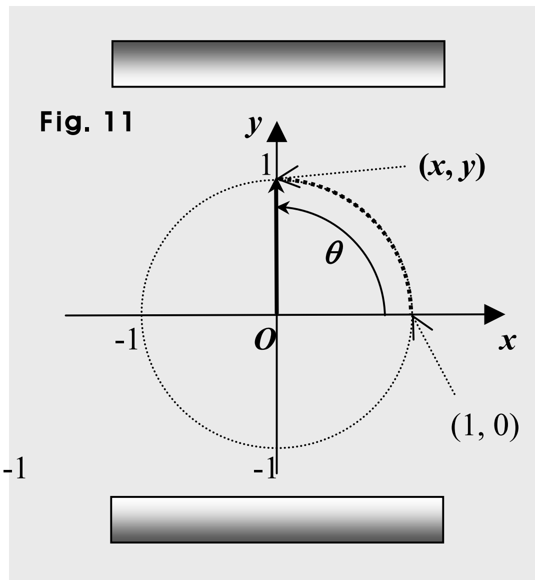


Fig. 11

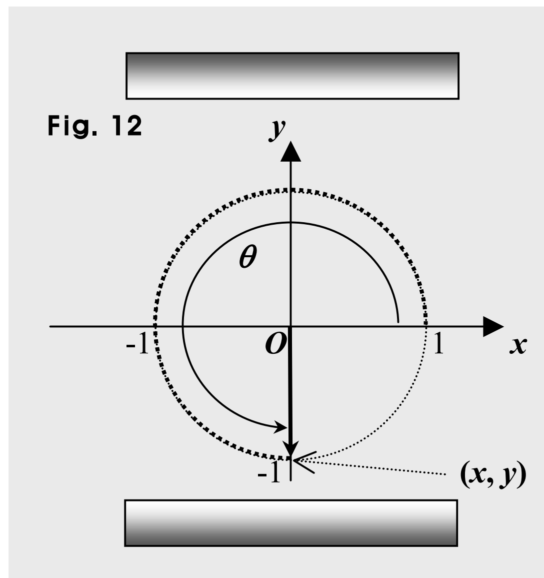


Fig. 12

So assuming ***f*** is a tangent function, its expression is **tan *x***, and the domain is a set of all angles except all odd multiples of 90°, we say that the domain is a set of all real numbers except $\frac{\boldsymbol{n\pi}}{2}$ where ***n*** is odd, and can put the tangent function ***f*** the way as follows.

$$\boldsymbol{y = f(x) = \tan x} \text{ for } \boldsymbol{x} \text{ real other than } \frac{\boldsymbol{n\pi}}{2} \text{ where } \boldsymbol{n} \text{ is odd.}$$

What if we simply define the tangent function this way: ***y* = *f*(*x*) = tan *x***?

If no domain is specified, we assume the largest domain, so we assume that the dome is a set of all real numbers other than $\frac{\boldsymbol{n\pi}}{2}$ where ***n*** is odd. What then about outputs?

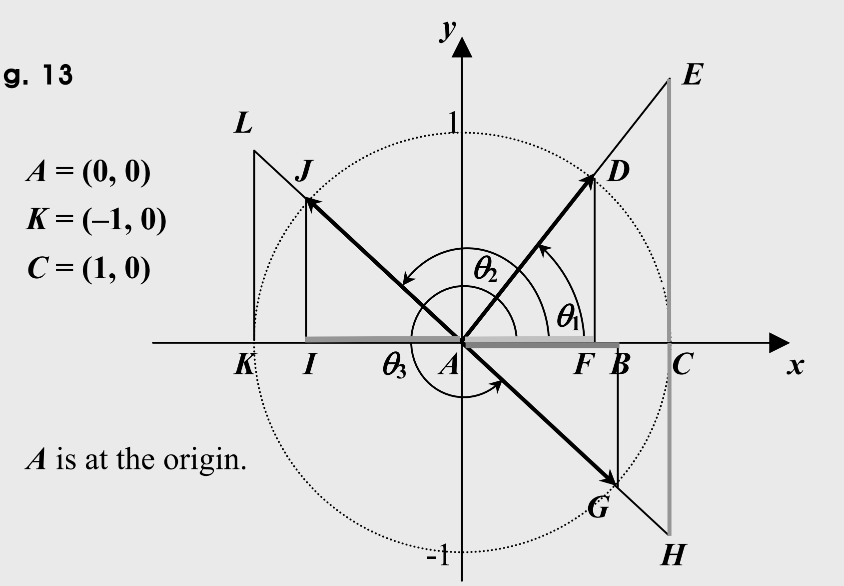
They are trig ratios called the tangents, because in the tangent function ***y* = *f*(*x*) = tan *x***, the ***x*** gets an angle, so each value of **tan *x***, that is, each value of ***f*(*x*)** is a trig ratio called a tangent, which is a real number. What then about the range?

Other than the sine function $g(x) = \sin x$ and the cosine function $h(x) = \cos x$, the range of the tangent function $f(x) = \tan x$ is not a set of all real numbers from –1 to 1.

If the domain of $f(x)$ is a set of all real numbers except $\frac{n\pi}{2}$ where n is odd, its range is a set of all real numbers, and each of the numbers is a trig ratio, which is an output, which is the value of $f(x)$, which is the value of y.
So the range of the tangent function f can be put this way, too: $-\infty < y < \infty$, or $|y| < \infty$.

Let's get back to the **x-y** plane where the projections of the vector of length 1 are made on the x and y-axes and see how the range can be put the way above.

Fig. 13



To begin with, we know the tangent is opposite over the adjacent, that is, $\frac{\text{the opposite}}{\text{the adjacent}}$.

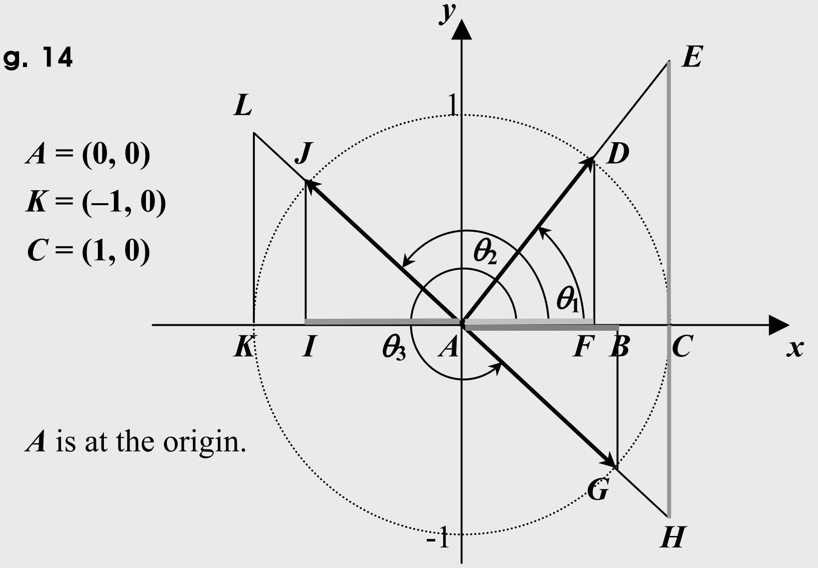
So first, in Fig. 13, we know $AC = 1$, so we can get $\tan\theta_1 = \frac{FD}{AF} = \frac{CE}{AC} = \frac{CE}{1} = CE$.

Next, we can get $\tan\theta_2 = \frac{IJ}{AI} = \frac{KL}{AK} = \frac{KL}{-1} = -KL$, which is negative.
Note that the opposite > 0, but the adjacent < 0 if the vector is in the second quadrant.
So if the vector is in the second quadrant, the tangent is negative.
If however, the vector is in the third quadrant, the tangent is positive, because both the opposite and the adjacent are negative.

Next, in Fig. 14 below, assuming $\boldsymbol{LH}$ is a line segment, we can see that $\boldsymbol{CH = -KL}$, since the two triangles $\boldsymbol{AKL}$ and $\boldsymbol{ACH}$ themselves are the same, and $\boldsymbol{CH}$ is below the $\boldsymbol{x}$-axis.

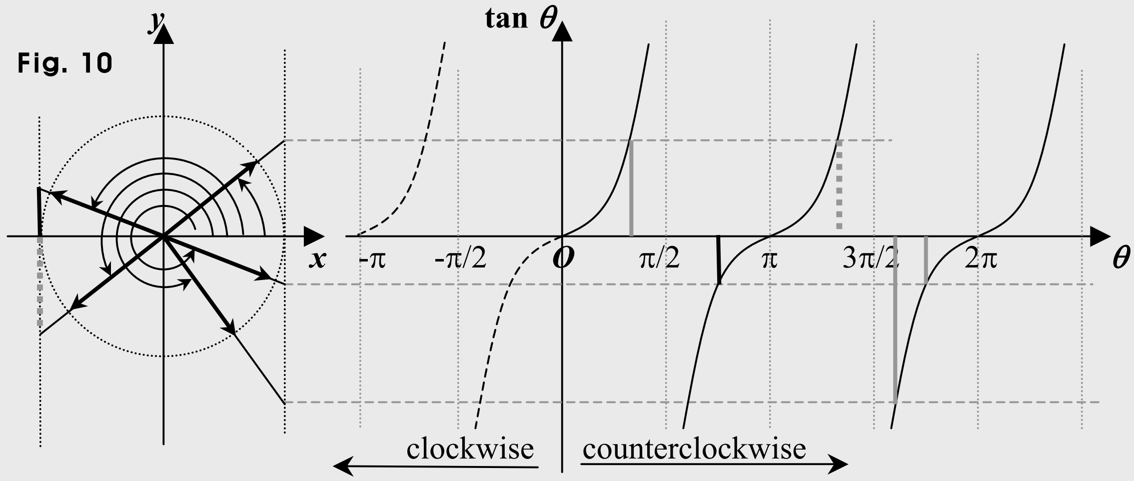
So next, we can get $\mathbf{tan}\ \theta_3 = \dfrac{BG}{AB} = \dfrac{CH}{AC} = \dfrac{CH}{1} = CH$.

Fig. 14



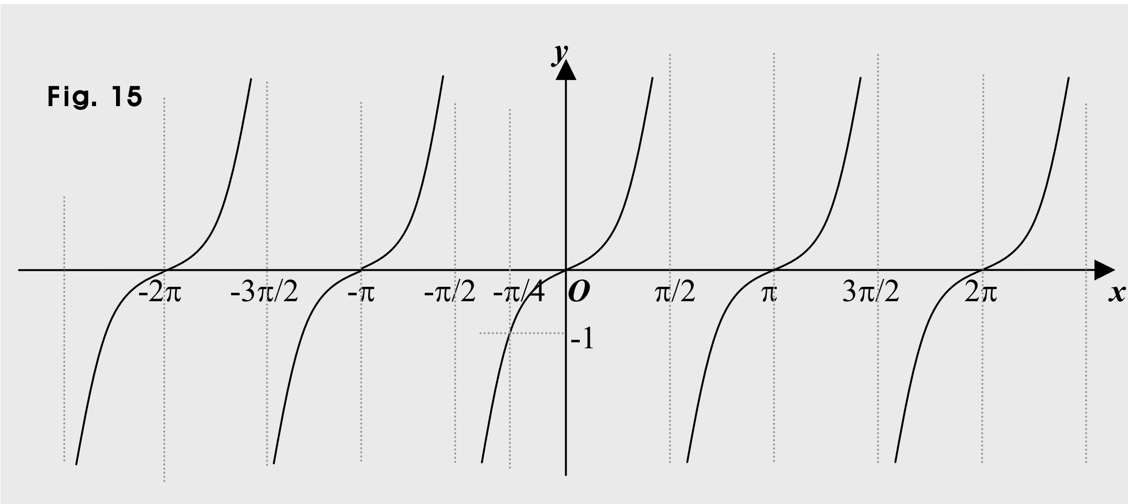
Thus, we can get the curve of $\mathbf{tan}\ \theta$ the way as follows. Note that $\mathbf{tan}\ \theta > \mathbf{0}$ in the first and the third quadrants, and that $\mathbf{tan}\ \theta < \mathbf{0}$ in the second and the fourth quadrants.

Fig. 10



And of course, we can put it in the ***x-y*** system.
Simply replacing ***c*** with ***y***, and ***θ*** with ***x***, we get $\boldsymbol{y = f(x) = \tan x}$.
And we can use numbers as angles. So assuming the domain is a set of all angles less $\boldsymbol{(2n + 1)90^o}$ for ***n*** integer, we can set $\boldsymbol{y = f(x) = \tan x}$ for $\boldsymbol{x} \neq \frac{(2\boldsymbol{n}+1)\pi}{2}$ for ***n*** integer.
Also, just setting $\boldsymbol{y = f(x) = \tan x}$, we mean that the domain is a set of all real numbers less $\frac{(2\boldsymbol{n}+1)\pi}{2}$ for ***n*** integer.
And we can put in the ***x-y*** plane, the curve of the tangent function above the way below.

Fig. 15



And we can see that the output is not bounded, and can be any real number. So the range is a set of all real numbers. And of course, even if the domain is $-\frac{\pi}{2} < \boldsymbol{x} < \frac{\pi}{2}$, the range is a set of all real numbers, too.
Also of course, the range can be other than a set of all real numbers.
If for instance, $\boldsymbol{y = g(x) = \tan x}$ for $-\frac{\pi}{4} \leq \boldsymbol{x} < \frac{\pi}{2}$, the range is $\boldsymbol{y \geq -1}$.

And the tangent function $\boldsymbol{y = f(x) = \tan x}$ can be called the prototype, and is thus, in the most basic form. Assuming ***F*** is a tangent function, too, and using a general form, we can put it the way as follows.

$\boldsymbol{y = F(x) = A \cdot \tan \{w(x + a)\} + b}$, where ***A***, ***w***, ***a***, and ***b*** are constant.

Note that $\boldsymbol{w}(\boldsymbol{x}+\boldsymbol{a})$ is the angle.

And just simply, we often put it this way, too: $\boldsymbol{y}=\boldsymbol{F}(\boldsymbol{x})=\boldsymbol{A}\cdot\mathbf{tan}\ \boldsymbol{w}(\boldsymbol{x}+\boldsymbol{a})+\boldsymbol{b}$, since in this case, we can assume that $\boldsymbol{A}$, $\boldsymbol{w}$, $\boldsymbol{a}$, and $\boldsymbol{b}$ are constant.

What then about the domain?

We know that the domain is the set of all the values that can be taken by the input variable, which is $\boldsymbol{x}$ in this case.
Thus first, finding all the values that cannot be the value of $\boldsymbol{x}$, and then, excluding all those values from a set of all real numbers, we can get the domain. So to begin with, what are the values that cannot be the value of $\boldsymbol{x}$?

We know that the tangent is ***not*** defined if the angle is $\frac{(2\boldsymbol{n}+1)\pi}{2}$ for $\boldsymbol{n}$ integer.
Also, we know that the angle in the function $\boldsymbol{F}$ is $\boldsymbol{w}(\boldsymbol{x}+\boldsymbol{a})$, and not just $\boldsymbol{x}$.

So we want to set $\boldsymbol{w}(\boldsymbol{x}+\boldsymbol{a})\neq\frac{(2\boldsymbol{n}+1)\pi}{2}$ for $\boldsymbol{n}$ integer.
Finding thus, the domain, we can get it the way as follows.

$$\boldsymbol{w}(\boldsymbol{x}+\boldsymbol{a})=\frac{(\mathbf{2}\boldsymbol{n}+\mathbf{1})\pi}{\mathbf{2}}\Rightarrow\boldsymbol{x}+\boldsymbol{a}=\frac{(\mathbf{2}\boldsymbol{n}+\mathbf{1})\pi}{\mathbf{2}\boldsymbol{w}}\Rightarrow\boldsymbol{x}=\frac{(\mathbf{2}\boldsymbol{n}+\mathbf{1})\pi}{\mathbf{2}\boldsymbol{w}}-\boldsymbol{a}.$$

So if $\boldsymbol{x}\neq\frac{(\mathbf{2}\boldsymbol{n}+\mathbf{1})\pi}{\mathbf{2}\boldsymbol{w}}-\boldsymbol{a}$, the function $\boldsymbol{F}$ is defined. Thus, the domain is a set of all real numbers other than $\frac{(\mathbf{2}\boldsymbol{n}+\mathbf{1})\pi}{\mathbf{2}\boldsymbol{w}}-\boldsymbol{a}$ where $\boldsymbol{n}$ is an integer.

And $|\boldsymbol{w}|$ is called the frequency, $\frac{\boldsymbol{\pi}}{|\boldsymbol{w}|}$ is the period, and $\boldsymbol{a}$ is called the phase.

So in the function $\boldsymbol{y}=\boldsymbol{f}(\boldsymbol{x})=\mathbf{tan}\ \boldsymbol{x}$, the phase is 0, and the frequency is 1, so the period is π (i.e., 180°). How though?

The prototype in tangent functions can be put this way: $\boldsymbol{y = f(x) = \tan x}$.
And assuming $\boldsymbol{F}$ is a tangent function, too, and using a general form, we can put it the way as follows.

$$\boldsymbol{y = F(x) = A \cdot \tan \{w(x + a)\} + b}, \text{ where } \boldsymbol{A}, \boldsymbol{w}, \boldsymbol{a}, \text{ and } \boldsymbol{b} \text{ are constant.}$$

So we can put the prototype into the general form, the way as follows.

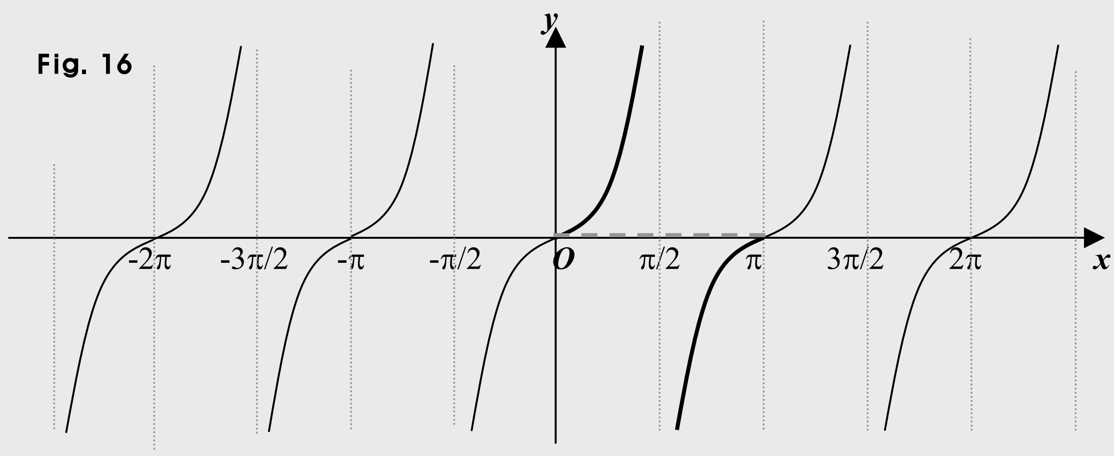
$$\boldsymbol{y = f(x) = 1 \cdot \tan 1(x + 0) + 0}$$

Thus, the phase $\boldsymbol{a}$ is 0, and the frequency $|\boldsymbol{w}|$ is 1, so the period $\dfrac{\pi}{|w|}$ is π.

What do we mean by the period?

Putting in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane, the curve of the tangent function $\boldsymbol{f}$ above, we can get this:

Fig. 16



Then, we can see that a part of the cuve repleats itself, which is in two pieces though.
And the part is in fact, the smallest part repeating itself. What then is the part?

It is the part from 0 to π. So the part repeats itself every π interval.
And we call such an interval a period.
So the interval π is the period in the function $\boldsymbol{y = f(x) = \tan x}$.

Thus, we call a tangent function a *periodic* function. And of course, the same is true, too, for sine and cosine functions, and other trig-functions such as cotangent functions.

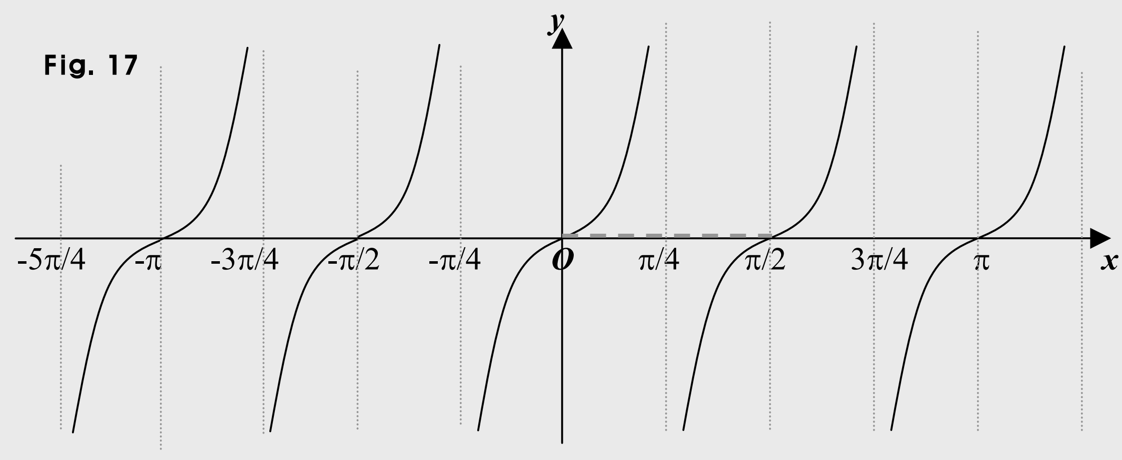
The tangent function ***f*** above is the prototype, and thus, is in the most basic form.

So the period π can be called the *basic period* in tangent functions.
In other words, the interval π can be called the *basic interval* in tangent functions.

And for instance, in the general form, setting ***w*** to 2, ***a*** and ***b*** to 0 each, and ***A*** to 1, we get a new function where $\boldsymbol{y = g(x) = \tan 2x}$ where $\boldsymbol{2x} \neq \frac{(2\boldsymbol{n}+1)\pi}{2}$ for ***n*** integer.

Then, putting in the ***x-y*** plane, the curve of the tangent function ***g*** above, we can get this:

Fig. 17



Then again, we can see that a part of the cuve repleats itself.
And the part is of course, the smallest part repeating itself. What then, is the part?

It is the part from 0 to $\frac{\pi}{2}$. So the part repeats itself every $\frac{\pi}{2}$ interval.

And we call such an interval a period. So the interval $\frac{\pi}{2}$ is the period in the function ***g***.

How many times does the smallest part stated above appear in the basic interval π?

It appears twice in the basic interval π, which is the basic period. Then, we say the frequency is 2 in the function ***g***. What then do we mean by the frequency?

It is the number of times the the period appears in the basic period π, which is the basic interval. In short, the frequency is the number of times the period appears in π.

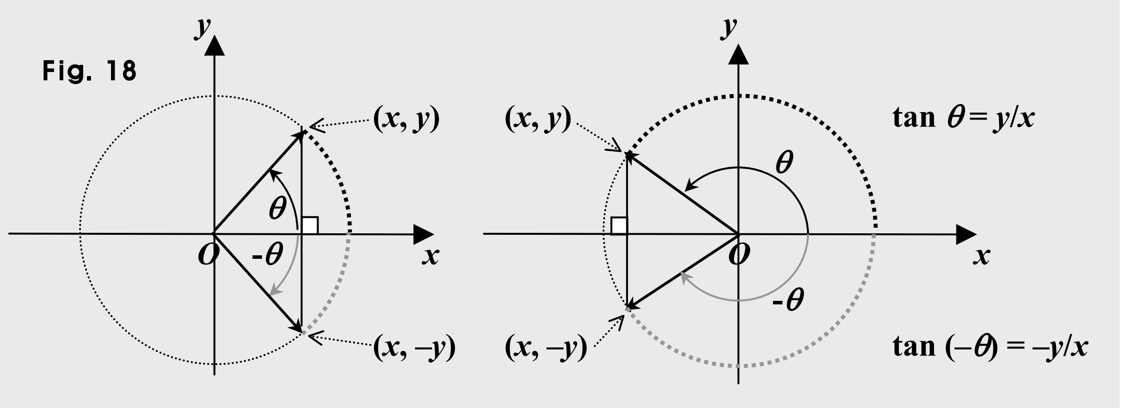
The basic period however, in sine and cosine functions is not π but 2π. So in general, the frequency in a trig function is the number of times its period appears in its basic period.

Thus, in the general form $\boldsymbol{A\cdot \tan w(x + a) + b}$, $\boldsymbol{|w|}$ is the frequency.
Then, given a tangent function in the general form, how can we get its period?

We know that $\boldsymbol{|w|}$ is the frequency, that is, the number of times the period appears in the basic period π. So using the frequency $\boldsymbol{|w|}$ and the basic period π, we can express the period this way: $\dfrac{\pi}{|w|}$. Why not just $\boldsymbol{w}$ but $\boldsymbol{|w|}$ though?

We can have a tangent function where $\boldsymbol{y = h(x) = \tan(-x)}$ for $\boldsymbol{x}$ real. Then, the frequency in the tangent function $\boldsymbol{h}$ is $|-1| = 1$, simply because a frequency is positive.
And we have $\boldsymbol{\tan(-x) = -\tan x}$. So assuming the vector is of length 1, we can get this:

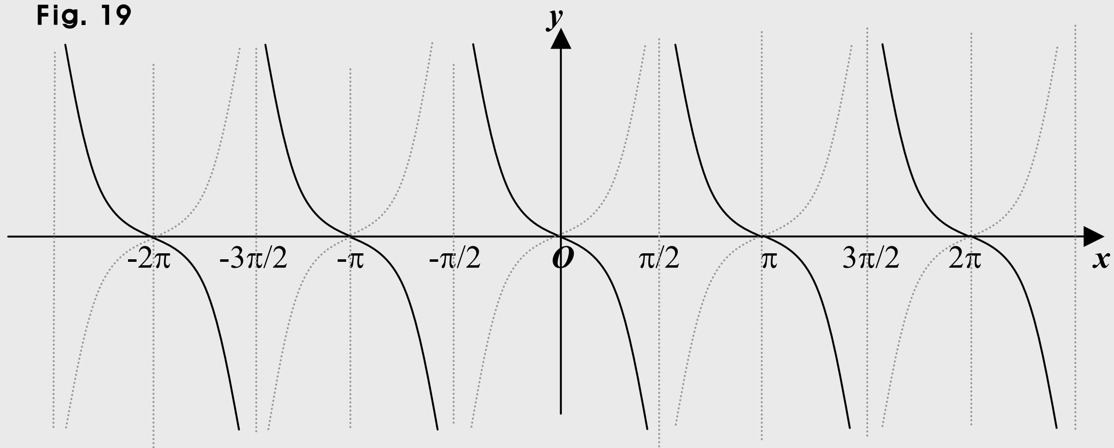
Fig. 18



So we get $\boldsymbol{\tan(-\theta) = -\tan \theta}$.

And putting in the $\boldsymbol{x\text{-}y}$ plane, the curve of $\boldsymbol{h}$, we can get the graph as follows.

Fig. 19



And next, what do we mean by the phase?

Using the general form, we can put a tangent function called $\boldsymbol{F}$ the way as follows.

$$y = F(x) = A\cdot\tan w(x + a) + b \text{ where } w(x + a) \neq \frac{(2n+1)\pi}{2} \text{ for } n \text{ integer}$$

Then, $\boldsymbol{a}$ is called the phase, $|\boldsymbol{w}|$ is called the frequency, and $\dfrac{\pi}{|w|}$ is the period.

Next, shifting by $-a$ in the direction of the $\boldsymbol{x}$-axis, the curve of $y = Q(x) = A\cdot\tan wx + b$ where $wx \neq \dfrac{(2n+1)\pi}{2}$ for $\boldsymbol{n}$ integer, we get the curve of the function $\boldsymbol{F}$.

So the two curves themselves of $\boldsymbol{F}$ and $\boldsymbol{Q}$ are the same, and moving the curve of $\boldsymbol{Q}$ in the amount of $-a$ along the $\boldsymbol{x}$-axis, we get the curve of $\boldsymbol{F}$.

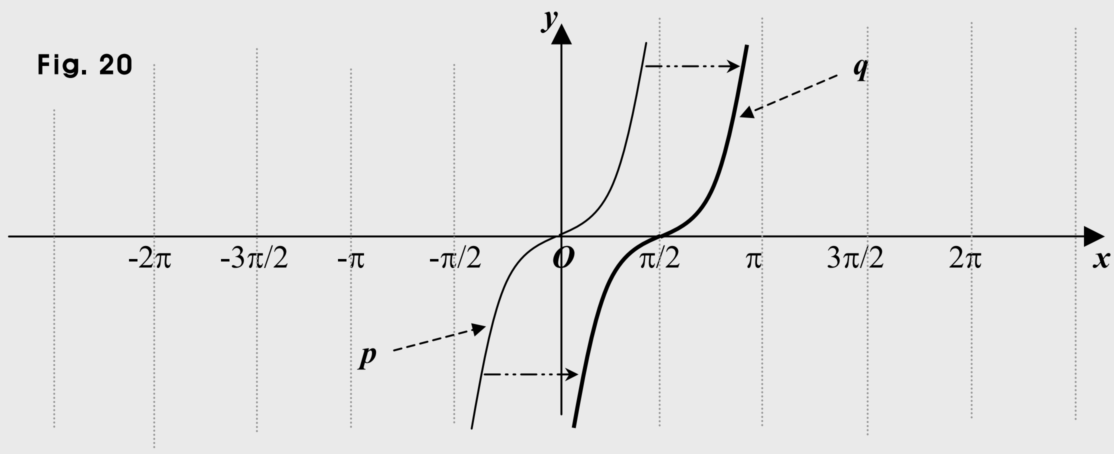
Thus, as in the case of sine or cosine functions, if the phase $\boldsymbol{a}$ is positive, the curve gets shifted to the left, and if negative, it gets shifted to the right.

Assuming for instance, shifting by $\frac{\pi}{2}$ in the direction of the $\boldsymbol{x}$-axis, that is, to the right, the curve of $\boldsymbol{p(x)} = \mathbf{tan}\ \boldsymbol{x}$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, we get the curve of a function as follows.

$$\boldsymbol{q(x)} = \mathbf{tan}\ (\boldsymbol{x} - \frac{\pi}{2}) \text{ for } \mathbf{0} < \boldsymbol{x} < \pi.$$

And putting the two curves in a graph, we can get this:

Fig. 20



So the curve of $\boldsymbol{p}$ gets moved to the right in the amount of $\frac{\pi}{2}$. And the new curve is the curve of $\boldsymbol{q}$. So we can call the phase a horizontal shift, too.
What then about the constant $\boldsymbol{b}$ in $\boldsymbol{y} = \boldsymbol{Q(x)} = \boldsymbol{A}\cdot\mathbf{tan}\ \boldsymbol{wx} + \boldsymbol{b}$?

It is called a vertical shift. So shifting the curve of $\boldsymbol{y} = \boldsymbol{V(x)} = \boldsymbol{A}\cdot\mathbf{tan}\ \boldsymbol{w(x + a)}$ by $\boldsymbol{b}$ in the direction of the $\boldsymbol{y}$-axis, we get the curve of the function $\boldsymbol{F}$ above.

So the two curves themselves of $\boldsymbol{F}$ and $\boldsymbol{V}$ are the same, and moving the curve of $\boldsymbol{V}$ in the amount of $\boldsymbol{b}$ along the $\boldsymbol{y}$-axis, we get the curve of $\boldsymbol{F}$.

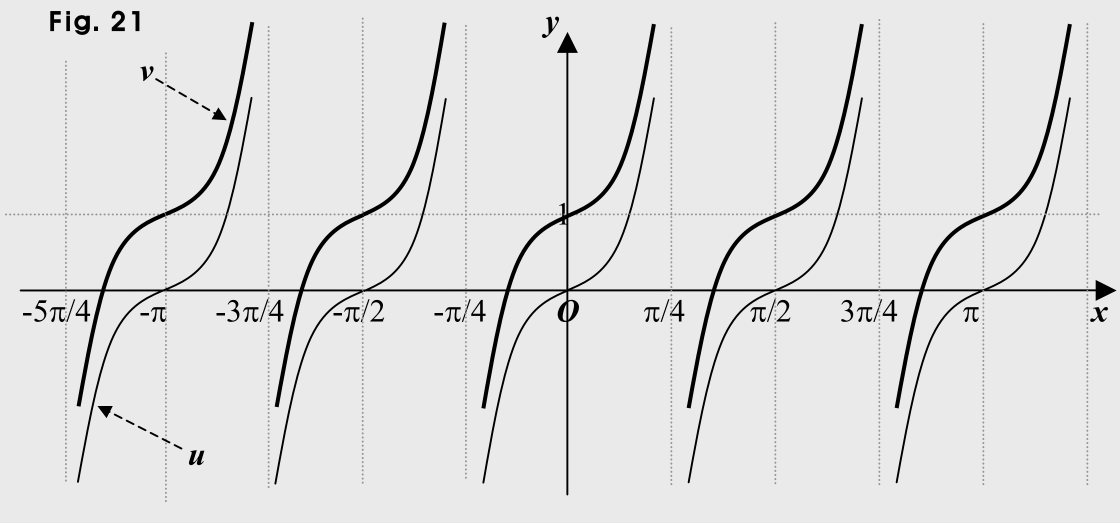
And if $\boldsymbol{b}$ is positive, the curve gets shifted upward, and if negative, it moves downward.

So assuming for instance, shifting the curve of $\boldsymbol{u(x)} = \mathbf{tan}\ \boldsymbol{x}$ by 1 in the direction of the $\boldsymbol{y}$-axis, that is, upward, we get the curve of a function as follows.

$$\boldsymbol{v(x)} = \mathbf{tan}\ \boldsymbol{x} + \mathbf{1}.$$

And putting ***u*** and ***v*** in a graph, we can get this:

Fig. 21



So the curve of ***u*** gets moved upward in the amount of 1. And the new curve is the curve of ***v***. So we can call ***b*** a vertical shift.

What about ***A*** though, in the general form $\boldsymbol{A}\cdot\mathbf{tan}\ \boldsymbol{w(x + a) + b}$?

In the cases of the sine and the cosine functions, ***A*** is called an amplitude, but in the case of tangent funcitons, it can amplify or reduce the output of a tangent function.

The tangent function has no amplitude. So ***A*** is not the amplitude, but can be an amplifier. If $|\boldsymbol{A}|$ is bigger, the output changes faster, but if $|\boldsymbol{A}|$ is smaller, the output change is slower. Let's see now how it works.

We know $\boldsymbol{A}$ is a constant.
So for instance, assuming $\boldsymbol{y = B(x) = \tan x}$, and multiplying by $\boldsymbol{A}$ the $\boldsymbol{y}$-value in the function $\boldsymbol{B}$ for an $\boldsymbol{x}$-value $\frac{\pi}{6}$, we get the $\boldsymbol{y}$-value in a function $\boldsymbol{y = D(x) = A\cdot\tan x}$ for the same $\boldsymbol{x}$-value $\frac{\pi}{6}$. That is to say that we get $\boldsymbol{D(x) = A\cdot B(x)}$.

If for instance, $\boldsymbol{A = 2}$, and $\boldsymbol{x} = \frac{\pi}{6}$, we get $\boldsymbol{D}(\frac{\pi}{6}) = \boldsymbol{2B}(\frac{\pi}{6}) = \frac{2\sqrt{3}}{3}$, since $\boldsymbol{\tan}\frac{\pi}{6} = \frac{\sqrt{3}}{3}$.

If for another instance, $\boldsymbol{A = 0.5}$, and $\boldsymbol{x} = \frac{\pi}{4}$, we get $\boldsymbol{D}(\frac{\pi}{4}) = \boldsymbol{0.5B}(\frac{\pi}{4}) = \boldsymbol{0.5}$, since $\boldsymbol{\tan}\frac{\pi}{4} = \boldsymbol{1}$.

And let's now put in a graph, for instance, the curve of a tangent function as follows.

$$\boldsymbol{y = T(x) = -2\tan(-2x + \tfrac{\pi}{2}) + 1} \text{ where } \boldsymbol{-2x} + \frac{\pi}{2} \neq \frac{(2\boldsymbol{n}+1)\pi}{2} \text{ for } \boldsymbol{n} \text{ integer}$$

Then. to begin with, we have $\boldsymbol{\tan(-\theta) = -\tan\theta}$.
So we can get $\boldsymbol{\tan(-2x + \tfrac{\pi}{2}) = \tan\{-(2x - \tfrac{\pi}{2})\} = -\tan(2x - \tfrac{\pi}{2})}$.

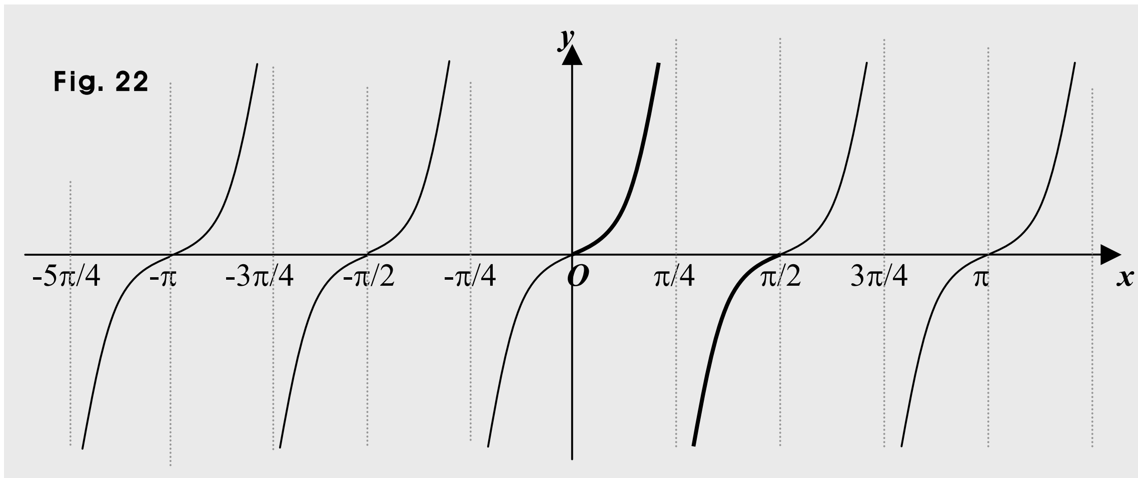
Thus, we get $\boldsymbol{-2\tan(-2x + \tfrac{\pi}{2}) + 1 = -2\tan(2x - \tfrac{\pi}{2}) + 1}$.

And putting it in the general form, we get $\boldsymbol{-2\tan 2(x - \tfrac{\pi}{4}) + 1}$.

So we can now see that the frequency is 2, the phase is $-\frac{\pi}{4}$, that is, shifting by $\frac{\pi}{4}$ to the right, and the vertical shift is 1.

And next, putting it in a graph, we may want to begin with the prototype $\boldsymbol{\tan x}$.

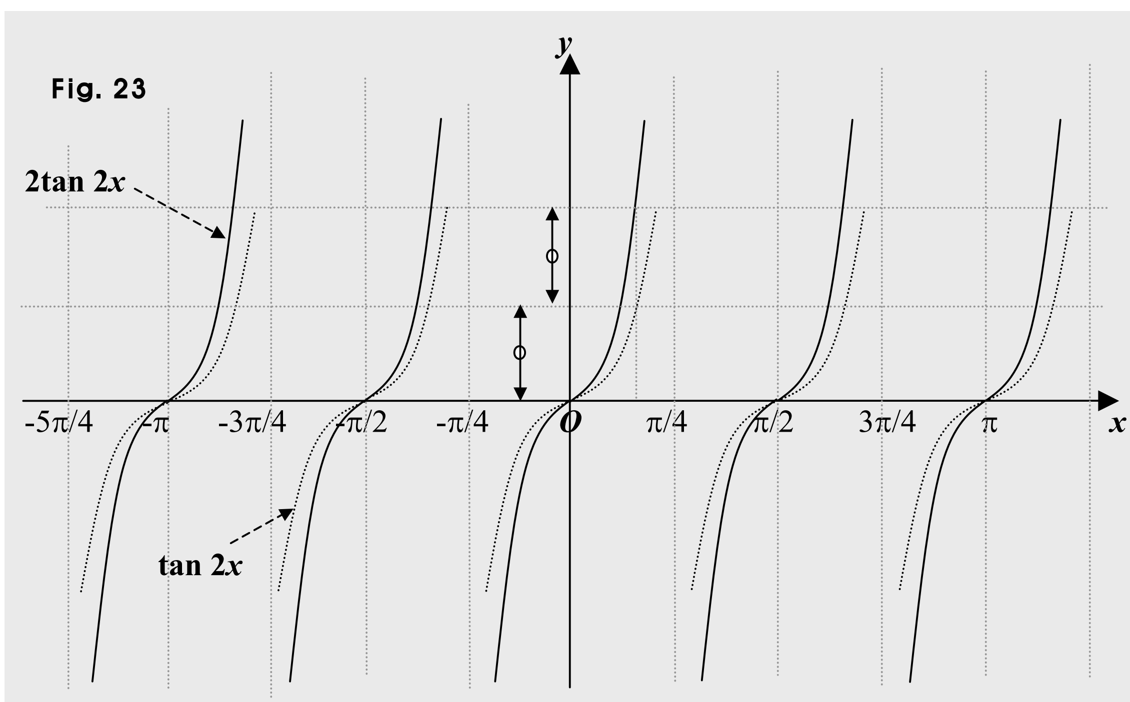
First, setting the frequency to 2, we get $\boldsymbol{\tan 2x}$. We know that the period is the basic period over the frequency, and that the basic period for the tangent is π.
So we can see that the period is $\frac{\pi}{2}$.
Thus, we can put the curve of $\boldsymbol{\tan 2x}$ the way as follows.



Next, setting the constant ***A*** to 2, we get **2 tan 2*x***.

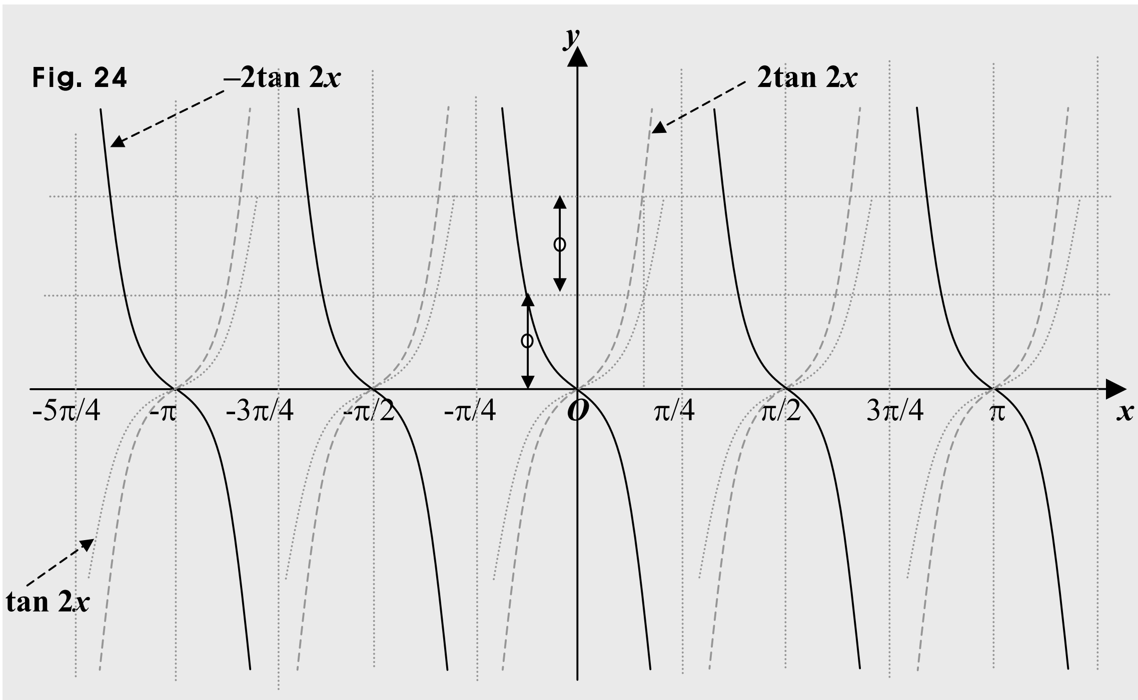
And assuming ***m*(*x*) = tan 2*x*** and ***n*(*x*) = 2 tan 2*x***, we get ***n*(*x*) = 2*m*(*x*)** for every ***x***-value.

Thus, we can put in a graph the curve of **2 tan 2*x*** the way as follows.



And we want to put a negative sign in front. So we get **–2 tan 2*x***.

Then, the curve is as follows.

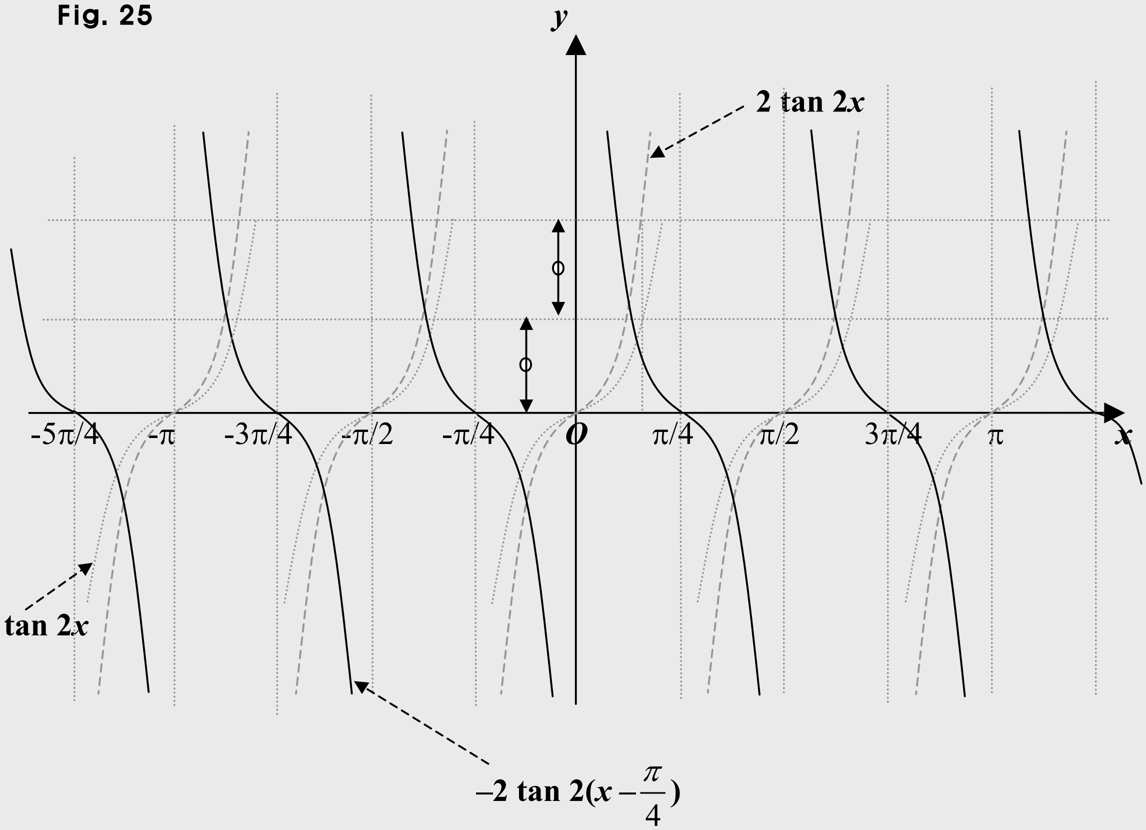


Next, setting the phase to $-\frac{\pi}{4}$, we get $\boldsymbol{-2\tan 2(x-\frac{\pi}{4})}$.

Then, shifting the curve of $\boldsymbol{-2\tan 2x}$ by $\frac{\pi}{4}$ in the direction of the ***x***-axis, we get the curve of $\boldsymbol{-2\tan 2(x-\frac{\pi}{4})}$.

Moving therefore, the curve of $\boldsymbol{-2\tan 2x}$ in the amount of $\frac{\pi}{4}$ to the right, we get the curve of $\boldsymbol{-2\tan 2(x-\frac{\pi}{4})}$, which is as follows.

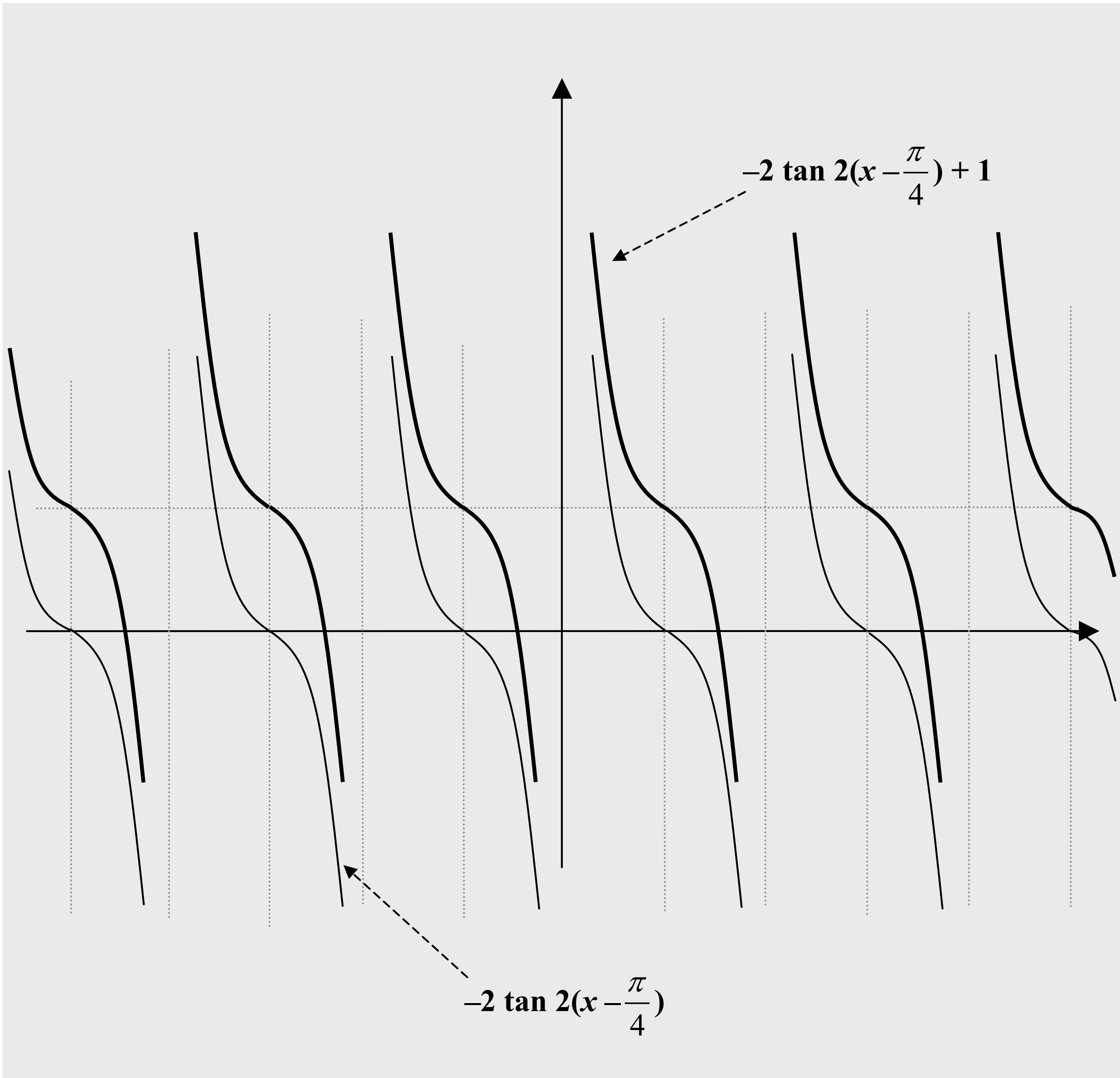
Fig. 25



And next, setting the vertical shift to 1, we get $\mathbf{-2tan\ (2}x-\frac{\pi}{4}\mathbf{)+1}$.

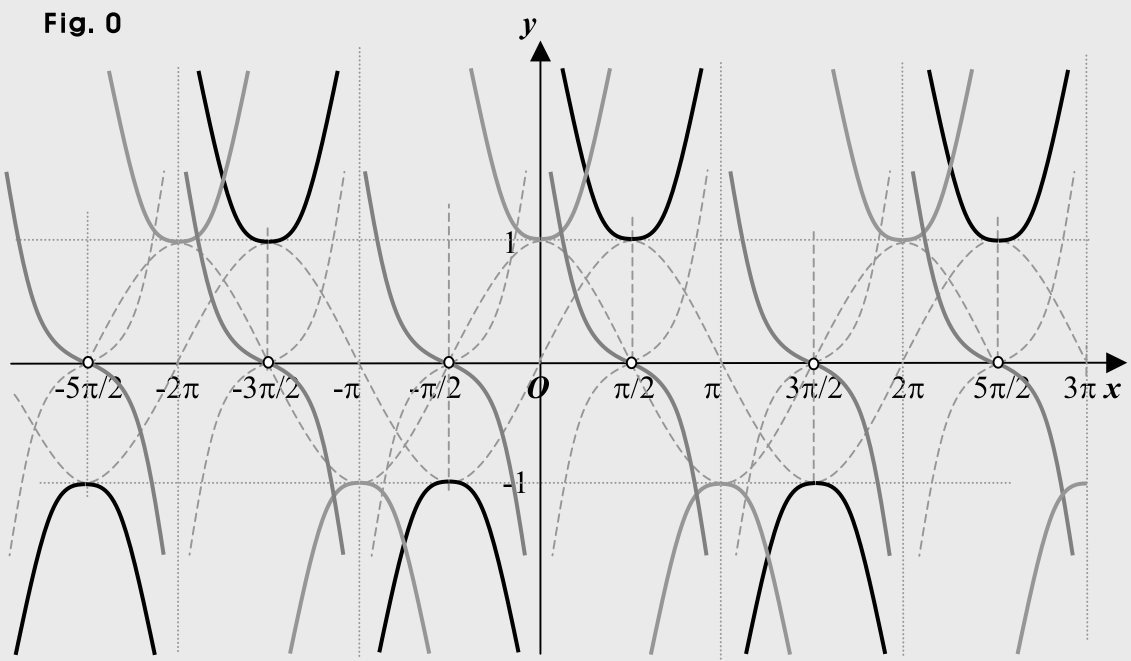
Then, we move upward the curve of $\mathbf{-2tan\ 2(}x-\frac{\pi}{4}\mathbf{)}$ in the amount of 1 to get the curve of $\mathbf{-2tan\ 2(}x-\frac{\pi}{4}\mathbf{)+1}$, which is the curve of the function ***T*** given.

Thus, we can put in a graph the curve of the function ***T*** the way as follows.


$-2 \tan 2(x - \frac{\pi}{4}) + 1$
$-2 \tan 2(x - \frac{\pi}{4})$

7. The Reciprocals

Fig. 0



7.0. Cosecant

What is cosecant?

It is the reciprocal of the sine.

We know the sine is $\frac{\text{the opposite}}{\text{the hypotenuse}}$.

So taking the reciprocal of the sine, we get $\frac{\text{the hypotenuse}}{\text{the opposite}}$.

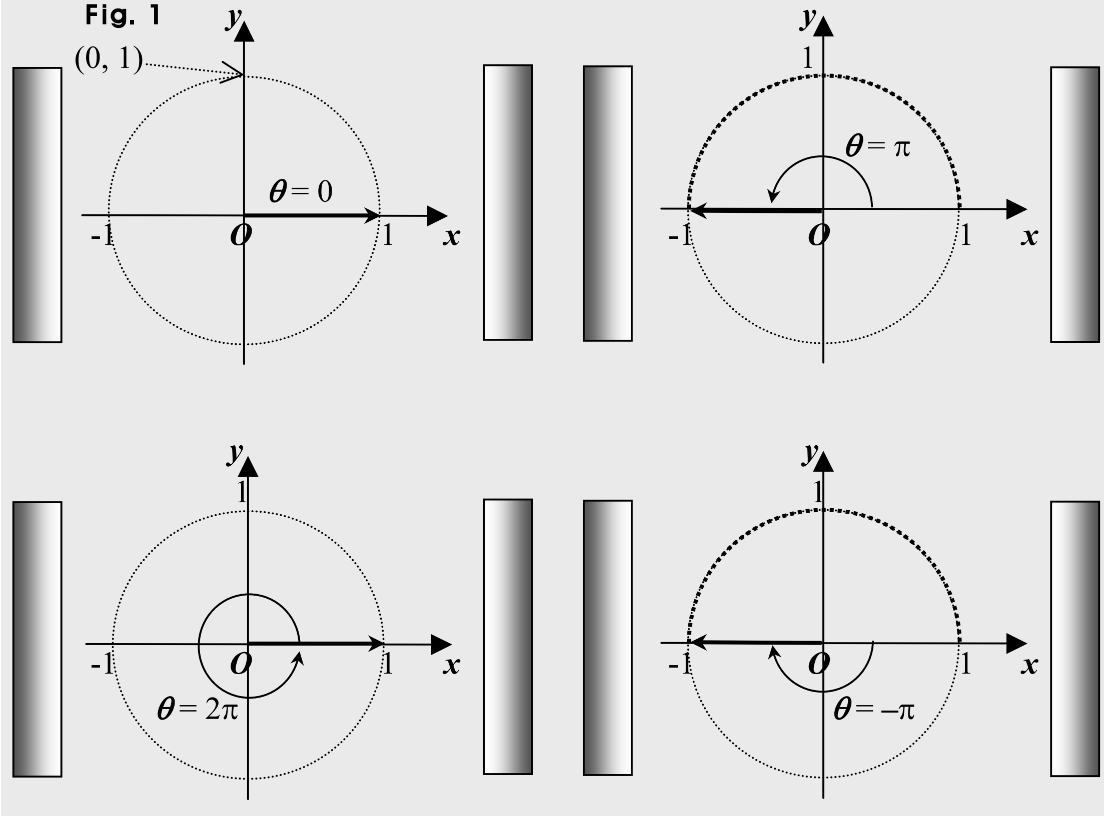
And taking the reciprocal of the sine, we get another trig ratio called the cosecant, denoted by **csc**. Taking thus, the reciprocal of **sin *x***, we get **csc *x***.

And we can take **sin x** as a function, can set $\boldsymbol{y = f(x) = \sin x}$, and call it a sine function. What then about a cosecant function?

We can make another trig function called a cosecant function using **csc x**, of course.
So for instance, we can make cosecant functions as $\boldsymbol{y = g(x) = \csc x}$ or $\boldsymbol{y = h(x) = 2\csc x}$.
However, the cosecant functions above cannot be defined for all angles. Why not?

We know that the sine is the opposite over the hypotenuse.
So the cosecant is the hypotenuse over the opposite, since it is the reciprocal of the sine.
And we know that the opposite can be 0 for some angles as 0, $-\pi$, and π.
In Fig. 1 below, each of the projections on the $\boldsymbol{y}$-axis is 0, and is the opposite.

Fig. 1



So the function **csc x** cannot be defined for $\boldsymbol{x = \pi}$. And in general, it cannot be defined for $\boldsymbol{x = n\pi}$ where $\boldsymbol{n}$ is an integer.

Thus, just setting $y = g(x) = \csc x$, we mean that $x \neq n\pi$ where n is an integer.

That is, the domain is a set of all real numbers less $n\pi$ for n integer.

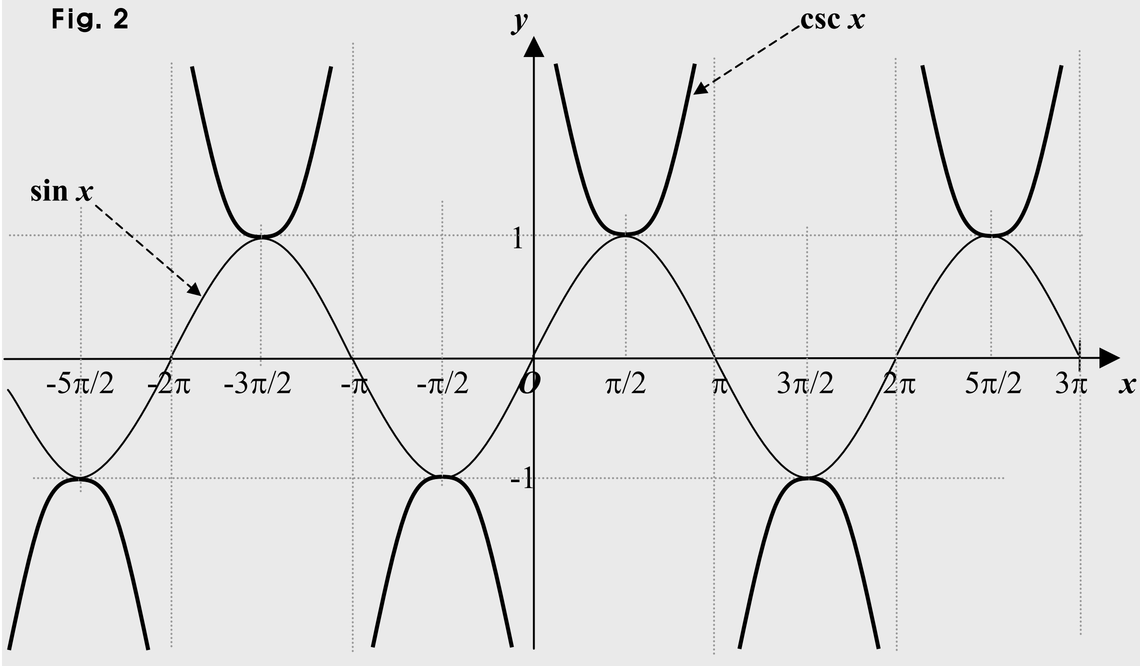
So we can define or specify the cosecant function the way as follows, too:

$y = g(x) = \csc x$ for $x \neq n\pi$ for n integer.

What then about its curve?

We know the cosecant is the reciprocal of the sine. So keeping the fact in mind, and using the curve of the sine function $y = f(x) = \sin x$, we can put in a graph, the curve of the cosecant function g the way as follows.

Fig. 2



So we can now see that the y-values, that is, the outputs are grater than or equal to 1 or less than or equal to –1. Thus, the range is $y \leq -1$ or $y \geq 1$, that is, $|y| \geq 1$.

And of course, if the domain is different, the range can be other than the one above. Assuming for instance, $y = h(x) = \csc x$ for $0 < x < \pi$, we get $y \geq 1$, which is the range.

And the cosecant function $y = g(x) = \csc x$ can be called the prototype, and is in the most basic form. Assuming G is a cosecant function, too, and the domain is a set of all real numbers less $n\pi$ for n integer, and using a general form, we can put it the way as follows.

$$y = G(x) = A\cdot\csc\{w(x + a)\} + b$$

where $w(x + a) \neq n\pi$ for n integer, and A, w, a, and b are constant.

And just simply, we often put it this way, too: $y = G(x) = A\cdot\csc w(x + a) + b$.

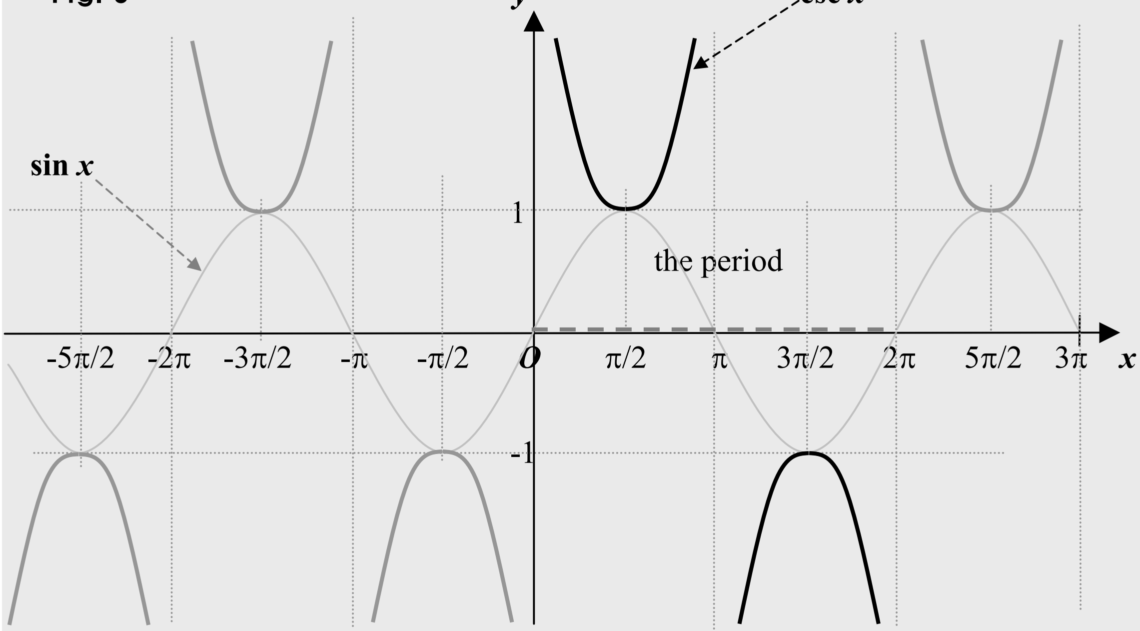
Then, $|w|$ is called the frequency, $\frac{2\pi}{|w|}$ is the period, and a is called the phase.

So in the function $y = g(x) = \csc x$, the phase is 0, and the frequency is 1, so the period is 2π. What do we mean by though, the period?

Looking at the curve above, we can see that a part of the cuve repleats itself. The part is in two pieces though.
And the part is in fact, the smallest part repeating itself. What then is the part?

It is the part from 0 to 2π. So the part repeats itself every 2π interval. And we call such an interval a period.

Fig. 3



So the interval 2π is the period in the cosecant function $y = g(x) = \csc x$.

Thus, we call a cosecant function a *periodic* function, too.
And the cosecant function $\boldsymbol{g}$ above is the prototype, and thus, is in the most basic form.
So the period $\mathbf{2\pi}$ can be called the *basic period* in cosecant functions.
In other words. the interval $\mathbf{2\pi}$ can be called the *basic interval* in cosecant functions.

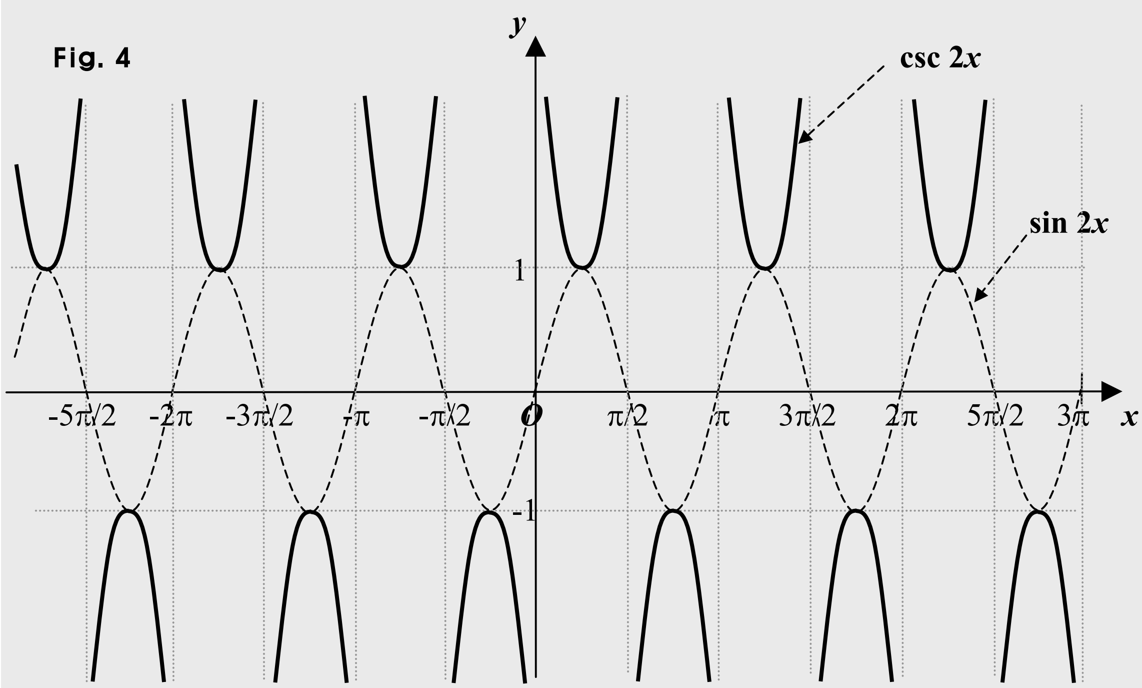
And for instance, in the general form, setting $\boldsymbol{w}$ to 2, $\boldsymbol{a}$ and $\boldsymbol{b}$ to 0 each, and $\boldsymbol{A}$ to 1, we get a new function where $\boldsymbol{y} = \boldsymbol{k}(\boldsymbol{x}) = \mathbf{csc}\ \mathbf{2}\boldsymbol{x}$. What then is its domain?

We know that $\mathbf{csc}\ \boldsymbol{\theta}$ is not defined for $\boldsymbol{\theta} = \boldsymbol{n}\pi$ where $\boldsymbol{n}$ is an integer.
So in $\mathbf{csc}\ \mathbf{2}\boldsymbol{x}$, we don't want $\mathbf{2}\boldsymbol{x}$ to be $\boldsymbol{n}\pi$.

That is, we want $\mathbf{2}\boldsymbol{x} \neq \boldsymbol{n}\pi \Rightarrow \boldsymbol{x} \neq \frac{n\pi}{2}$, which means $\boldsymbol{x}$ is not an integer multiple of 90°.

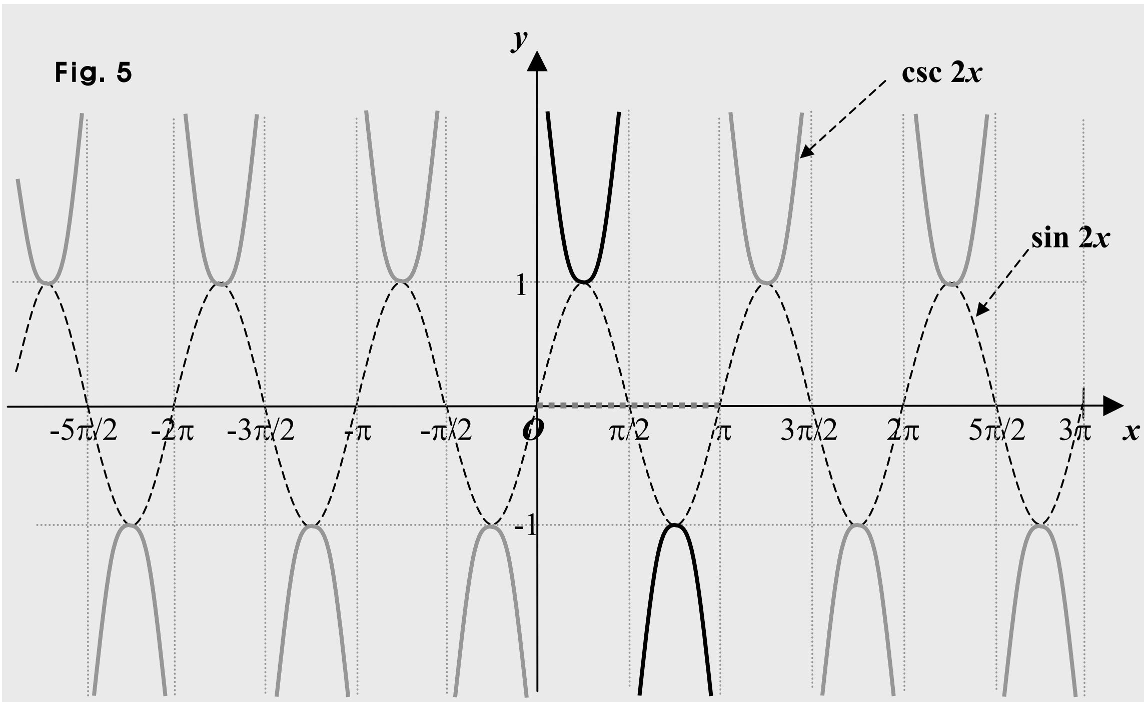
Thus, the domain of the function $\boldsymbol{k}(\boldsymbol{x}) = \mathbf{csc}\ \mathbf{2}\boldsymbol{x}$ can be a set of all real numbers less $\frac{n\pi}{2}$ for $\boldsymbol{n}$ integer. And of course, if $\boldsymbol{E}$ is the domain just described, $\boldsymbol{E}$ is the biggest domain that the cosecant function $\mathbf{csc}\ \mathbf{2}\boldsymbol{x}$ can have. So we can put in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane, the curve of the cosecant function $\boldsymbol{y} = \boldsymbol{k}(\boldsymbol{x}) = \mathbf{csc}\ \mathbf{2}\boldsymbol{x}$ the way as follows.

Fig. 4



(Note that all the curved lines in solid black form the curve of $\mathbf{csc}\ \mathbf{2}\boldsymbol{x}$.)

Then again, we can see that a part of the cuve repleats itself, and is in two pieces.
And the part is of course, the smallest part repeating itself. What then is the part?



It is the part from 0 to π. So the part repeats itself every π interval.

And we call such an interval a period. So the interval π is the period in the function ***k***.
And the smallest part stated above appears twice in the basic interval 2π, which is the basic period.
Then, we say that the frequency is 2 in the function ***k***.

So the frequency is the number of times the smallest part repeating itself appears in the basic inverval 2π, which is the basic period.

In short, the frequency is the number of times the smallest part appears in 2π.

More precisely though, we want to put the frequency the way as follows.
In case of cosecant functions, the frequency is the number of times the samllest part repeating itself appears in the basic period or the basic interval 2π.

Thus, in the general form $\boldsymbol{A}\cdot\mathbf{csc}\ \boldsymbol{w}(\boldsymbol{x}+\boldsymbol{a})+\boldsymbol{b}$, $|\boldsymbol{w}|$ is the frequency.

Then, given a cosecant function in the general form, how can we get its period?

We know that $|w|$ is the frequency, that is, the number of times the smallest part repeating itself appears in the basic interval (period) 2π, and that the interval that fits the smallest part is the period.

So using the frequency $|w|$ and the basic interval 2π, we can put the period this way: $\dfrac{2\pi}{|w|}$.

And we can have a cosecant function where $y = h(x) = \csc(-x)$ for $x \neq \dfrac{n\pi}{2}$ for n integer.

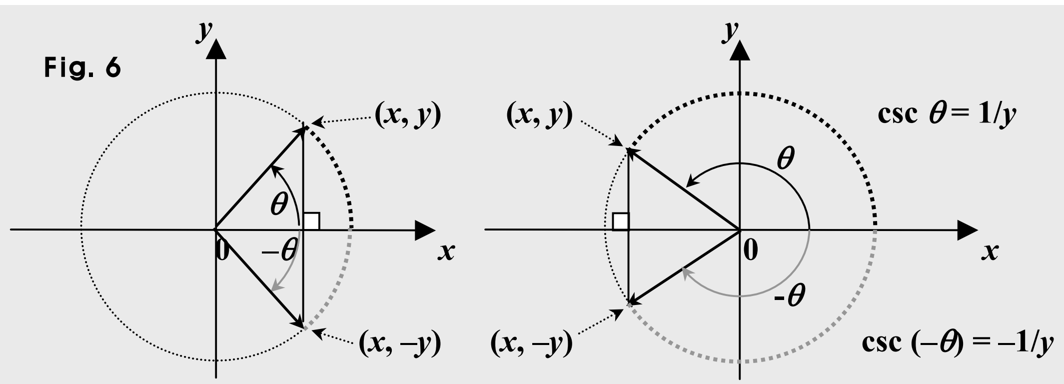
Then, the frequency in the cosecant function h is $|-1|$, simply because a frequency is positive.

And we can put h this way, too: $y = h(x) = -\csc x$ for $x \neq \dfrac{n\pi}{2}$ for n integer.

It's because $\csc(-x) = -\csc x$.

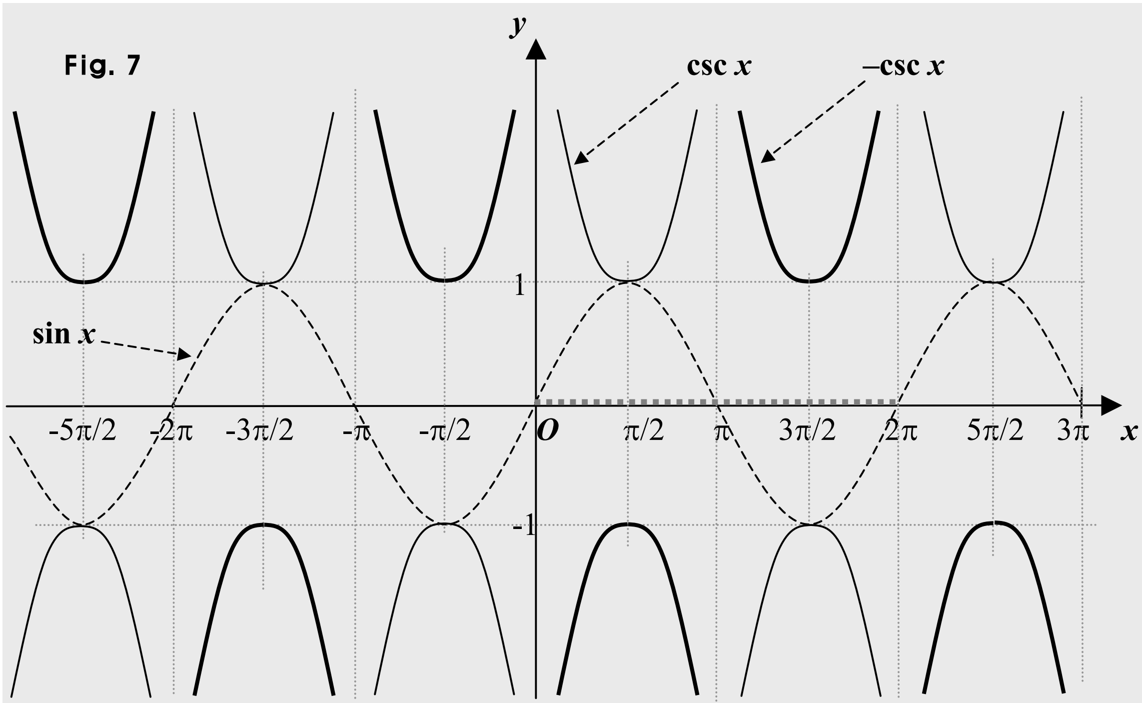
And we know the cosecant is the hypotenuse over the opposite.
So assuming the vector, that is, the hypotenuse is of length 1, we can get the graph as follows.

Fig. 6



So we get $\csc(-\theta) = -\csc\theta$, and can put in the x-y plane, the curve of $-\csc x$ the way as follows.

Fig. 7



And we can see every 2π interval, the smallest part repeating itself, which is in two pieces though. So the period of $h(x) = -\csc x$ is 2π, too.
And next, what do we mean by the phase?

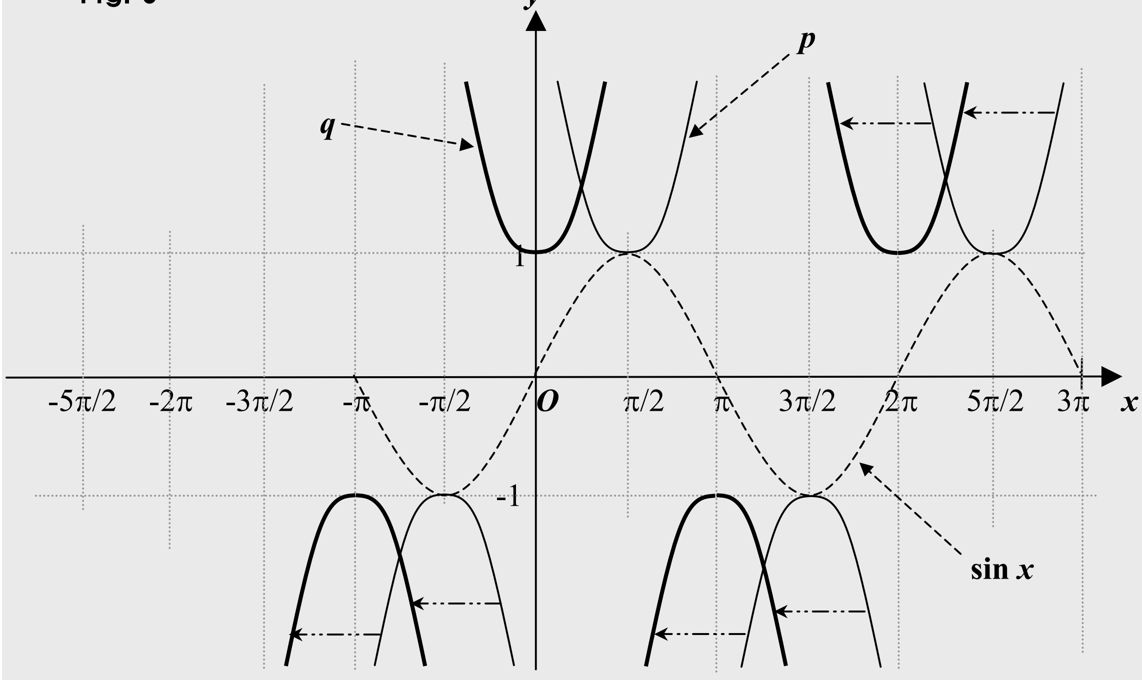
Using the general form, we can put a cosecant function called G the way as follows.

$$y = G(x) = A\cdot\csc w(x + a) + b, \text{ where } w(x + a) \neq n\pi \text{ for } n \text{ integer.}$$

Then, a is called the phase, $|w|$ is called the frequency, and $\dfrac{2\pi}{|w|}$ is the period.
Now, shifting by $-a$ in the direction of the x-axis, the curve of $y = Q(x) = A\cdot\csc wx + b$ where $wx \neq n\pi$ for n integer, we get the curve of the function G above.

So the two curves themselves of G and Q are the same, and moving the curve of Q in the amount of $-a$ along the x-axis, we get the curve of G. And if the phase is positive, the curve gets shifted to the left, and if negative, it gets shifted to the right.
Assuming for instance, shifting the curve of $p(x) = \csc x$ for $-\pi < x < 3\pi$ by $-\dfrac{\pi}{2}$ in the direction of the x-axis, that is, to the left, we get the curve of a function as follows.
$q(x) = \csc\left(x + \dfrac{\pi}{2}\right)$ for $-\dfrac{3\pi}{2} < x < \dfrac{5\pi}{2}$. So we can put the two curves in a graph this way:

Fig. 8



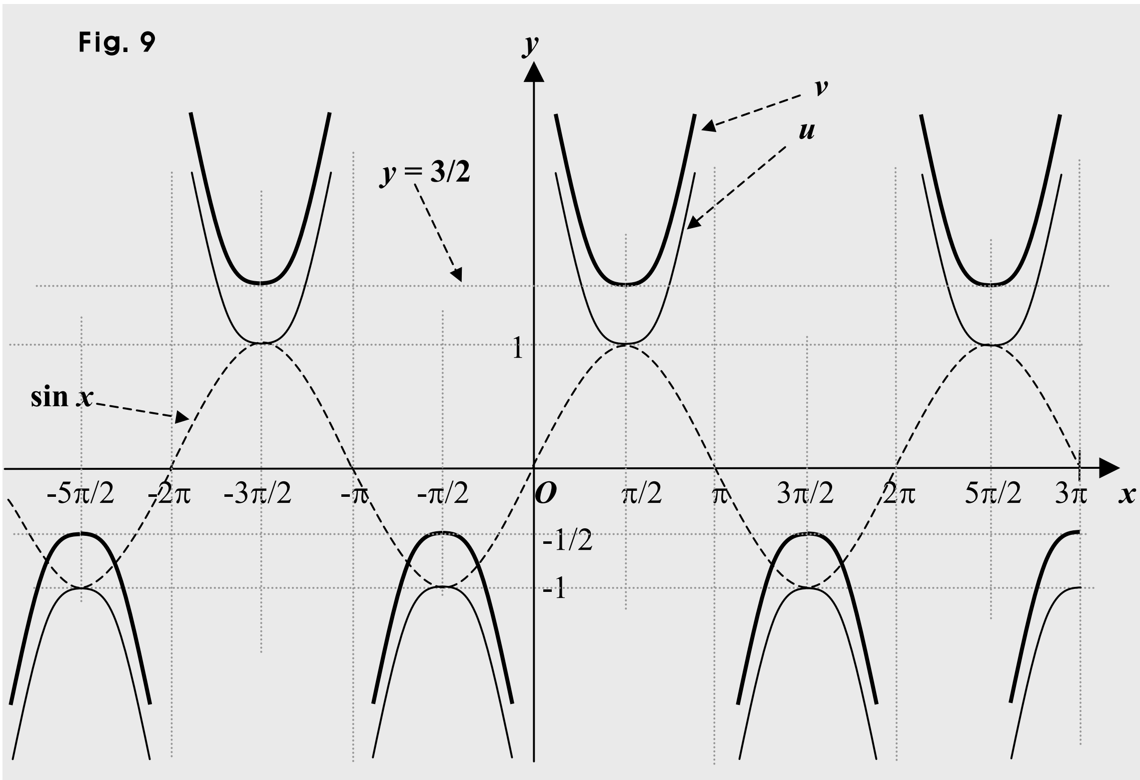
Thus, the curve of $\boldsymbol{p}$ gets moved to the left in the amount of $\frac{\pi}{2}$. And the new curve is the curve of $\boldsymbol{q}$. So we can call the phase a horizontal shift, too.
What then about the constant $\boldsymbol{b}$ in $\boldsymbol{y = Q(x) = A \cdot \csc wx + b}$?

It is called a vertical shift.
So shifting the curve of a function $\boldsymbol{y = V(x) = A \cdot \csc w(x + a)}$ by $\boldsymbol{b}$ in the direction of the $\boldsymbol{y}$-axis, we get the curve of the function $\boldsymbol{G}$, which is: $\boldsymbol{y = G(x) = A \cdot \csc w(x + a) + b}$.

Thus, the two curves themselves of $\boldsymbol{G}$ and $\boldsymbol{V}$ are the same, and moving the curve of $\boldsymbol{V}$ in the amount of $\boldsymbol{b}$ along the $\boldsymbol{y}$-axis, we get the curve of $\boldsymbol{G}$.

And if $\boldsymbol{b}$ is positive, the curve is shifted upward, and if negative, it moves downward.
Assuming for instance, shifting by $\frac{1}{2}$ in the direction of the $\boldsymbol{y}$-axis, that is, upward, the curve of $\boldsymbol{u(x) = \csc x}$, we get the curve of a function as follows: $\boldsymbol{v(x) = \csc x} + \frac{1}{2}$.

Putting therefore, the two curves in a graph, we get this:



So the curve of ***u*** gets moved upward in the amount of $\frac{1}{2}$. And the new curve is the curve of ***v***. So we can call the constant ***b*** a vertical shift.

And let's now put in a graph, for instance, the curve of a cosecant function below:

$$\boldsymbol{y = S(x) = -\frac{1}{2}\csc(-2x + \pi) + 1}, \text{ where } \boldsymbol{-2x + \pi \neq n\pi} \text{ for } \boldsymbol{n} \text{ integer.}$$

To begin with, we have $\boldsymbol{\csc(-\theta) = -\csc\theta}$.

So we can get $\boldsymbol{\csc(-2x + \pi) = \csc\{-(2x - \pi)\} = -\csc(2x - \pi)}$.

Thus, we get $\boldsymbol{-\frac{1}{2}\csc(-2x + \pi) + 1 = \frac{1}{2}\csc(2x - \pi) + 1}$.

And putting it in the general form, we get $\boldsymbol{\frac{1}{2}\csc 2(x - \frac{\pi}{2}) + 1}$.

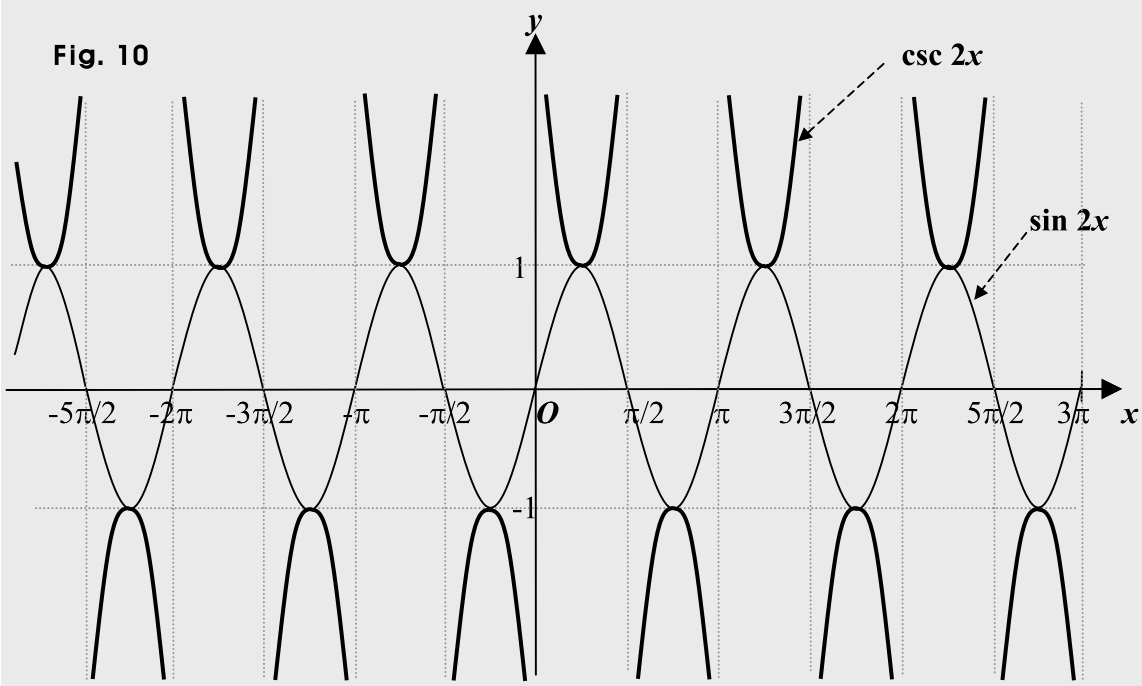
So we can now see that the frequency is 2, the phase is $-\frac{1}{2}$, that is, shifting by $\frac{1}{2}$ to the right, and the vertical shift is 1.

And next, putting it in a graph, we may want to begin with the prototype **csc *x***.
First, setting the frequency to 2, we get **csc 2*x***.

We know that the period is the basic period (interval) over the frequency, and that the basic interval (period) for the cosecant is 2π. So we can see that the period is $\frac{2\pi}{2} = \pi$.

Thus, we can put in a graph the curve with **csc 2*x*** the way as follows.

Fig. 10

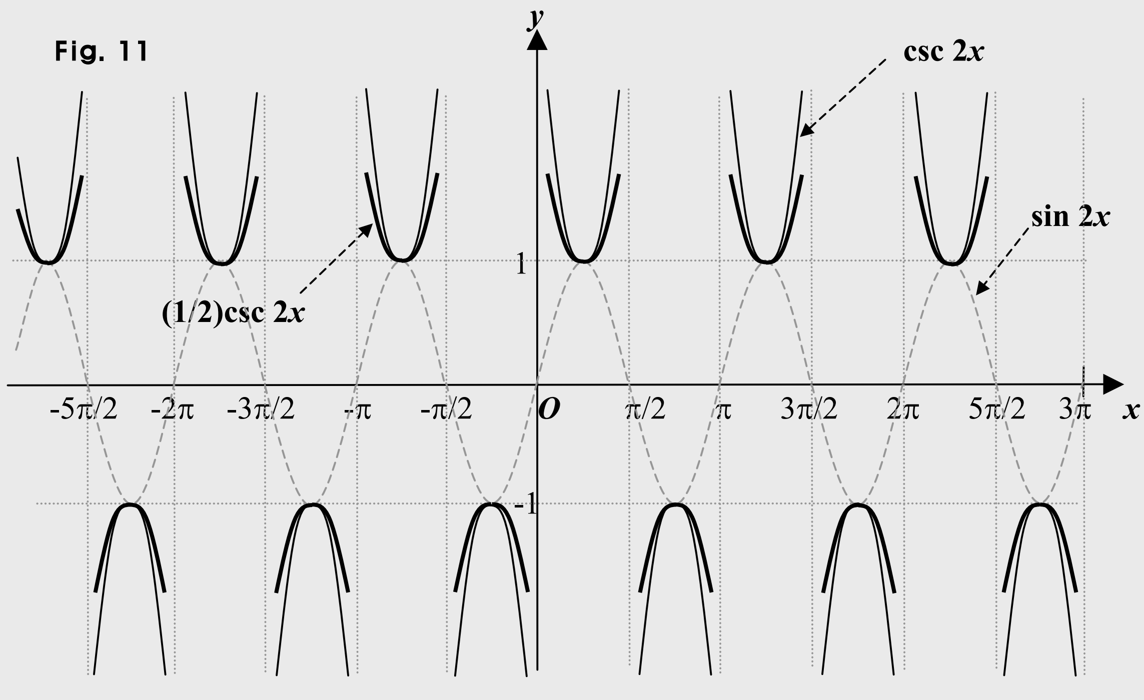


Next, setting the constant ***A*** to $\frac{1}{2}$, we get $\frac{1}{2}$ **csc 2*x***.

And assuming ***m*(*x*) = csc 2*x*** and ***n*(*x*)** = $\frac{1}{2}$ **csc 2*x***, we get ***n*(*x*)** = $\frac{1}{2}$ ***m*(*x*)** for every ***x***-value.

That is, for each and every ***x***-value, the ***y***-value in the function ***n*** is half the ***y***-value in the function ***m***. Thus, we can put in a graph the curve with $\frac{1}{2}$ **csc 2*x*** the way as follows.

Fig. 11



Next, setting the phase to $-\frac{\pi}{2}$, we get $\frac{1}{2}$ **csc 2(*x* $-\frac{\pi}{2}$).**

We know shifting the curve of with **csc *x*** by $-\frac{\pi}{2}$ in the direction of the ***x***-axis, we move the curve by $\frac{\pi}{2}$ to the left. Then, we get the curve with **csc (*x* $+\frac{\pi}{2}$).**

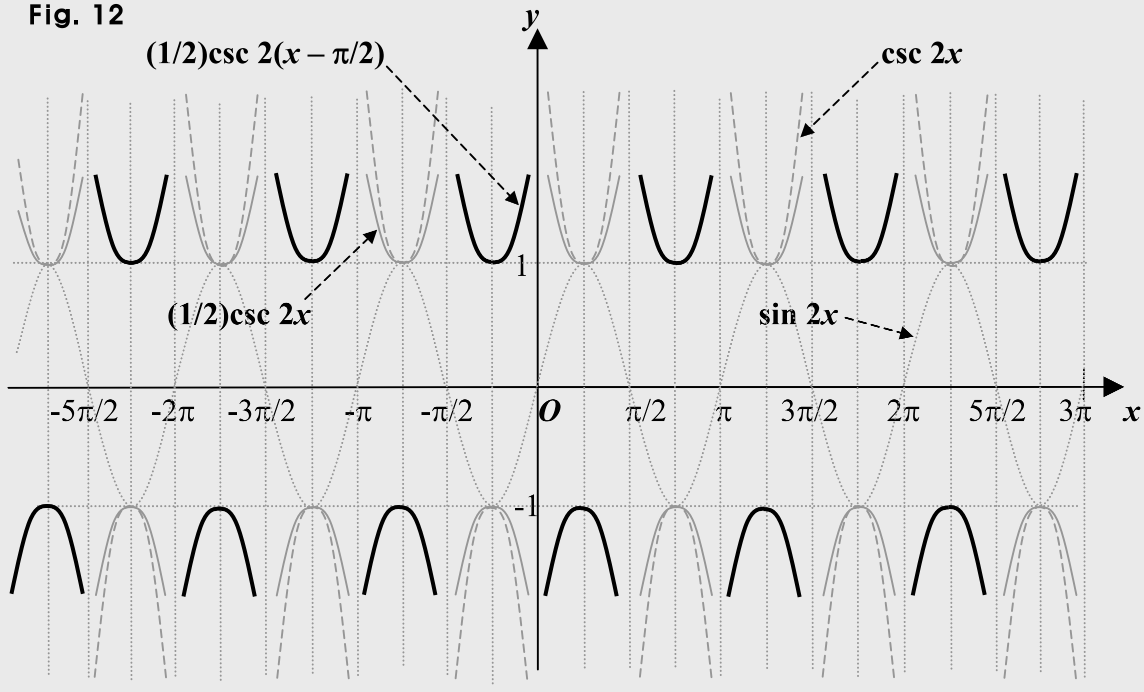
So shifting the curve with $\frac{1}{2}$ **csc 2*x*** by not $-\frac{\pi}{2}$ but $\frac{\pi}{2}$ in the direction of the ***x***-axis, we get the curve with $\frac{\pi}{2}$ **csc 2(*x* $-\frac{\pi}{2}$).** That is, moving the curve with $\frac{\pi}{2}$ **csc 2*x*** in the amount of $\frac{\pi}{2}$ to the right, we get the curve with $\frac{\pi}{2}$ **csc 2(*x* $-\frac{\pi}{2}$).**

So we do a horizontal shifting to the right in the amount of $\frac{\pi}{2}$.

Note however, if the domain is a set of all real numbers less $n\pi$ for n integer, the curve of $\frac{1}{2}$ **csc 2(*x* −** $\frac{\pi}{2}$**)** is in fact, no other than the curve of $\frac{1}{2}$ **csc 2(*x* +** $\frac{\pi}{2}$**)**. If however, the phase is not a multiple of $\frac{\pi}{2}$, the two curves are different.

Thus, we can put in a graph, the curve of $\frac{1}{2}$ **csc 2(*x* −** $\frac{\pi}{2}$**)** the way as follows.

Fig. 12



7.1. Secant

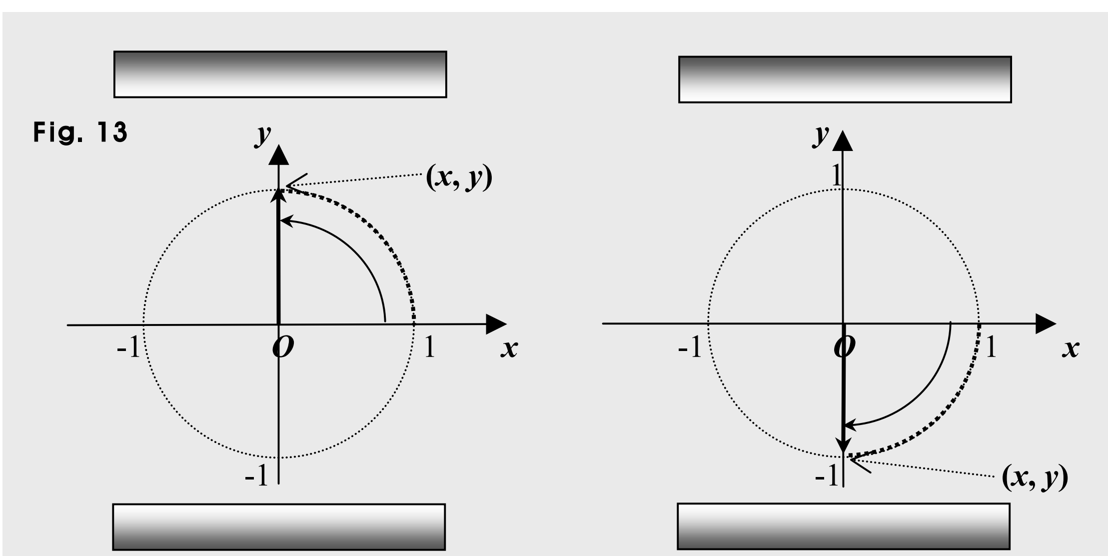
Taking next, the reciprocal of the cosine, we get another trig ratio called the secant, denoted by **sec**. Thus, taking the reciprocal of the cosine, that is, **cos *x***, we get **sec *x***.

And we can take **cos *x*** as a function, set $\boldsymbol{y = f(x) = \cos x}$, and call it a cosine function. What then about a secant function?

We can make another trig-function called a secant function using **sec *x***, of course.
So for instance, we can make secant functions as $\boldsymbol{y = g(x) = \sec x}$ or $\boldsymbol{y = q(x) = \sec 3x^2}$.
As in the case of a cosecant function as $\boldsymbol{y = h(x) = \csc x}$ however, the secant function above cannot be defined for all angles. Why not?

We know that the cosine is the adjacent over the hypotenuse.
So the secant is the hypotenuse over the adjacent, since it is the reciprocal of the cosine.
And we know that the adjacent can be 0 for some angles as $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. The projection on the $\boldsymbol{x}$-axis is 0, so the adjacent is 0 if the angle is an odd multiple of $\frac{\pi}{2}$ as in Fig. 14.

Fig. 13



So for instance, the secant **sec *x*** cannot be defined for $\boldsymbol{x} = \frac{\pi}{2}$. And in general, it cannot be defined for $\boldsymbol{x} = \frac{(2\boldsymbol{n}+1)\pi}{2}$ for ***n*** integer, that is, $\boldsymbol{x} = \frac{\boldsymbol{k}\pi}{2}$ where ***k*** is an odd integer.

So just setting: ***y*** **=** ***g*****(*x*) = sec *x***, we mean $\boldsymbol{x} \neq \frac{\boldsymbol{n}\pi}{2}$ for ***n*** odd.

That is, the domain is a set of all real numbers less $\frac{\boldsymbol{n}\pi}{2}$ where ***n*** is an odd integer.

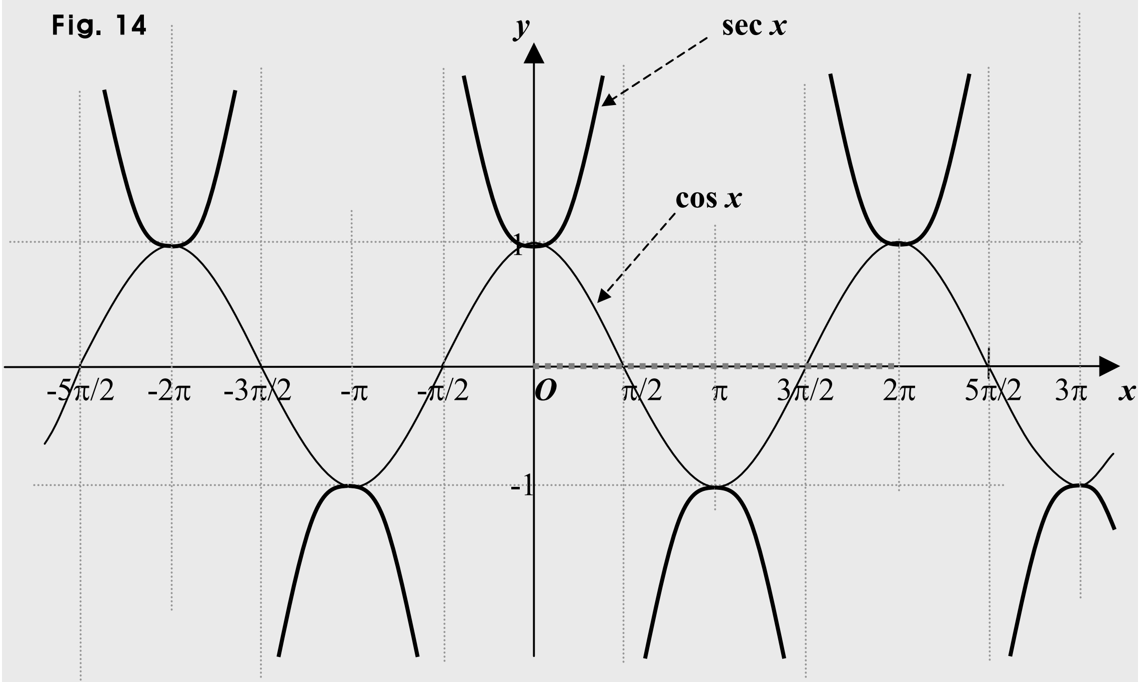
So we can define or specify the secant function the way as follows, too.

$$\boldsymbol{y} = \boldsymbol{g}(\boldsymbol{x}) = \mathbf{sec}\ \boldsymbol{x} \text{ for } \boldsymbol{x} \neq \frac{\boldsymbol{n}\pi}{2} \text{ for } \boldsymbol{n} \text{ odd.}$$

What then about its curve?

We know that the secant is the reciprocal of the cosine. So keeping the fact in mind, and using the curve of the cosine function ***y*** **=** ***f*****(*x*) = cos *x***, we can put in a graph, the curve of the secant function ***g*** the way as follows.

Fig. 14



So we can now see that the $\boldsymbol{y}$-values, that is, the outputs are grater than or equal to 1 or less than or equal to –1. Thus, the range is: $\boldsymbol{y \leq -1}$ or $\boldsymbol{y \geq 1}$, that is, $\boldsymbol{|y| \geq 1}$.
And of course, if the domain is different, the range can be other than the one above.
Assuming for instance, $\boldsymbol{y = h(x) = \sec x}$ for $\boldsymbol{0 < x <} \frac{\pi}{2}$, we get $\boldsymbol{y > 1}$, which is the range.

And the secant function $\boldsymbol{y = g(x) = \sec x}$ can be called the prototype, and is thus, in the most basic form. Assuming $\boldsymbol{G}$ is a secant function, too, and the domain is a set of all real numbers less $\frac{\boldsymbol{n}\pi}{2}$ for $\boldsymbol{n}$ odd, and using a general form, we can put it the way as follows.

$$\boldsymbol{y = G(x) = A \cdot \sec \{w(x + a)\} + b}$$

where $\boldsymbol{w(x + a)} \neq \frac{\boldsymbol{n}\pi}{2}$ for $\boldsymbol{n}$ odd, and $\boldsymbol{A}$, $\boldsymbol{w}$, $\boldsymbol{a}$, and $\boldsymbol{b}$ are constant

And we can just put it this way, too: $\boldsymbol{y = G(x) = A \cdot \sec w(x + a) + b}$.

Then, $\boldsymbol{|w|}$ is called the frequency, $\frac{\boldsymbol{2\pi}}{|\boldsymbol{w}|}$ is the period, and $\boldsymbol{a}$ is called the phase.
So in the function $\boldsymbol{y = g(x) = \sec x}$, the phase is 0, and the frequency is 1.

And looking at the curve above, we can see that a part of the cuve repleats itself. And the part is in fact, the smallest part, in three pieces though, repeating itself.
What then is the part?

It is the part from 0 to 2π. So the part repeats itself every 2π interval.
So the interval 2π is the period in the secant function $\boldsymbol{y = g(x) = \sec x}$.

Thus, we call a secant function a *periodic* function, too.
And the secant function $\boldsymbol{g}$ above is the prototype, and thus, is in the most basic form.

So the period **2π** can be called the *basic period* in secant functions.
In other words, the interval **2π** can be called the *basic interval* in secant functions.

And for instance, in the general form, setting $\boldsymbol{w}$ to 2, $\boldsymbol{a}$ and $\boldsymbol{b}$ to 0 each, and $\boldsymbol{A}$ to 1, we get a new function where $\boldsymbol{y = k(x) = \sec 2x}$. What then is the domain?

We know that **sec** $\boldsymbol{\theta}$ is not defined for $\boldsymbol{\theta} = \frac{\boldsymbol{n\pi}}{2}$ where $\boldsymbol{n}$ is an odd integer.

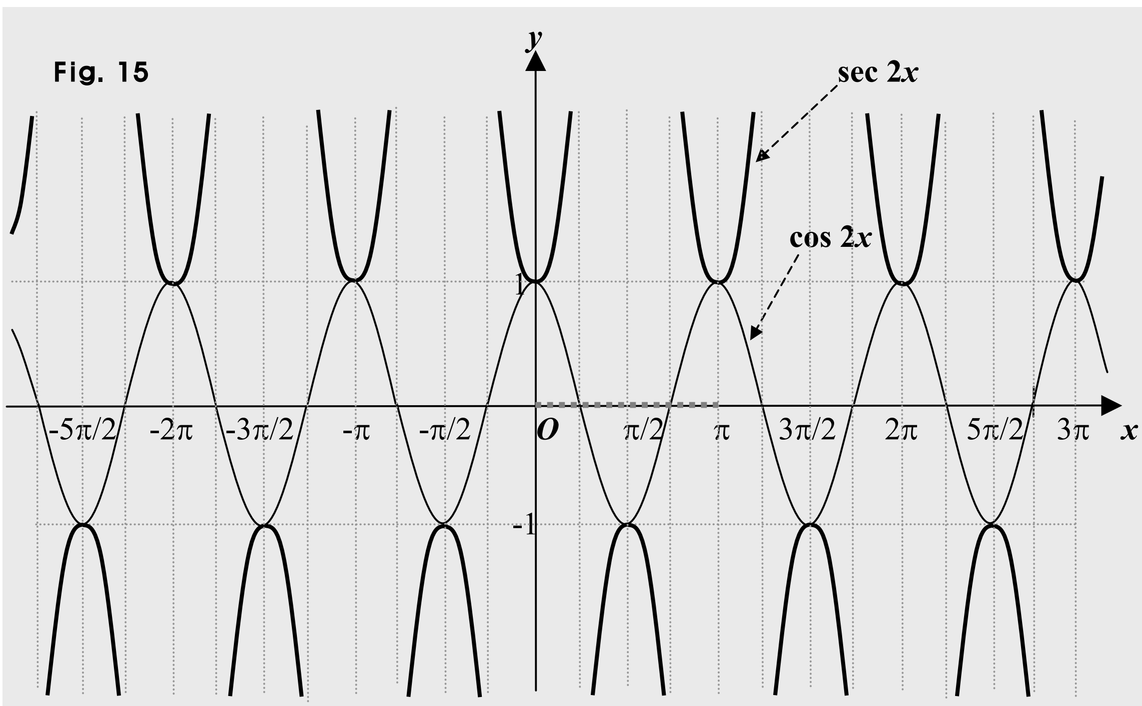
So in **sec 2*x***, we don't want **2*x*** to be $\frac{\boldsymbol{n\pi}}{2}$ for $\boldsymbol{n}$ odd.

That is, we want $\mathbf{2}\boldsymbol{x} \neq \frac{\boldsymbol{n\pi}}{2} \Rightarrow \boldsymbol{x} \neq \frac{\boldsymbol{n\pi}}{4}$ for $\boldsymbol{n}$ odd.

So the domain in the function **sec 2*x*** can be a set of all real numbers less $\frac{\boldsymbol{n\pi}}{4}$ for $\boldsymbol{n}$ odd.

And of course, if ***E*** is the domain just described, ***E*** is the biggest domain that the secant function **sec 2*x*** can have.

Then, putting in the ***x-y*** plane, the curve of the secant function ***k*(*x*) = sec 2*x***, we get this:



Then again, we can see that a part of the cuve repleats itself, and is in three pieces.

And the part is of course, the smallest part repeating itself. What then is the part?

It is the part from 0 to π. So the part repeats itself every π interval. Thus, the interval π is the period in the function ***k***. And the smallest part stated above appears twice in the basic interval 2π, which is the basic period.

Then, we say the frequency is 2 in the function $\boldsymbol{k}$.

So in case of secant functions, the frequency is the number of times the samllest part repeating itself appears in the basic period or the basic interval 2π.
Thus, in the general form $\boldsymbol{A \cdot \sec w(x + a) + b}$, $|\boldsymbol{w}|$ is the frequency.

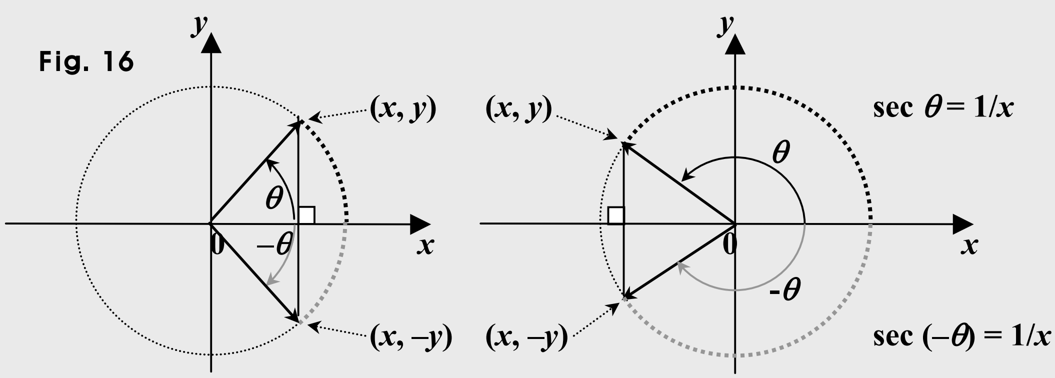
And we know that the interval that fits the smallest part is the period. So using the frequency $|\boldsymbol{w}|$ and the basic interval 2π, we can put the period this way: $\dfrac{2\pi}{|w|}$

And we can have a secant function where $\boldsymbol{y = h(x) = \sec(-x)}$. Then, the frequency in the secant function $\boldsymbol{h}$ is $|-1|$, simply because a frequency is positive.

And we can put $\boldsymbol{h}$ this way, too: $\boldsymbol{y = h(x) = \sec x}$, because $\boldsymbol{\sec(-x) = \sec x}$. Why though?

We know the secant is: the hypotenuse over the adjacent.
So assuming the vector, that is, the hypotenuse is of length 1, we can get this:

Fig. 16



So we get $\boldsymbol{\sec(-\theta) = \sec\theta}$. Thus, putting in a graph, the curve of $\boldsymbol{y = h(x) = \sec(-x)}$, we get the same curve as that of $\boldsymbol{y = g(x) = \sec x}$, and the period of $\boldsymbol{h}$ is 2π, too.

Next, shifting by $\boldsymbol{-a}$ in the direction of the $\boldsymbol{x}$-axis, the curve of $\boldsymbol{y = Q(x) = A \cdot \sec wx + b}$ where $\boldsymbol{wx} \neq \dfrac{n\pi}{2}$ for $\boldsymbol{n}$ odd, we get the curve of a function as follows.

$$y = G(x) = A \cdot \sec w(x + a) + b \text{ where } w(x + a) \neq \frac{n\pi}{2} \text{ for } n \text{ odd.}$$

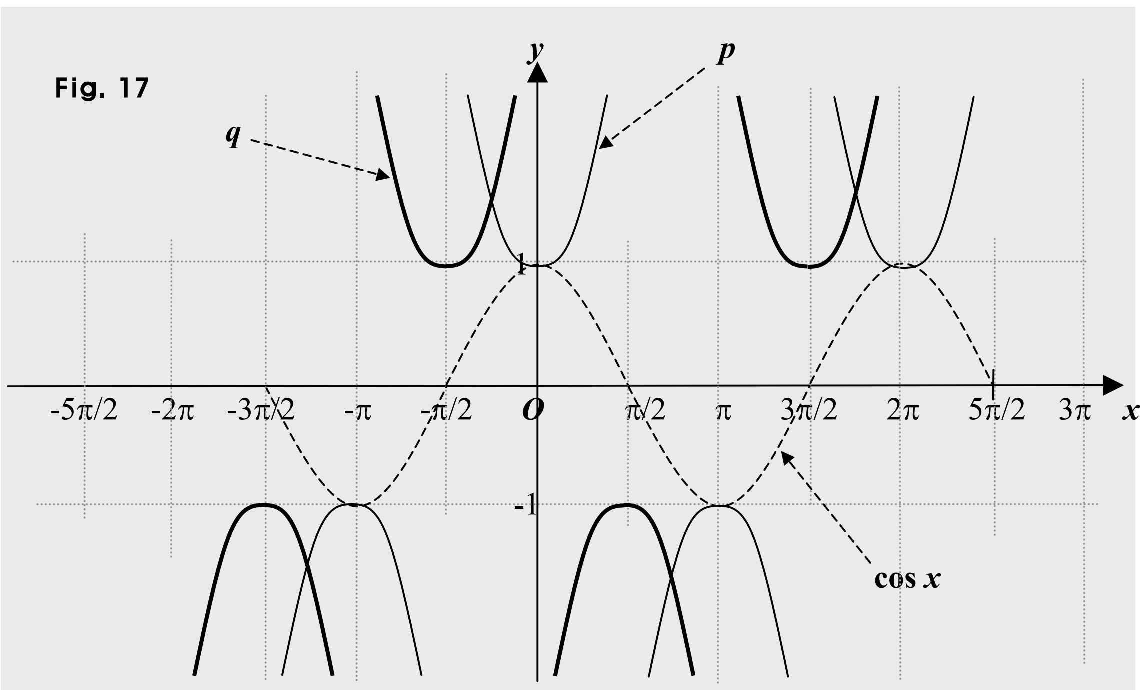
So the two curves themselves of ***G*** and ***Q*** are the same, and moving the curve of ***Q*** in the amount of ***–a*** along the ***x***-axis, we get the curve of ***G***.

And we call ***a*** the phase, and if ***a*** is positive, the curve gets shifted to the left, and if negative, it moves to the right.

Assuming for instance, shifting the curve of ***p***(***x***) = **sec *x*** for $-\frac{3\pi}{2} < x < \frac{5\pi}{2}$ by $-\frac{\pi}{2}$ in the direction of the ***x***-axis, that is, to the left, we get the curve of a function as follows.

$$q(x) = \sec\left(x + \frac{\pi}{2}\right) \text{ for } -2\pi < x < 2\pi$$

So putting the two curves in one graph, we can get this:



Thus, the curve of ***p*** gets moved to the left in the amount of $\frac{\pi}{2}$. And the new curve is the curve of ***q***. So we can call the phase a horizontal shift, too.

What then about the constant ***b*** in ***y*** = ***Q***(***x***) = ***A***·**sec *wx*** + ***b***?

It is called a vertical shift.

So shifting the curve of $\boldsymbol{y = V(x) = A \cdot \sec w(x + a)}$ where $\boldsymbol{w(x + a)} \neq \frac{\boldsymbol{n\pi}}{2}$ for $\boldsymbol{n}$ odd, by $\boldsymbol{b}$ in the direction of the $\boldsymbol{y}$-axis, we get the curve of the function $\boldsymbol{G}$ as follows.

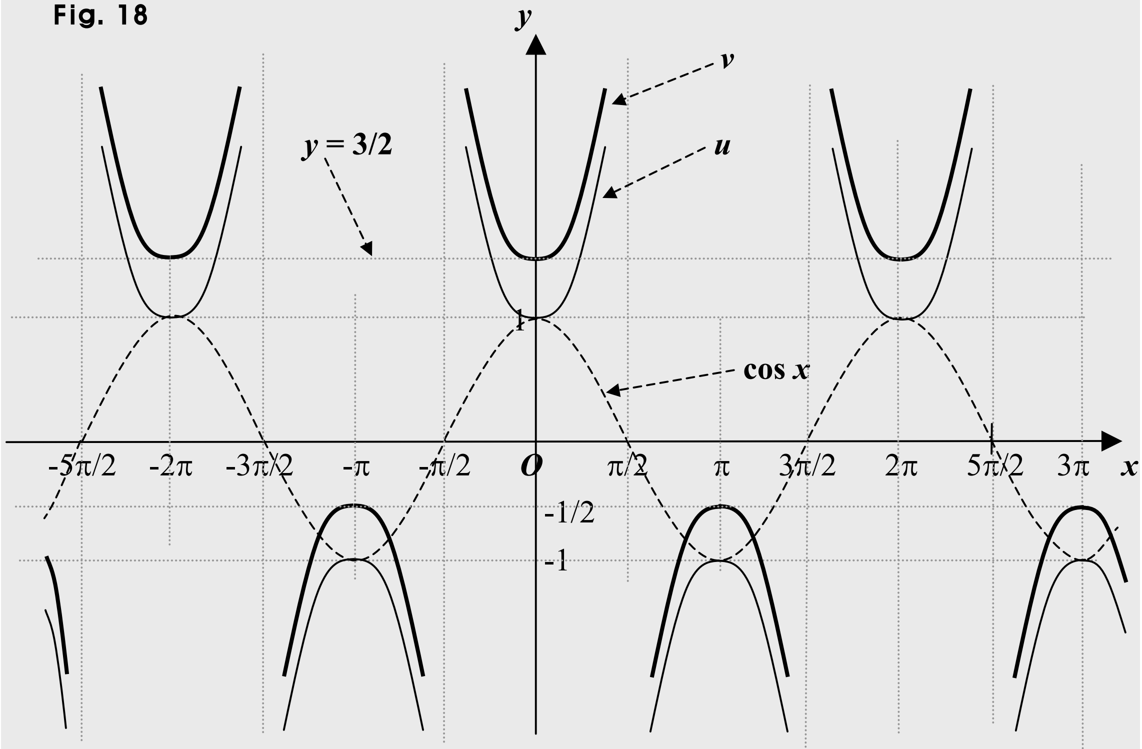
$$\boldsymbol{y = G(x) = A \cdot \sec w(x + a) + b} \text{ where } \boldsymbol{w(x + a)} \neq \frac{\boldsymbol{n\pi}}{2} \text{ for } \boldsymbol{n} \text{ odd.}$$

So the two curves themselves of $\boldsymbol{G}$ and $\boldsymbol{V}$ are the same, and moving the curve of $\boldsymbol{V}$ in the amount of $\boldsymbol{b}$ along the $\boldsymbol{y}$-axis, we get the curve of $\boldsymbol{G}$.
And if $\boldsymbol{b}$ is positive, the curve is shifted upward, and if negative, it moves downward.
Assuming for instance, shifting by $\frac{1}{2}$ in the direction of the $\boldsymbol{y}$-axis, that is, upward, the curve of $\boldsymbol{u(x) = \sec x}$, we get the curve of a function as follows.

$$\boldsymbol{v(x) = \sec x} + \frac{1}{2}$$

Putting therefore, the two curves in a graph, we can get this:

Fig. 18



So the curve of $\boldsymbol{u}$ gets moved upward in the amount of $\frac{1}{2}$. And the new curve is the curve of $\boldsymbol{v}$. So we can call the $\boldsymbol{b}$ a vertical shift.

Suppose now, we put in a graph, for instance, the curve of a secant function as follows.

$$\boldsymbol{y = S(x) = -\frac{1}{2}\sec(-2x + \pi) + 1} \text{ where } \boldsymbol{-2x + \pi \neq \frac{n\pi}{2}} \text{ for } \boldsymbol{n} \text{ odd}$$

Then, to begin with, we have $\mathbf{sec}\ (-\theta) = \mathbf{sec}\ \theta$.
So we can get $\mathbf{sec}\ (-2x + \pi) = \mathbf{sec}\ \{-(2x - \pi)\} = \mathbf{sec}\ (2x - \pi)$.
Thus, we get $-\frac{1}{2}\mathbf{sec}\ (-2x + \pi) + 1 = -\frac{1}{2}\mathbf{sec}\ (2x - \pi) + 1$.

And putting it in the general form, we get $-\frac{1}{2}\mathbf{sec}\ 2(x - \frac{\pi}{2}) + 1$.

So we can now see that the frequency is 2, the phase is $-\frac{\pi}{2}$, that is, shifting by $\frac{\pi}{2}$ to the right, and the vertical shift is 1.

And next, putting it in a graph, we may want to begin with the prototype $\mathbf{sec}\ x$.
First, setting the frequency to 2, we get $\mathbf{sec}\ 2x$.

We know that the period is the basic period (interval) over the frequency, and that the basic interval (period) for the secant is 2π. So we can see that the period is $\frac{2\pi}{2} = \pi$.

Next, setting the constant A to $\frac{1}{2}$, we get $\frac{1}{2}\mathbf{sec}\ 2x$.

And assuming $m(x) = \mathbf{sec}\ 2x$ and $n(x) = \frac{1}{2}\mathbf{sec}\ 2x$, we get $n(x) = \frac{1}{2}m(x)$ for every x-value.
That is, for each and every x-value, the y-value in the function n is half the y-value in the function m.
Next, setting the phase to $-\frac{\pi}{2}$, we get $\frac{1}{2}\mathbf{sec}\ 2(x - \frac{\pi}{2})$.

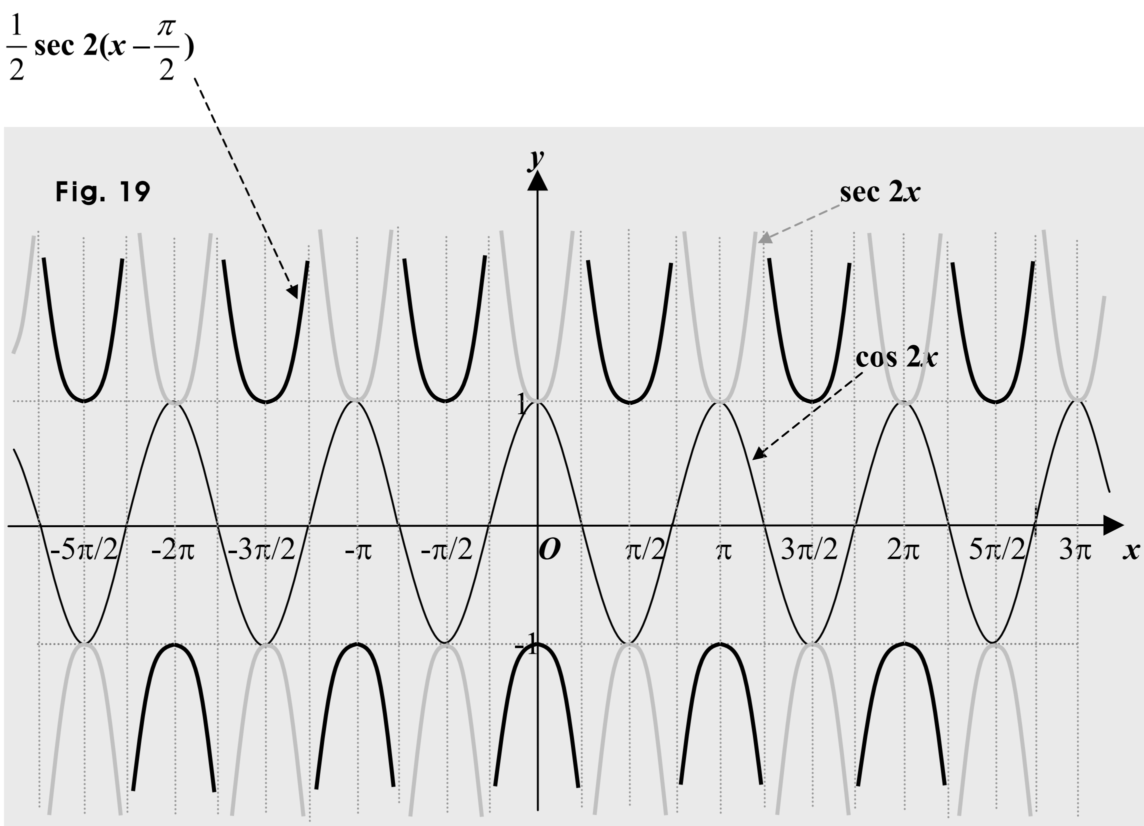
We know shifting the curve of with $\mathbf{sec}\ x$ by $-\frac{\pi}{2}$ in the direction of the x-axis, we move the curve by $\frac{\pi}{2}$ to the left. Then, we get the curve with $\mathbf{sec}\ (x + \frac{\pi}{2})$.

So shifting the curve of $\frac{1}{2}$ **sec 2*x*** by not $-\frac{\pi}{2}$ but $\frac{\pi}{2}$ in the direction of the ***x***-axis, we get the curve of $\frac{1}{2}$ **sec 2(*x* $-\frac{\pi}{2}$)**. That is, moving the curve of $\frac{1}{2}$ **sec 2*x*** in the amount of $\frac{\pi}{2}$ to the right, we get the curve of $\frac{1}{2}$ **sec 2(*x* $-\frac{\pi}{2}$)**.

So we do a horizontal shifting to the right in the amount of $\frac{\pi}{2}$.

Note however, if the domain is a set of all real numbers less $\frac{n\pi}{2}$ for ***n*** odd, the curve of $\frac{1}{2}$ **sec 2(*x* $-\frac{\pi}{2}$)** is in fact, no other than the curve of $\frac{1}{2}$ **sec 2(*x* $+\frac{\pi}{2}$)**. If however, the phase is not a multiple of $\frac{\pi}{2}$, the two curves are different.

And the curve is as follows.



7.2. Cotangent

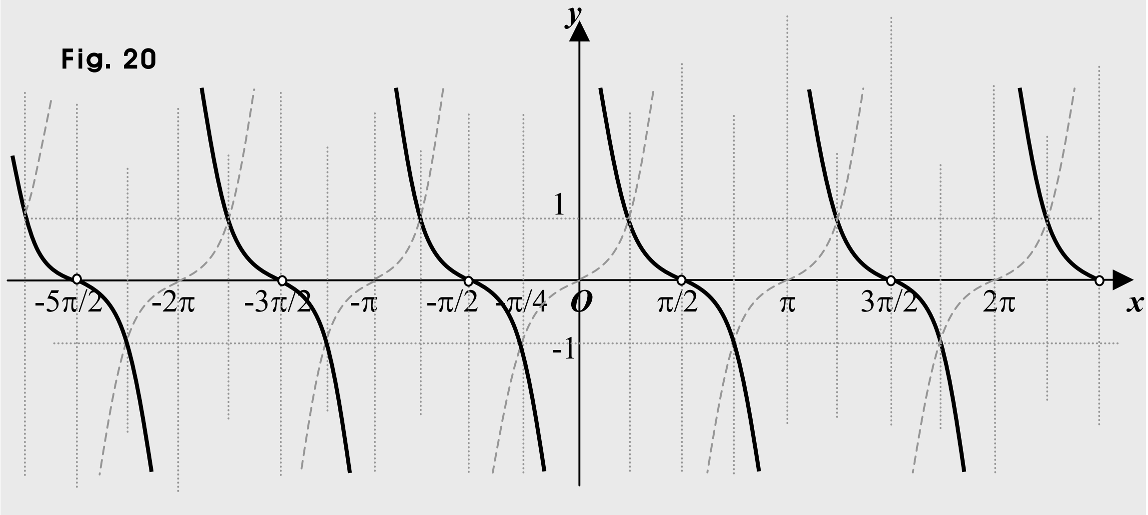


Fig. 20

Taking next, the reciprocal of the tangent, we get another trig ratio called the cotangent, denoted by **cot**.
So taking the reciprocal of the tangent, that is, **tan *x***, we get **cot *x***.
And taking **tan *x*** as a function, we can set $\boldsymbol{y = f(x) = \tan x}$, and call it a tangent function.

So we can make another trig function called a cotangent function using **cot *x***, of course.
For instance, we can make a cotangent function as $\boldsymbol{y = g(x) = \cot x}$ or $\boldsymbol{y = h(x) = 2\cot x^3}$.

And the cotangent function **cot *x*** cannot be defined for all angles. Why not?

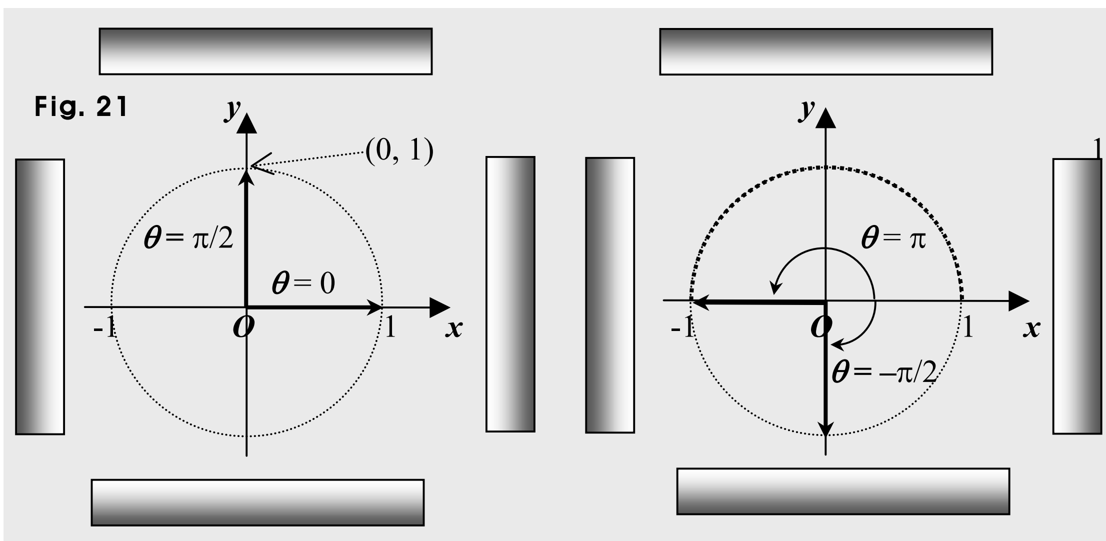
We know that the tangent is the opposite over the adjacent.
So the cotangent is the adjacent over the opposite, since it is the reciprocal of the tangent.

And we know that the opposite can be 0 for some angles as 0, -π, and π. So for instance, the cotangent **cot *x*** cannot be defined for $\boldsymbol{x = \pi}$. And in general, it cannot be defined for $\boldsymbol{x = 2n\pi}$ where ***n*** is an integer. That's not all though. What else then?

Unlike the tangent, taking the cotangent, **cot** $\boldsymbol{x}$, we take not just the ratio of the adjacent to the opposite but the reciprocal of the tangent, **tan** $\boldsymbol{x}$, too. So taking **cot** $\boldsymbol{x}$, we want to make sure that the reciprocal of **tan** $\boldsymbol{x}$ is defined. And in fact, the same is true, too, for **csc** $\boldsymbol{x}$ and **sec** $\boldsymbol{x}$, the reciprocals of **sin** $\boldsymbol{x}$ and **cos** $\boldsymbol{x}$. Why though?

We know $\mathbf{sin}\ \boldsymbol{x} = \mathbf{0}$ for $\boldsymbol{x} = \boldsymbol{n}\pi$ where $\boldsymbol{n}$ is an integer.

Fig. 21



The projection on $\boldsymbol{y}$-axis is 0 if $\boldsymbol{\theta}$ is $\boldsymbol{n}\pi$ where $\boldsymbol{n}$ is integer, so the opposite is 0; thus, the sine is 0, and the projection on the $\boldsymbol{x}$-axis is 0 if $\boldsymbol{\theta}$ is $\dfrac{\boldsymbol{n\pi}}{2}$ where $\boldsymbol{n}$ is odd, so the adjacent is 0; thus, the cosine is 0.

So for $\boldsymbol{x} = \boldsymbol{n}\pi$ where $\boldsymbol{n}$ is an integer, we get $\mathbf{csc}\ \boldsymbol{x} = \dfrac{1}{\sin \boldsymbol{x}} = \dfrac{1}{0}$, which is not allowed.

Thus, **csc** $\boldsymbol{x}$ cannot be defined for $\boldsymbol{x} = \boldsymbol{n}\pi$ where $\boldsymbol{n}$ is an integer.

Next, we know $\mathbf{cos}\ \boldsymbol{x} = \mathbf{0}$ for $\boldsymbol{x} = \dfrac{\boldsymbol{n\pi}}{2}$ where $\boldsymbol{n}$ is an odd integer.

So for $\boldsymbol{x} = \dfrac{\boldsymbol{n\pi}}{2}$ for $\boldsymbol{n}$ odd, we get $\mathbf{sec}\ \boldsymbol{x} = \dfrac{1}{\cos \boldsymbol{x}} = \dfrac{1}{0}$, which is not allowed. Thus, **sec** $\boldsymbol{x}$ cannot be defined for $\boldsymbol{x} = \dfrac{\boldsymbol{n\pi}}{2}$ where $\boldsymbol{n}$ is an integer odd. In fact, for $\boldsymbol{x} = \dfrac{\boldsymbol{n\pi}}{2}$ where $\boldsymbol{n}$ is odd, the adjacent is 0, so **sec** $\boldsymbol{x}$ cannot be defined. What then about **cot** $\boldsymbol{x}$?

We know **tan** $\boldsymbol{x}$ cannot be defined for $\boldsymbol{x} = \dfrac{\boldsymbol{n}\pi}{2}$ where $\boldsymbol{n}$ is odd, because $\tan \boldsymbol{x} = \dfrac{\sin \boldsymbol{x}}{\cos \boldsymbol{x}}$ and **cos** $\boldsymbol{x}$ **= 0** if $\boldsymbol{x} = \dfrac{\boldsymbol{n}\pi}{2}$ where $\boldsymbol{n}$ is odd. So for $\boldsymbol{x} = \dfrac{\boldsymbol{n}\pi}{2}$ for $\boldsymbol{n}$ odd, **cot** $\boldsymbol{x}$ cannot be defined, because $\cot \boldsymbol{x} = \dfrac{1}{\tan \boldsymbol{x}}$ and **tan** $\boldsymbol{x}$ is not defined. Also, **cot** $\boldsymbol{x}$ cannot be defined either for $\boldsymbol{x} = \boldsymbol{n}\pi$ for $\boldsymbol{n}$ integer, because $\cot \boldsymbol{x} = \dfrac{\cos \boldsymbol{x}}{\sin \boldsymbol{x}}$ and **sin** $\boldsymbol{x}$ **= 0** for $\boldsymbol{x} = \boldsymbol{n}\pi$ for $\boldsymbol{n}$ integer.

So in sum, **cot** $\boldsymbol{x}$ cannot be defined for $\boldsymbol{x} = \boldsymbol{n}\pi$ or $\dfrac{(2\boldsymbol{n}+1)\pi}{2}$ for $\boldsymbol{n}$ integer.

Thus, just setting $\boldsymbol{y} = \boldsymbol{g}(\boldsymbol{x}) = \mathbf{cot}\ \boldsymbol{x}$, we mean that $\boldsymbol{x} \neq \boldsymbol{n}\pi$ or $\dfrac{(2\boldsymbol{n}+1)\pi}{2}$ for $\boldsymbol{n}$ integer.

That is, the domain is a set of all real numbers less $\boldsymbol{n}\pi$ and $\dfrac{(2\boldsymbol{n}+1)\pi}{2}$ for $\boldsymbol{n}$ integer.

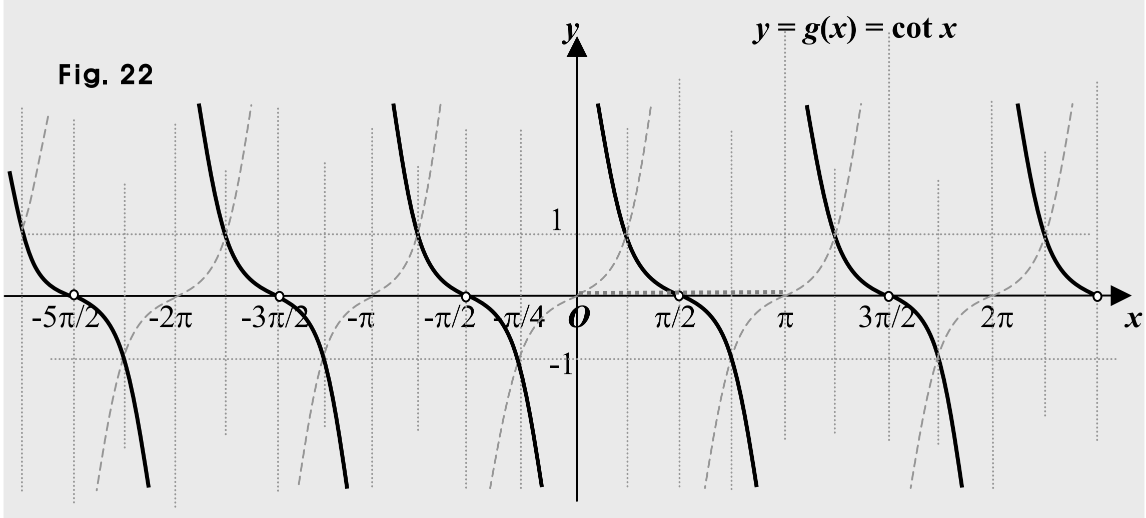
So we can define or specify the cotangent function the way as follows, too.

$$\boldsymbol{y} = \boldsymbol{g}(\boldsymbol{x}) = \mathbf{cot}\ \boldsymbol{x} \text{ for } \boldsymbol{x} \neq \boldsymbol{n}\pi \text{ or } \frac{(2\boldsymbol{n}+1)\pi}{2} \text{ for } \boldsymbol{n} \text{ integer.}$$

What then about its curve?

We know that the cotangent is the reciprocal of the tangent. So keeping the fact in mind, and using the curve of the tangent function $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x}) = \mathbf{tan}\ \boldsymbol{x}$, we can put in a graph, the curve of the cotangent function $\boldsymbol{y} = \boldsymbol{g}(\boldsymbol{x}) = \mathbf{cot}\ \boldsymbol{x}$ the way as follows.

Fig. 22



Note that there is a point missing at every $\frac{\boldsymbol{n}\pi}{2}$ for ***n*** odd.

It's because the cotangent function ***g*** is *not* defined not only for $\boldsymbol{x} = \boldsymbol{n}\pi$ for ***n*** integer but for $\boldsymbol{x} = \frac{\boldsymbol{n}\pi}{2}$ for ***n*** odd, too.

So we can now see that the ***y***-values, that is, the outputs are all real numbers other than 0. Thus, the range is $\boldsymbol{y} \neq 0$.

And of course, if the domain is different, the range can be other than the one above.

Assuming for instance, $\boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}) = \mathbf{cot}\,\boldsymbol{x}$ for $\mathbf{0} < \boldsymbol{x} < \frac{\pi}{2}$, we get $\boldsymbol{y} > \mathbf{0}$, which is the range.

And the cotangent function $\boldsymbol{y} = \boldsymbol{g}(\boldsymbol{x}) = \mathbf{cot}\,\boldsymbol{x}$ can be called the prototype.
Assuming ***G*** is a cotangent function, too, and the domain is a set of all real numbers less $\mathbf{2}\boldsymbol{n}\pi$ and $\frac{(2\boldsymbol{n}+1)\pi}{2}$ for ***n*** integer, and using a general form, we can put it this way:

$$\boldsymbol{y} = \boldsymbol{G}(\boldsymbol{x}) = \boldsymbol{A}\cdot\mathbf{cot}\,\{\boldsymbol{w}(\boldsymbol{x} + \boldsymbol{a})\} + \boldsymbol{b}$$

where $\boldsymbol{w}(\boldsymbol{x} + \boldsymbol{a}) \neq \mathbf{2}\boldsymbol{n}\pi$ or $\frac{(2\boldsymbol{n}+1)\pi}{2}$ for ***n*** integer, and ***A***, ***w***, ***a***, and ***b*** are constant.

And just simply, we can put it this way, too: $\boldsymbol{y} = \boldsymbol{G}(\boldsymbol{x}) = \boldsymbol{A}\cdot\mathbf{cot}\,\boldsymbol{w}(\boldsymbol{x} + \boldsymbol{a}) + \boldsymbol{b}$.

Then, $|\boldsymbol{w}|$ is called the frequency, $\frac{\pi}{|\boldsymbol{w}|}$ is the period, and ***a*** is called the phase.

So in the function $\boldsymbol{y} = \boldsymbol{g}(\boldsymbol{x}) = \mathbf{cot}\,\boldsymbol{x}$, the phase is 0, and the frequency is 1.
And looking at the curve in Fig. 22 above, we can see that a part of the cuve repleats itself. And the part is in fact, the smallest part, in two pieces though, repeating itself. What then is the part?

It is the part between 0 and π. So the part repeats itself every π interval.
So the interval π is the period in the cotangent function $\boldsymbol{y} = \boldsymbol{g}(\boldsymbol{x}) = \mathbf{cot}\,\boldsymbol{x}$.
Thus, we call a cotangent function a *periodic* function, too.
And the secant function ***g*** above is the prototype, and thus, is in the most basic form.

So the period **π** can be called the *basic period* in cotangent functions.
In other words, the interval **π** can be called the *basic interval* in cotangent functions.

And for instance, in the general form, setting $\boldsymbol{w}$ to 2, $\boldsymbol{a}$ and $\boldsymbol{b}$ to 0 each, and $\boldsymbol{A}$ to 1, we get a new function where $\boldsymbol{y = k(x) = \cot 2x}$. What is the domain though?

We know that $\boldsymbol{\cot\theta}$ is not defined for $\boldsymbol{\theta = 2n\pi}$ and $\dfrac{(2\boldsymbol{n}+1)\pi}{2}$ where $\boldsymbol{n}$ is an integer.

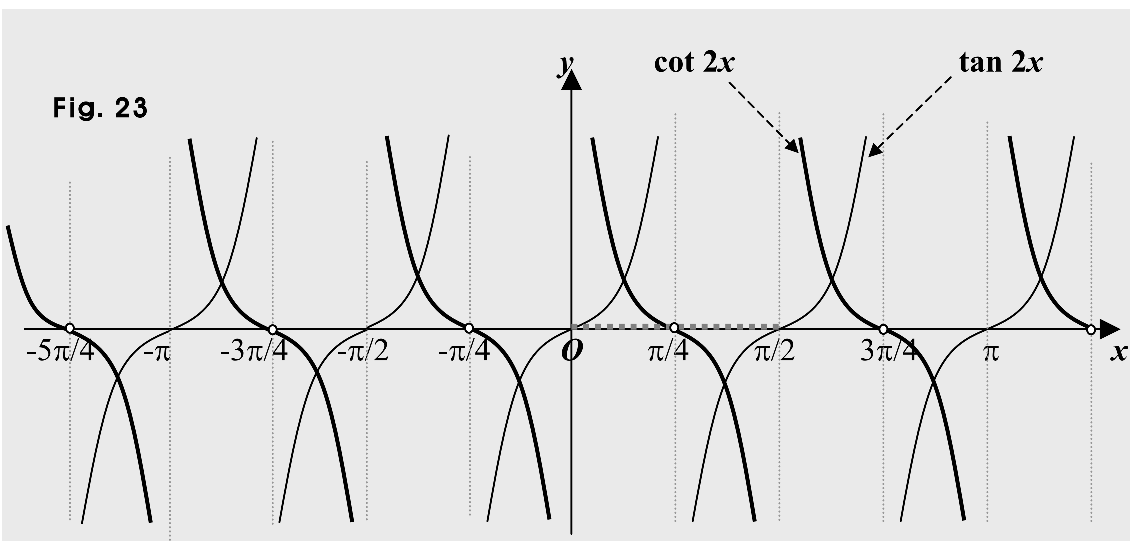
So in $\boldsymbol{\cot 2x}$, we don't want $\boldsymbol{2x}$ to be $\boldsymbol{2n\pi}$ or $\dfrac{(2\boldsymbol{n}+1)\pi}{2}$ for $\boldsymbol{n}$ integer.

That is, we wan: $\boldsymbol{2x \neq 2n\pi}$ or $\dfrac{(2\boldsymbol{n}+1)\pi}{2}$ $\Rightarrow \boldsymbol{x \neq n\pi}$ or $\dfrac{(2\boldsymbol{n}+1)\pi}{4}$ for $\boldsymbol{n}$ integer.

Thus, the domain in the function $\boldsymbol{k(x) = \cot 2x}$ can be a set of all real numbers less $\boldsymbol{n\pi}$ and $\dfrac{(2\boldsymbol{n}+1)\pi}{4}$ for $\boldsymbol{n}$ integer.

And of course, if $\boldsymbol{E}$ is the domain just described above, $\boldsymbol{E}$ is the biggest domain that the cotangent function $\boldsymbol{\cot 2x}$ can have.
Then, putting in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane, the curve of $\boldsymbol{k(x) = \cot 2x}$, we can get this:

Fig. 23



Then again, we can see that a part of the cuve repleats itself, and is in two pieces.

And the part is of course, the smallest part repeating itself. What then, is the part?

It is the part from 0 to $\frac{\pi}{2}$. So the part repeats itself every $\frac{\pi}{2}$ interval. Thus, the interval $\frac{\pi}{2}$ is the period in the function $\boldsymbol{k(x)} = \textbf{cot } \boldsymbol{2x}$. And the smallest part stated above appears twice in the basic interval π, which is called the basic period, also.
So we say that the frequency is 2 in the function $\boldsymbol{k}$.
So in case of cotangent functions, the frequency is the number of times the samllest part repeating itself appears in the basic period or the basic interval π.
Thus, in the general form $\boldsymbol{A}\cdot\textbf{cot } \boldsymbol{w(x + a) + b}$, $|\boldsymbol{w}|$ is the frequency.

And we know that the interval that fits the smallest part is the period. So using the frequency $|\boldsymbol{w}|$ and the basic interval π, we can put the period this way: $\frac{\pi}{|w|}$.

Thus, the period of the function $\boldsymbol{y} = \boldsymbol{k(x)} = \textbf{cot } \boldsymbol{2x}$ is $\frac{\pi}{2}$.

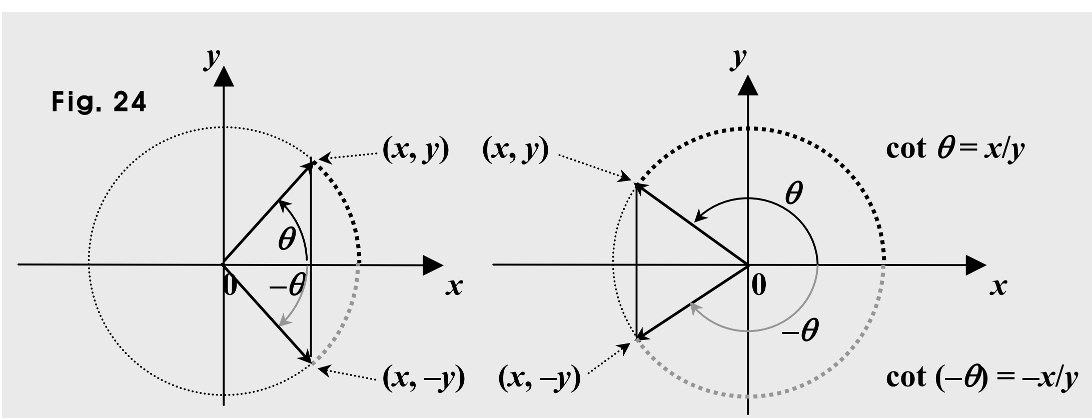
And we can have a function $\boldsymbol{y} = \boldsymbol{h(x)} = \textbf{cot } \boldsymbol{(-x)}$ for $\boldsymbol{x} \neq \boldsymbol{n}\pi$ or $\frac{(2\boldsymbol{n}+1)\pi}{4}$ for $\boldsymbol{n}$ integer.

Then, the frequency in the function $\boldsymbol{h}$ is $|-1|$, simply because a frequency is positive.

And we can put $\boldsymbol{h}$ this way, too: $\boldsymbol{y} = \boldsymbol{h(x)} = -\textbf{cot } \boldsymbol{x}$ for $\boldsymbol{x} \neq \boldsymbol{n}\pi$ or $\frac{(2\boldsymbol{n}+1)\pi}{4}$ for $\boldsymbol{n}$ integer.

It's because $\textbf{cot } \boldsymbol{(-x)} = -\textbf{cot } \boldsymbol{x}$.   How though?

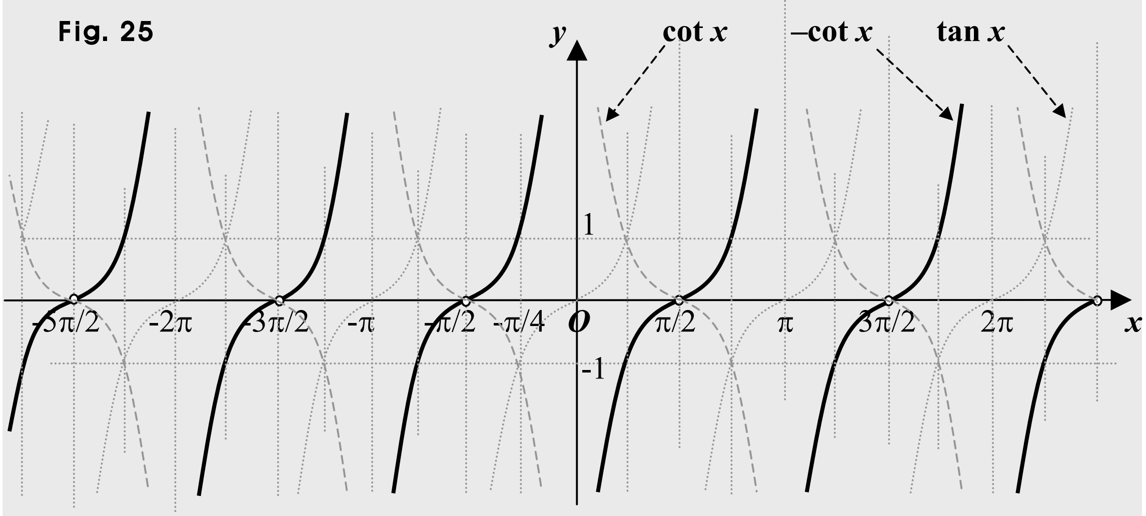
We know the cotangent is the adjacent over the opposite.
So assuming the vector, that is, the hypotenuse is of length 1, we can get this:



So we get: $\textbf{cot } (-\boldsymbol{\theta}) = -\textbf{cot } \boldsymbol{\theta}$.

Thus, putting in a graph, the curve of $\boldsymbol{y = h(x) = \cot(-x)}$, we get the curve as follows.

Fig. 25



And we can see that the period of $\boldsymbol{h(x) = \cot(-x)}$ is 2π, too.

Next, shifting by $\boldsymbol{-a}$ in the direction of the $\boldsymbol{x}$-axis, the curve of $\boldsymbol{y = Q(x) = A \cdot \cot wx + b}$, we get the curve of $\boldsymbol{y = G(x) = A \cdot \cot w(x + a) + b}$.

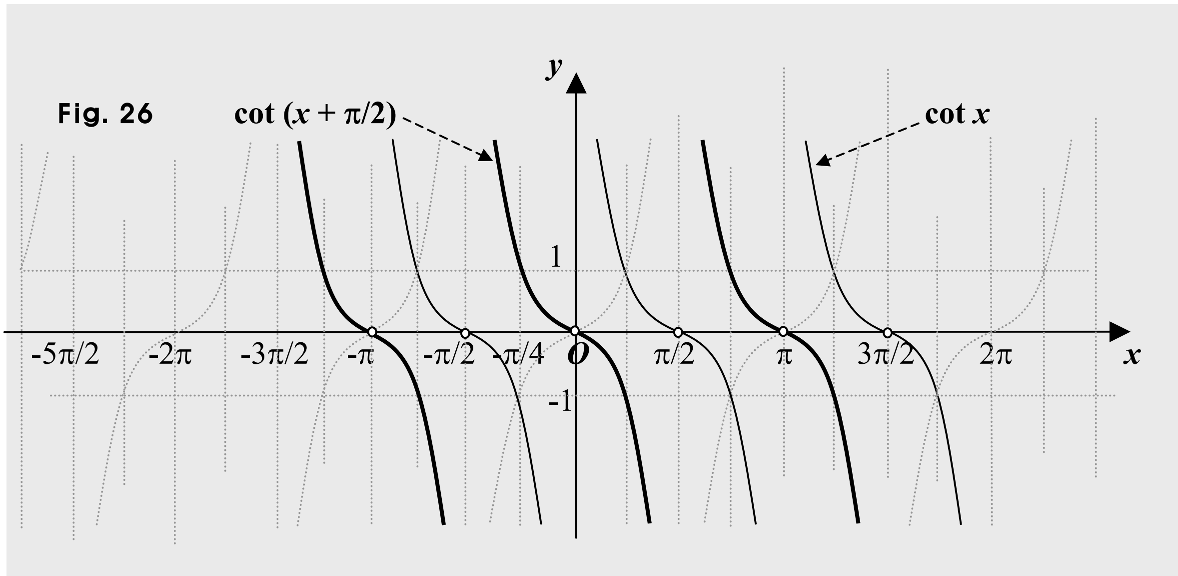
So the two curves themselves of $\boldsymbol{G}$ and $\boldsymbol{Q}$ are the same, and moving the curve of $\boldsymbol{Q}$ in the amount of $\boldsymbol{-a}$ along the $\boldsymbol{x}$-axis, we get the curve of $\boldsymbol{G}$.

And we call $\boldsymbol{a}$ the phase, and if $\boldsymbol{a}$ is positive, the curve gets shifted to the left, and if negative, it moves to the right.

Assuming for instance, shifting the curve of $\boldsymbol{p(x) = \cot x}$ for $\boldsymbol{-\pi < x < 2\pi}$ by $-\dfrac{\pi}{2}$ in the direction of the $\boldsymbol{x}$-axis, that is, to the left, we get the curve of a function as follows.

$$\boldsymbol{q(x) = \cot\left(x + \frac{\pi}{2}\right)} \text{ for } \boldsymbol{-2\pi < x < 2\pi}$$

So putting the two curves in a graph, we can get this:



Thus, the curve of $\boldsymbol{p}$ gets moved to the left in the amount of $\frac{\pi}{2}$. And the new curve is the curve of $\boldsymbol{q}$. So we can call the phase a horizontal shifting, too.

What then, about the constant $\boldsymbol{b}$ in $\boldsymbol{y = Q(x) = A \cdot \cot wx + b}$?

It can be called a vertical shift.

So shifting the curve of $\boldsymbol{y = V(x) = A \cdot \cot w(x + a)}$ by $\boldsymbol{b}$ in the direction of the $\boldsymbol{y}$-axis, we get the curve of $\boldsymbol{y = G(x) = A \cdot \cot w(x + a) + b}$.

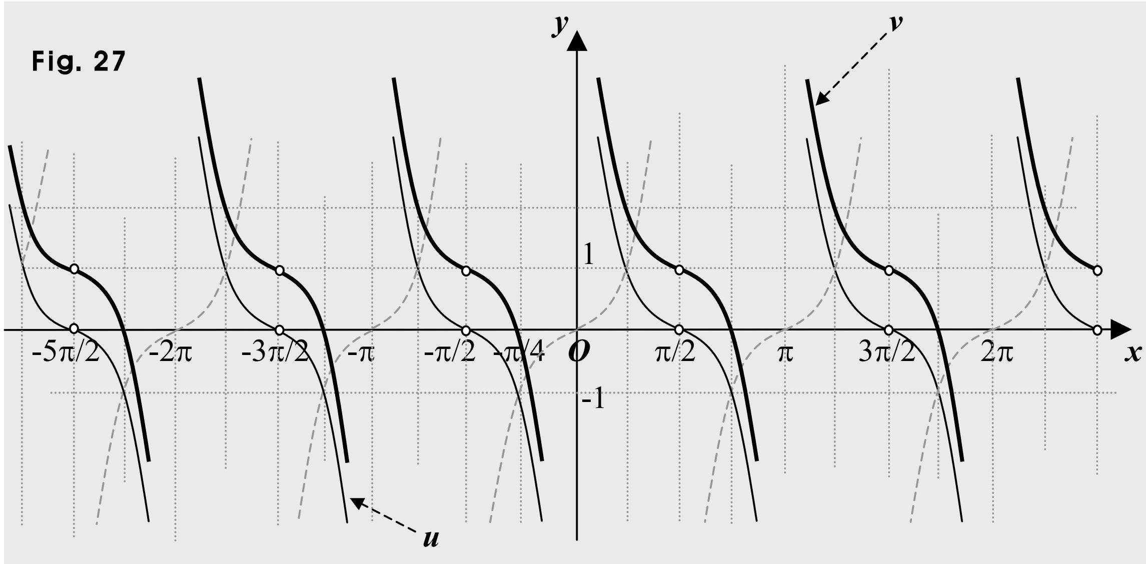
So the two curves themselves of $\boldsymbol{G}$ and $\boldsymbol{V}$ are the same, and moving the curve of $\boldsymbol{V}$ in the amount of $\boldsymbol{b}$ along the $\boldsymbol{y}$-axis, we get the curve of $\boldsymbol{G}$.

And if $\boldsymbol{b}$ is positive, the curve is shifted upward, and if negative, it moves downward.

Assuming for instance, shifting by 1 in the direction of the $\boldsymbol{y}$-axis, that is, upward, the curve of $\boldsymbol{u(x) = \cot x}$, we get the curve of a function as follows: $\boldsymbol{v(x) = \cot x + 1}$.

Putting therefore, the two curves in a graph, we can get this:

Fig. 27



So the curve of ***u*** gets moved upward in the amount of 1. And the new curve is the curve of ***v***. So we can call the ***b*** a vertical shift.

And let's now put in a graph, for instance, the curve of a cotangent function as follows.

$$\boldsymbol{y = S(x) = -2 \cot (-2x + \pi) + 1}$$

Then, to begin with, we have $\boldsymbol{\cot(-\theta) = -\cot \theta}$.

So we can get $\boldsymbol{\cot(-2x + \pi) = \cot\{-(2x - \pi)\} = -\cot(2x - \pi)}$.

Thus, we get $\boldsymbol{-2 \cot(-2x + \pi) + 1 = 2 \cot(2x - \pi) + 1}$.

So the given function can be put this way: $\boldsymbol{y = S(x) = 2 \cot(2x - \pi) + 1}$.

The domain is important.

So next, let's get the domain. ***Note that*** **cot *x*** is not defined if $\boldsymbol{x = n\pi}$ or $\frac{(2\boldsymbol{n}+1)\pi}{2}$ for ***n*** integer. So the domain of **cot *x*** is a set of all real numbers less $\boldsymbol{n\pi}$ or $\frac{(2\boldsymbol{n}+1)\pi}{2}$ for ***n*** integer. Note that in the function ***S***, $\boldsymbol{2x - \pi}$ is the angle. Getting thus, the domain, we get

$\boldsymbol{2x - \pi \neq n\pi}$ or $\frac{(2\boldsymbol{n}+1)\pi}{2} \quad \Rightarrow \boldsymbol{x} \neq \frac{(\boldsymbol{n}+1)\pi}{2}$ or $\frac{(2\boldsymbol{n}+3)\pi}{4}$.

So the domain is a set of all real numbers less $\frac{(\boldsymbol{n}+1)\pi}{2}$ or $\frac{(2\boldsymbol{n}+3)\pi}{4}$ for ***n*** integer.

And putting the function in the general form, we get $\boldsymbol{2 \cot 2(x - \frac{\pi}{2}) + 1}$. Why and how?

It's easier to interpret a function if the function is in its general form.
Note that putting a cotangent function in general, we can put it the way as follows.

$$y = G(x) = A \cdot \cot w(x + a) + b.$$

Then, $|w|$ is the frequency, $\dfrac{\pi}{|w|}$ is the period, a is the phase, and b is the vertical shift.

So we can now see that the frequency is 2, the phase is $-\dfrac{\pi}{2}$, that is, shifting by $\dfrac{\pi}{2}$ to the right, so it's a horizontal shift, and the vertical shift is 1.

And next, putting it in a graph, we may want to begin with the prototype, which is $\cot x$.

Next, setting the frequency to 2, we get $\cot 2x$.

We know that the period is the basic period (interval) over the frequency, and that the basic interval (period) for the cotangent is π. So we can see that the period is $\dfrac{\pi}{2}$.

Next, setting the constant A to 2, we get $2 \cot 2x$.

And assuming $m(x) = \cot 2x$ and $n(x) = 2 \cot 2x$, we get $n(x) = 2m(x)$ for every x-value. That is, for each and every x-value, the y-value in the function n is twice the y-value in the function m.

Next, setting the phase to $-\dfrac{\pi}{2}$, we get $2 \cot 2(x - \dfrac{\pi}{2})$.

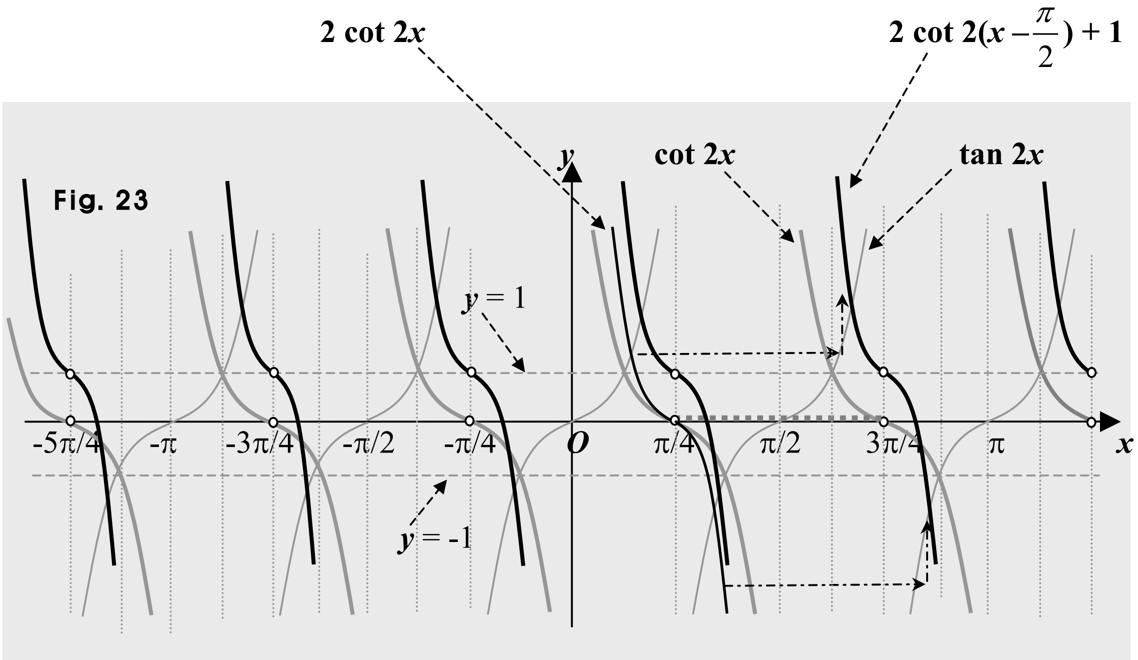
We know shifting the curve of with $\cot x$ by $-\dfrac{\pi}{2}$ in the direction of the x-axis, we get the curve with $\cot (x + \dfrac{\pi}{2})$.

So shifting the curve with $2 \cot 2x$ by not $-\dfrac{\pi}{2}$ but $\dfrac{\pi}{2}$ in the direction of the x-axis, we get the curve with $2 \cot 2(x - \dfrac{\pi}{2})$.

That is, moving the curve with $2 \cot 2x$ in the amount of $\dfrac{\pi}{2}$ to the right, we get the curve with $2 \cot 2(x - \dfrac{\pi}{2})$.

So we do a horizontal shifting to the right in the amount of $\dfrac{\pi}{2}$.

Now, putting in a graph, the curve of $\mathbf{2\cot 2(x - \frac{\pi}{2}) + 1}$, we can get the graph as follows.



Expanding vertically twice the curve of $\mathbf{\cot 2x}$, shifting the expanded curve by $\frac{\pi}{2}$ to the right, and then, shifting the shifted curve upward by 1, we get the curve of the function as follows.

$\boldsymbol{y = S(x) = -2\cot(-2x + \pi) + 1}$, which is equivalent to

$\boldsymbol{y = S(x) = 2\cot 2(x - \frac{\pi}{2}) + 1}$ for $\boldsymbol{x} \neq \frac{(\boldsymbol{n}+1)\pi}{2}$ or $\frac{(2\boldsymbol{n}+3)\pi}{4}$ for $\boldsymbol{n}$ integer.

It's because $\mathbf{2(x - \frac{\pi}{2}) \neq n\pi}$ or $\frac{(2\boldsymbol{n}+1)\pi}{4}$ for $\boldsymbol{n}$ integer.

It's because the function $\boldsymbol{S}$ is not defined if $\mathbf{2(x - \frac{\pi}{2}) = n\pi}$ or $\frac{(2\boldsymbol{n}+1)\pi}{4}$ for $\boldsymbol{n}$ integer.

Note however, if the domain is a set of all real numbers less $\frac{(\boldsymbol{n}+1)\pi}{2}$ or $\frac{(2\boldsymbol{n}+3)\pi}{4}$ for $\boldsymbol{n}$ integer, the curve of $\mathbf{2\cot 2(x - \frac{\pi}{2})}$ is in fact, no other than the curve of $\mathbf{2\cot 2(x + \frac{\pi}{2})}$.

If however, the phase is not a multiple of $\frac{\pi}{2}$, the two curves are different.

8. The Inverse Functions

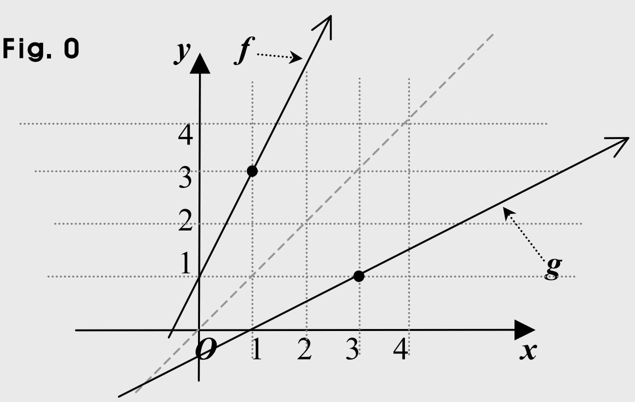
In this section, we'll cover mainly the inverses of the basic trig functions. For details on inverse functions, you may want to refer to the book **Basic Functions**.

The inverse of a function is called an inverse function, and is often just called the inverse. What then is an inverse function?

If a function gets an input, it produces an output, which is the output for the input.
For instance, putting an input 1 into the input variable $\boldsymbol{x}$ in a function $\boldsymbol{y = f(x) = 2x + 1}$, we get $\boldsymbol{f(1) = 2 + 1 = 3}$, which is an output, which is the output for the input 1.
What then about the inverse?

The inverse of a function is a function that produces an input for an output, which is the output for the input.
So for instance, assuming $\boldsymbol{g}$ is the inverse of $\boldsymbol{f}$, we get $\boldsymbol{g(3) = 1}$, because $\boldsymbol{f(1) = 3}$.

Fig. 0



Thus, we can say that an inverse function is a function that produces the input using the output. In short, the inverse produces the input using the output.
How then can we get the inverse?

There are two conditions.

One is that the inverse produces the domain using the range.
In other words, the domain of the inverse is the range of the function we take the inverse of, and the range of the inverse is the domain of the function we take the inverse of.

So the domain and the range are switched, which is however, a necessary condition, and is not a sufficient condition.
It's because not every function where the domain and range are switched is the inverse.
What then is the other condition?

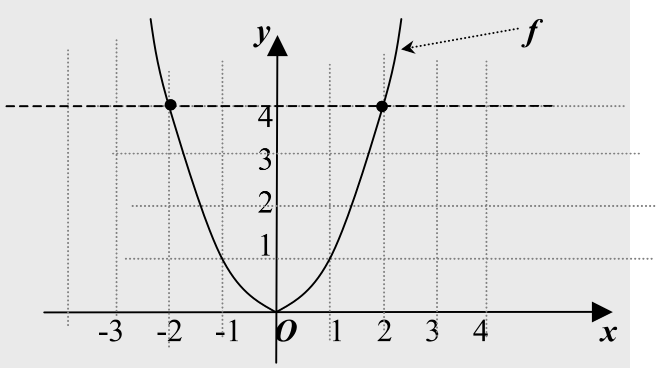
The inverse is a function, too.

So the function we take the inverse of has to be a function one-to-one.
In short, the original function has to be one-to-one. Why though?

First, there is no function that is one-to-many.
So next, if the original is not one-to-one, that is, many-to-one, then the inverse would be one-to-many, and thus, is not a function. So no inverse function exists.

For instance, a quadratic function $\boldsymbol{y = f(x) = x^2}$ for $\boldsymbol{x}$ real is many-to-one, because two different inputs can produce the same output. So if there were the inverse, it would not be able to produce the one unique input only.

Fig. 1



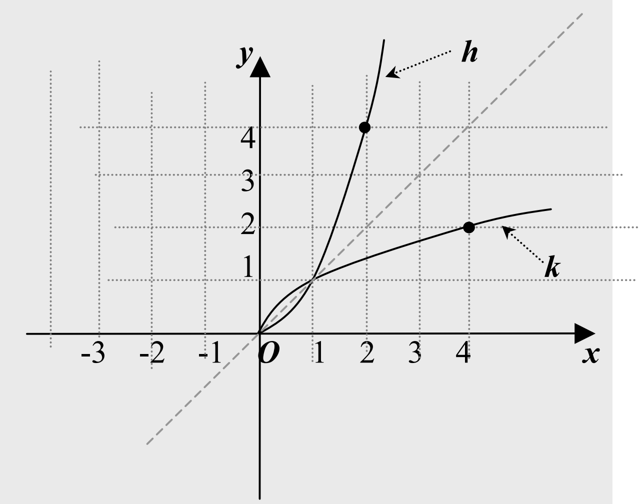
For instance, we can have $f(2) = 4$ and $f(-2) = 4$, and if the inverse were a function g, what would be $g(4)$? It would be 2 or –2, or it would be 2 and –2, none of which is not allowed for any function. So we cannot take the inverse of the function f.

If however, we have another quadratic function $y = h(x) = x^2$ for $x \geq 0$, the function h is one-to-one, so we can take the inverse, and the inverse can be as follows.

$$y = k(x) = \sqrt{x}$$

Since no domain is specified, and x is inside the square root sign, it must be that $x \geq 0$, which is therefore, the domain of k.

Fig. 2



And the same is true for the inverse of a trig function, too.
We know that the domain of a trig function as $\sin x$ is a set of angles, and the range is a set of trig ratios.

That is, each input of the inverse of a trig function is not an angle but a trig ratio, and each output is not a trig ratio but an angle.

So the domain of the inverse is the set of trig ratios, and the range is the set of angles.
And of course, the trig function we take the inverse of has to be one-to-one.
For instance however, a sine function as $y = f(x) = \sin x$ for x real is not one-to-one, because like the quadratic function $q(x) = x^2$ for x real, the sine function f can produce the same output for many different inputs.

For instance, $f(0) = \sin 0 = 0$, $f(\pi) = \sin \pi = 0$, $f(2\pi) = \sin 2\pi = 0$, …
And the same is true, also, for any other output as 1. We can use many different inputs to get the same out put. The same output for different inputs as shown in Fig. 3 below.

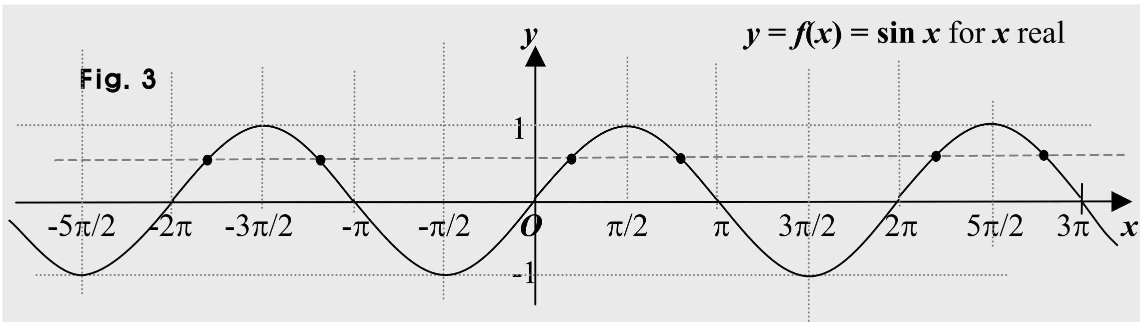


Fig. 3

And the same is true, too, for a cosine function as $y = g(x) = \cos x$ for x real.

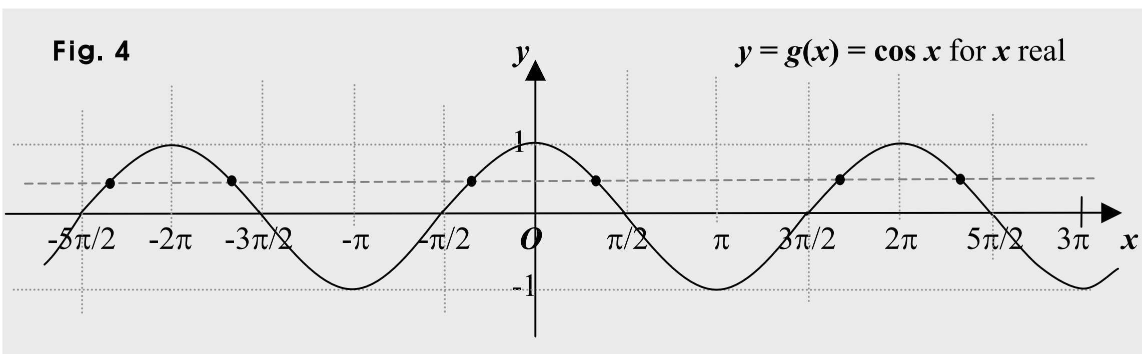


Fig. 4

The same output gets produced for different inputs.
So what is the inverse of a trig function?

Math begins with definitions. So we may want to begin with some definitions.
We can define the inverses of the three basic trig functions the way as follows.

$y = f(x) = \sin x \Leftrightarrow x = f^{-1}(y) = \sin^{-1} y$,

$y = g(x) = \cos x \Leftrightarrow x = g^{-1}(y) = \cos^{-1} y$, and

$y = h(x) = \tan x \Leftrightarrow x = h^{-1}(y) = \tan^{-1} y$, where x is an angle, and y is a trig ratio.

We call $\mathbf{sin^{-1}}\ \boldsymbol{y}$ the arc sine of $\boldsymbol{y}$, call $\mathbf{cos^{-1}}\ \boldsymbol{y}$ the arc cosine of $\boldsymbol{y}$, and call $\mathbf{tan^{-1}}\ \boldsymbol{y}$ the arc tangent of $\boldsymbol{y}$. And the superscript –1 is not an exponent, but is just the indicator that says the inverse.
And specifying a particular inverse function, we usually use $\boldsymbol{x}$ as the input variable, and use $\boldsymbol{y}$ as the output variable. So we usually specify the inverses the way as follows.

$$y = f^{1}(x) = \sin^{-1} x, \quad y = g^{-1}(x) = \cos^{-1} x, \quad \text{and } y = h^{-1}(x) = \tan^{-1} x,$$

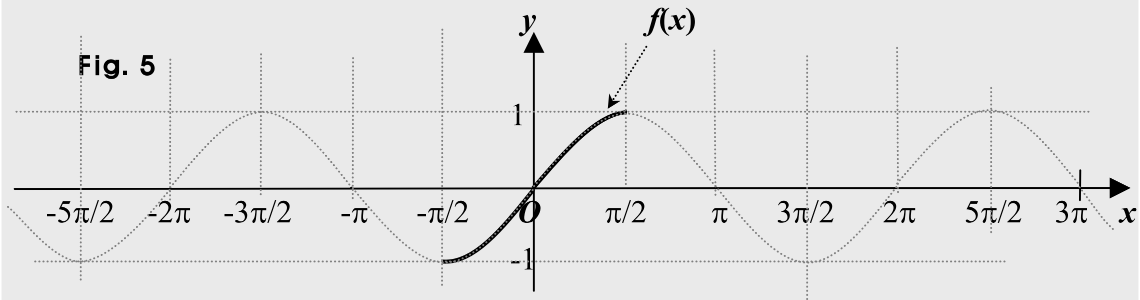
where $\boldsymbol{x}$ is an angle, and $\boldsymbol{y}$ is a trig ratio.

Let's now start with a sine function as follows.

$$y = f(x) = \sin x \text{ for } -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

Then, the function $\boldsymbol{f}$ is one-to-one.

Fig. 5



We know the domain of the sine function $\boldsymbol{f}$ is $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, and the range is $|y| \le 1$.

So assuming $\boldsymbol{g}$ is the inverse of the sine function $\boldsymbol{f}$, we get $y = g(x) = \sin^{-1} x$ for $|x| \le 1$,

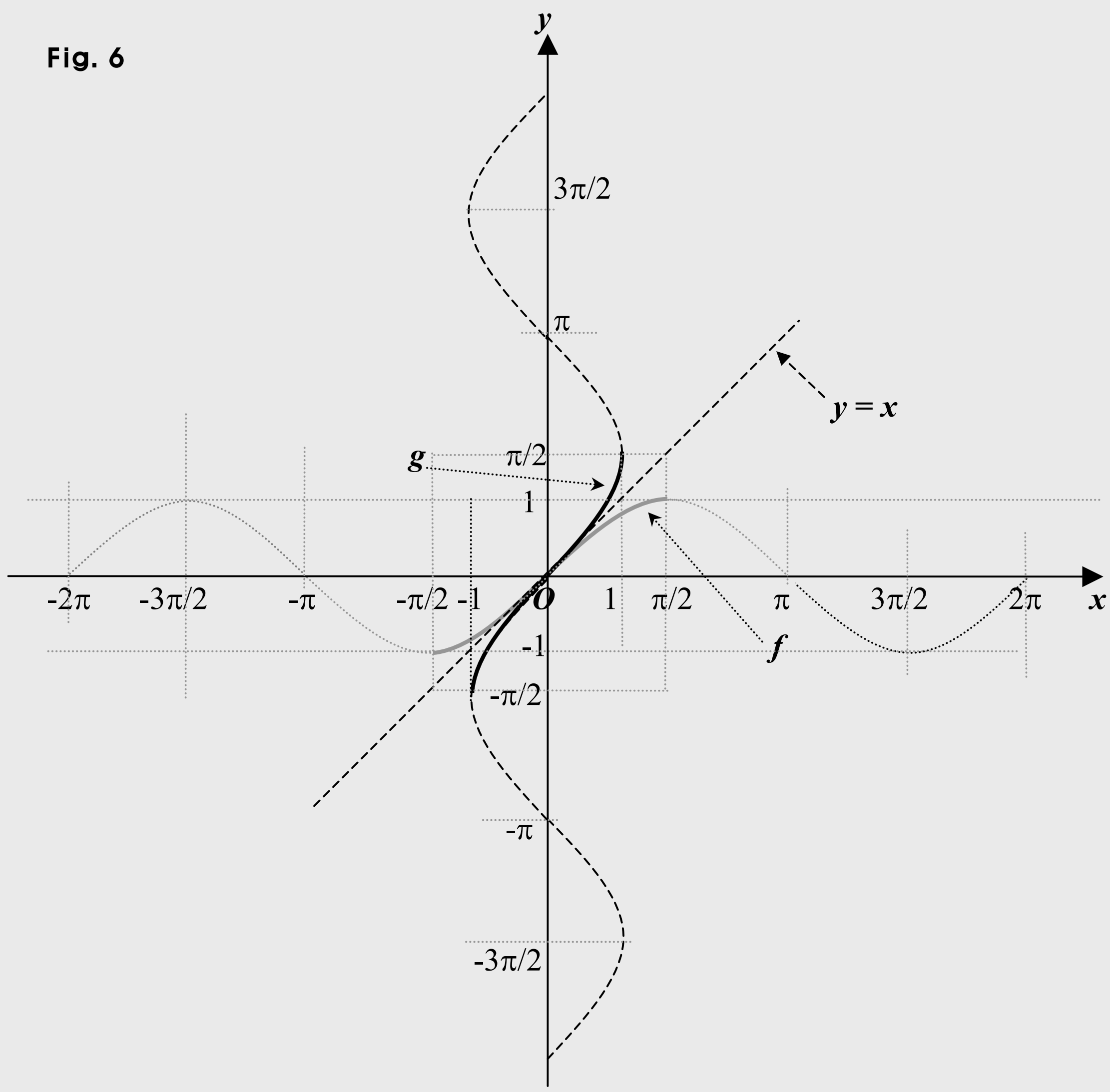
where $\boldsymbol{x}$ is a trig ratio, and $\boldsymbol{y}$ is an angle. And the range of $\boldsymbol{g}$ is $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

And we can call $\boldsymbol{g}$ an arc sine function, too, since $\mathbf{sin^{-1}}\ \boldsymbol{x}$ is called the arc sine of $\boldsymbol{x}$.
What then is the curve of the arc sine function $\boldsymbol{g}$, which is the inverse of the sine function $y = f(x) = \sin x$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$?

The curve of the inverse of a function is symmetric to the curve of the function about the line $y = x$. So the curve of $\boldsymbol{g}$ is symmetric to the curve of $\boldsymbol{f}$ about the line $y = x$, too.

Thus, putting both the two functions ***f*** and ***g*** in a graph, we can get this:

Fig. 6



So for instance, $\boldsymbol{g(1) = \sin^{-1} 1} = \frac{\pi}{2}$. And $\boldsymbol{g(-1) = \sin^{-1} (-1)} = -\frac{\pi}{2}$.

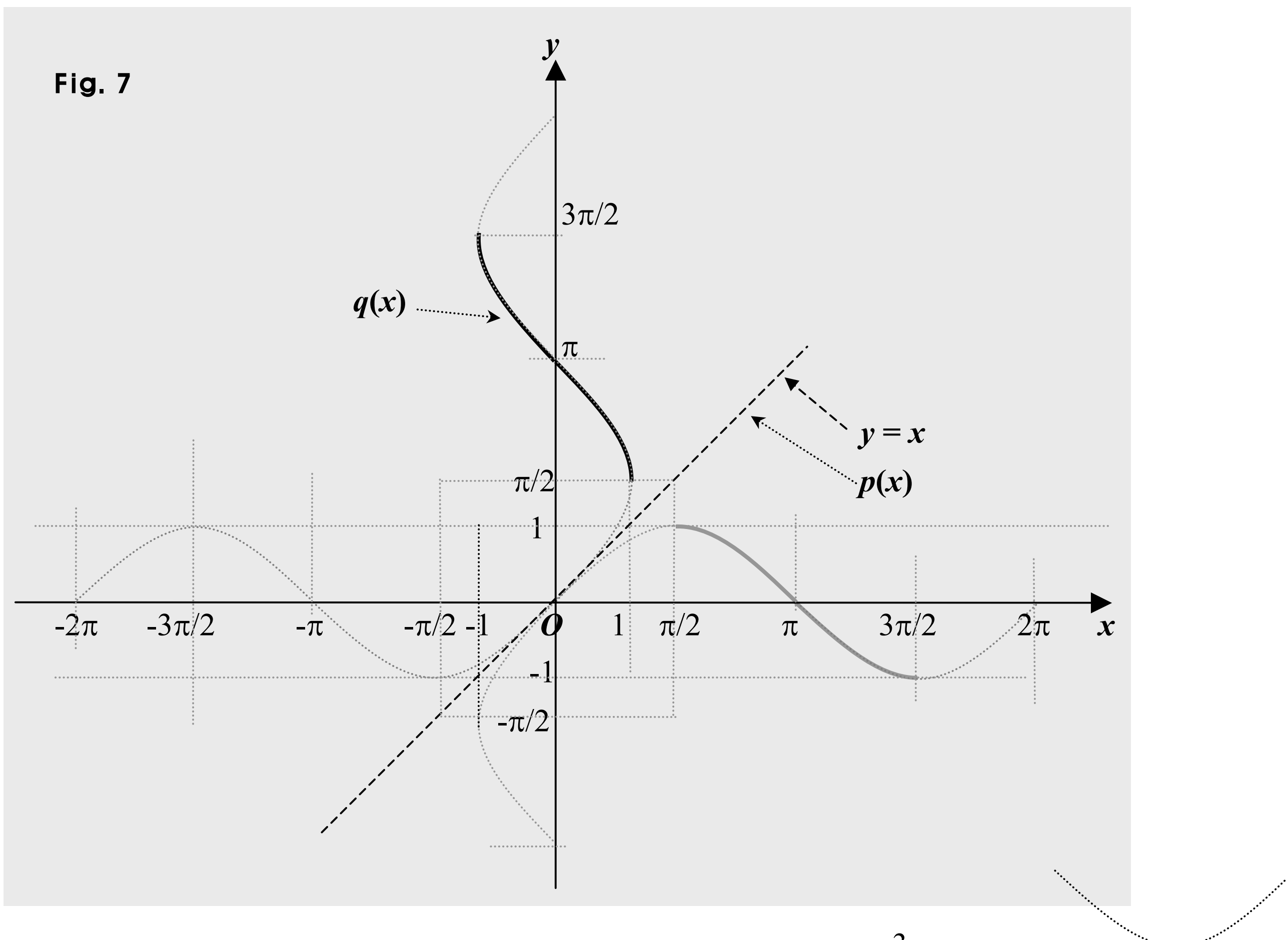
What then about the inverse of another sine function $\boldsymbol{y = p(x) = \sin x}$ for $\frac{\pi}{2} \le \boldsymbol{x} \le \frac{3\pi}{2}$?

We know $\boldsymbol{\sin^{-1} (-1)} = -\frac{\pi}{2}$, and $\boldsymbol{\sin^{-1} 1} = \frac{\pi}{2}$.

So assuming q is the inverse, we can put q this way: $y = q(x) = -\sin^{-1} x + \pi$ for $|x| \le 1$.

So for instance, $q(1) = -\sin^{-1} 1 + \pi = -\frac{\pi}{2} + \pi = \frac{\pi}{2}$.

And we get $q(-1) = -\sin^{-1}(-1) + \pi = -(-\frac{\pi}{2}) + \pi = \frac{3\pi}{2}$.

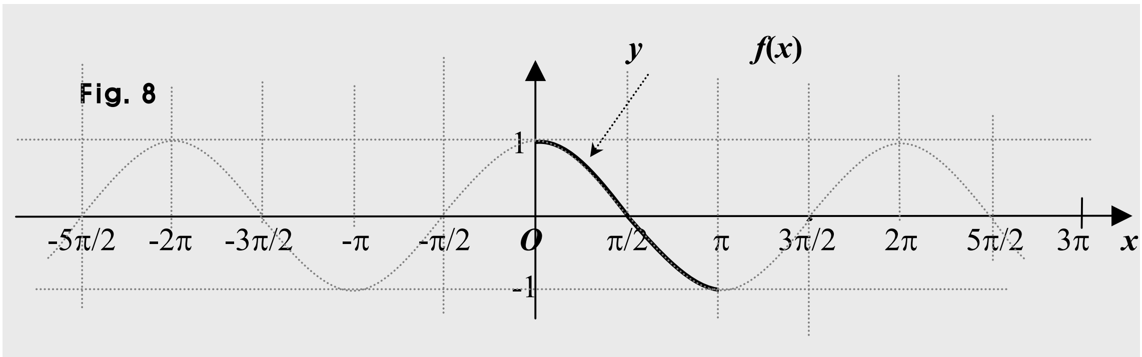


By the same token, assuming v is the inverse of $y = u(x) = \sin x$ for $-\frac{3\pi}{2} \le x \le -\frac{\pi}{2}$, we can put v this way: $y = v(x) = -(\sin^{-1} x + \pi)$ for $|x| \le 1$.

So for instance, $v(1) = -(\sin^{-1} 1 + \pi) = -(\frac{\pi}{2} + \pi) = -\frac{3\pi}{2}$.

And we get $v(-1) = -(\sin^{-1}(-1) + \pi) = -(-\frac{\pi}{2} + \pi) = -\frac{\pi}{2}$.

Next, a cosine function where $y = f(x) = \cos x$ for $0 \le x \le \pi$ is one-to-one.



We know the domain in f is $0 \le x \le \pi$, and the range is $|y| \le 1$.

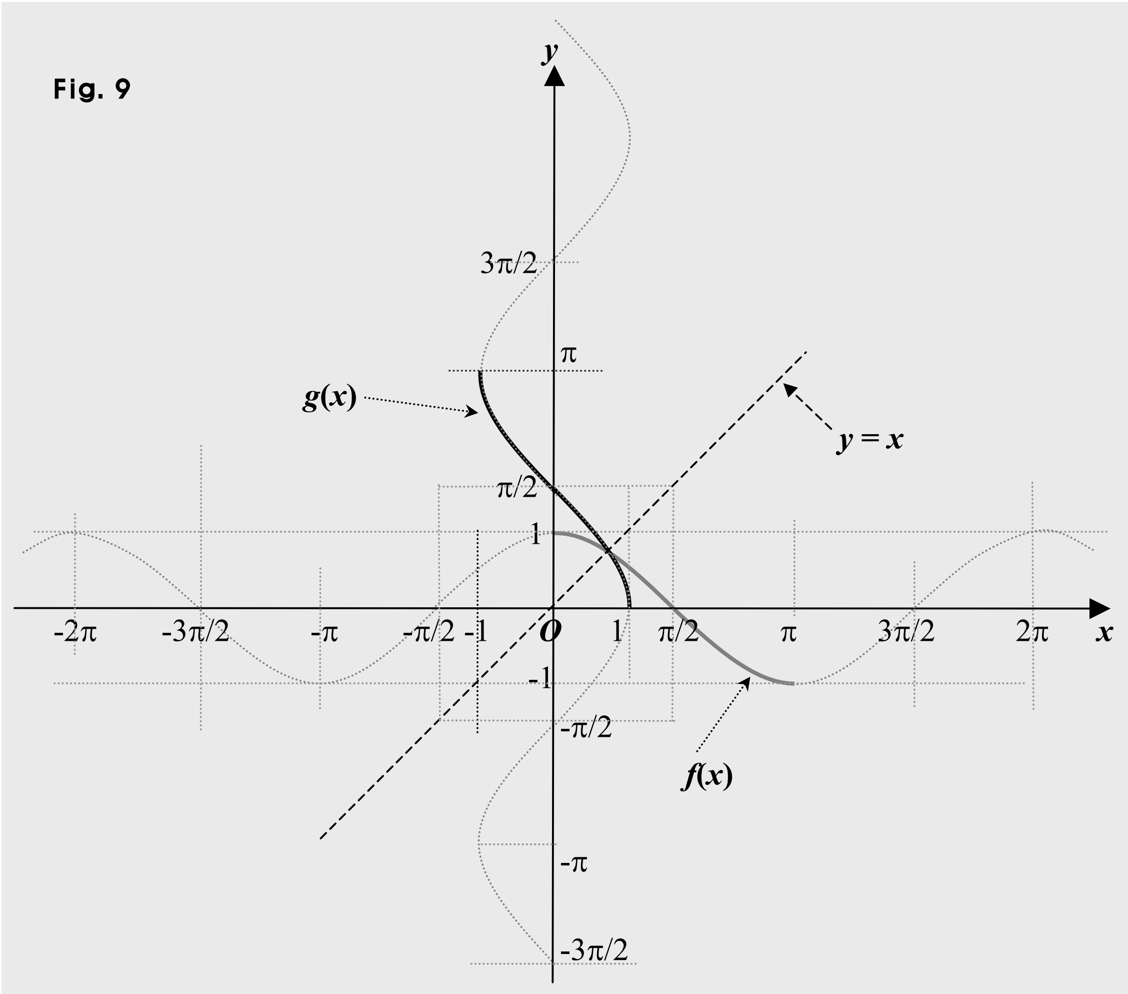
So assuming g is the inverse of the cosine function f, we get $y = g(x) = \cos^{-1} x$ for $|x| \le 1$, where x is a trig ratio, and y is an angle. And the range of g is $0 \le y \le \pi$.

And we can call g an arc cosine function, too, since $\cos^{-1} x$ is called the arc cosine of x.

What then is the curve of the arc cosine function g, which is the inverse of the cosine function $y = f(x) = \cos x$ for $0 \le x \le \pi$?

The curve of the inverse of a function is symmetric to the curve of the function about the line $y = x$.

So the curve of g is symmetric to the curve of f about the line $y = x$, too.
Thus, putting both the two functions f and g in a graph, we can get this:



So for instance, $\boldsymbol{g(1) = \cos^{-1} 1 = 0}$. And $\boldsymbol{g(-1) = \cos^{-1} (-1) = \pi}$.

What then about the inverse of another cosine function $\boldsymbol{y = p(x) = \cos x}$ for $\boldsymbol{\pi \le x \le 2\pi}$?

We know $\boldsymbol{\cos^{-1} (-1) = \pi}$, and $\boldsymbol{\cos^{-1} 1 = 0}$.

So assuming $\boldsymbol{q}$ is the inverse, we can put $\boldsymbol{q}$ the way below:

$\boldsymbol{y = q(x) = \cos^{-1} x + \pi}$ for $\boldsymbol{|x| \le 1}$.

So for instance, $\boldsymbol{p(1) = \cos^{-1} 1 + \pi = 0 + \pi = \pi}$. And $\boldsymbol{p(-1) = \cos^{-1} (-1) + \pi = \pi + \pi = 2\pi}$.
And by the same token, assuming $\boldsymbol{v}$ is the inverse of $\boldsymbol{y = u(x) = \cos x}$ for $\boldsymbol{-\pi \le x \le 0}$, we can put $\boldsymbol{v}$ this way: $\boldsymbol{y = v(x) = -\cos^{-1} x}$ for $\boldsymbol{|x| \le 1}$.

And next, a tangent function where $y = f(x) = \tan x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$ is one-to-one.

We know the domain in f is $-\frac{\pi}{2} < x < \frac{\pi}{2}$, and the range is a set of all real numbers.

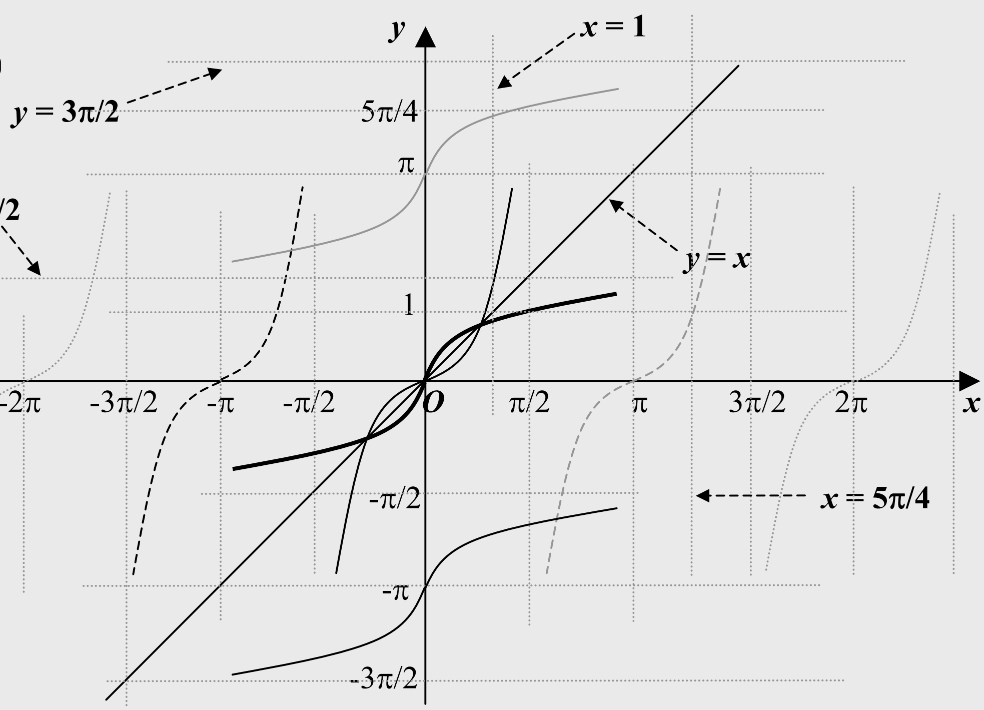
So assuming g is the inverse of the tangent function f, we get $y = g(x) = \tan^{-1} x$ for x real, where x is a trig ratio, and y is an angle. And the range of g is $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

And we can call g an arc tangent function, too, since $\tan^{-1} x$ is called the arc tangent of x.

What then is the curve of the arc tangent function g, which is the inverse of the tangent function $y = f(x) = \tan x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$?

The curve of the inverse of a function is symmetric to the curve of the function about the line $y = x$. So the curve of g is symmetric to the curve of f about the line $y = x$, too. And thus, putting both the two functions f and g in a graph, we can get this:

Fig. 10



So for instance, $\boldsymbol{g(1) = \tan^{-1} 1} = \frac{\pi}{4}$.

And $\boldsymbol{g(-1) = \tan^{-1} (-1)} = -\frac{\pi}{4}$.

What then about the inverse of a tangent function $\boldsymbol{y = p(x) = \tan x}$ for $\frac{\pi}{2} < x < \frac{3\pi}{2}$?

We know $\boldsymbol{\tan^{-1} (-1)} = -\frac{\pi}{4}$, and $\boldsymbol{\tan^{-1} 1} = \frac{\pi}{4}$.

So assuming $\boldsymbol{q}$ is the inverse, we can put $\boldsymbol{q}$ the way as follows.

$$\boldsymbol{y = q(x) = \tan^{-1} x + \pi} \text{ for } \boldsymbol{x} \text{ real.}$$

So for instance, we can get $\boldsymbol{p(1) = \cos^{-1} 1 + \pi} = \frac{\pi}{4} + \pi = \frac{5\pi}{4}$.

And we get $\boldsymbol{p(-1) = \tan^{-1} (-1) + \pi} = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}$.

And by the same token, assuming $\boldsymbol{v}$ is the inverse of $\boldsymbol{y = u(x) = \tan x}$ for $-\frac{3\pi}{2} \le x \le -\frac{\pi}{2}$,

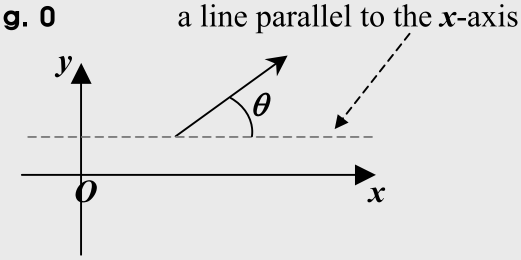
we can put $\boldsymbol{v}$ this way: $\boldsymbol{y = v(x) = \tan^{-1} x - \pi}$ for $\boldsymbol{x}$ real.

9. Samples on Trig Algebra

What can we do with **sin** θ, **cos** θ, etc.?

Taking care of objects that have to do with angles, we often use trig ratios.
A vector is a good example.
A vector is an object with an angle.
So taking care of vectors, we often need to work with trig ratios.
What do we mean by though, an object with an angle?

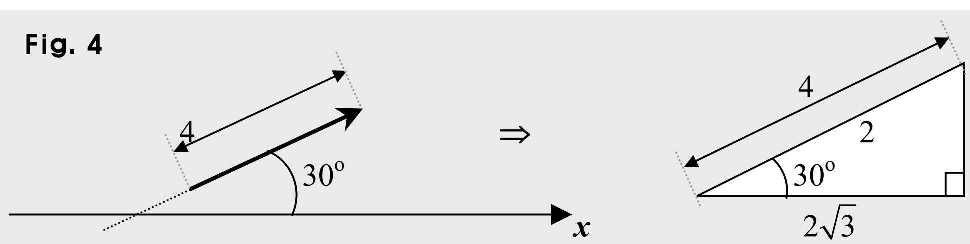
Fig. 0



A vector is an object that has a magnitude and an angle indicating a direction. So a vector has a magnitude and a direction. And usually a length shows the magnitude. Thus, a vector has not only a length but a direction, too.

So putting for instance, an arrow of length 4 with an angle of 30^o against the x-axis, we can put it the way as follows.

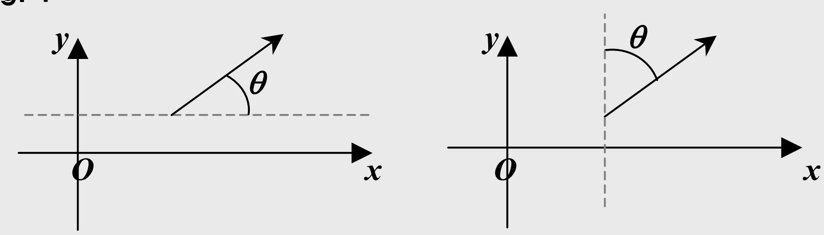
Fig. 4



And among vectors, we have forces, velocities, etc. So for instance, a force has its magnitude and its angle (or direction). How then do we describe a particular vector?

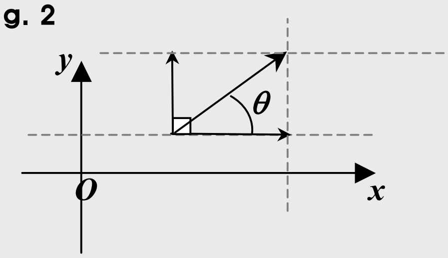
Describing a particular vector, we usually use an arrow, the length of which is the magnitude of the vector. And as in Fig. 1 below, the angle of the vector is the angle the arrow makes with a line, usually the ***x***-axis or the ***y***-axis in the ***x-y*** plane.

Fig. 1



Also, we can put a vector this way, too:
A vector is said to have its components, perpendicular to each other.
If for instance, a vector is in a 2-D space as the ***x-y*** plane, it can be said to have two components. One is called the horizontal component, usually parallel to the ***x***-axis, and the other is called the vertical component, often parallel to the ***y***-axis.

Fig. 2



And getting each component, we can use one of trig ratios as **sin** θ, **cos** θ, etc.
We can use it finding such a component stated above.
Assuming that θ is the angle between the vector and the ***x***-axis, we can get the vertical component using **sin** θ, and the horizontal component using **cos** θ.

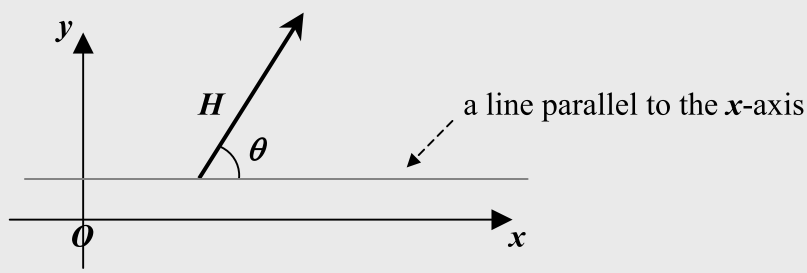
Also, using **tan** θ, we can specify the slope of the vector, because **tan** θ is the slope of the arrow, which is a line segment.
So we often take the adjacent as the horizontal component, and usually take the opposite as the vertical component. How then can we get the components?

Multiplying a vector by a trig ratio, we can get a component of the vector. So extracting each component of a vector, we multiply the vector by a trig ratio. Finding thus, each component of a vector, we multiply the vector by the sine, the cosine, or the tangent, and alternatively, we can multiply it by the cosecant, the secant, or the cotangent.

(Note that normally, specifying a vector, we put a small arrow on top of a letter saying the name of the vector as $\vec{H}$. Just for simplicity however, it is omitted in this book.)

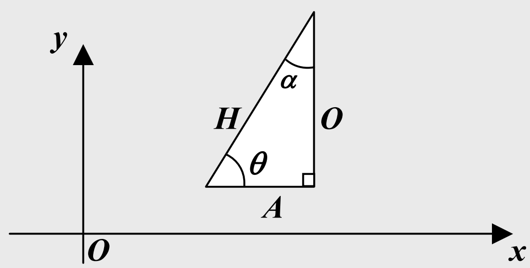
Assuming H is a vector, and its angle is θ, we can put it in the x-y plane the way below:

Fig. 3



Then, finding the components of H, we can put H in a right triangle the way below:

Fig. 4



Then, H is the hypotenuse, and A is the adjacent, because the angle we use is θ,
How then, can we get the adjacent A, the horizontal component in the setting above?

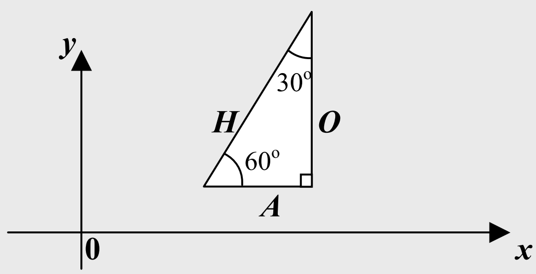
We know that the cosine is the adjacent over the hypotenuse. So we get $\cos\theta = \dfrac{A}{H}$.

Thus, we get $H \cdot \dfrac{A}{H} = A$, which is the adjacent, which is in this case, the horizontal component. So we get $H \cdot \cos\theta = H \cdot \dfrac{A}{H} = A$.

So multiplying the hypotenuse $\boldsymbol{H}$ by $\mathbf{cos}\ \theta$, we get $\boldsymbol{H}\cdot\mathbf{cos}\ \theta = \boldsymbol{H}\cdot\frac{\boldsymbol{A}}{\boldsymbol{H}} = \boldsymbol{A}$, the adjacent.
Thus, multiplying the hypotenuse by the cosine, we get the adjacent.

Assuming for instance, the angle θ is 60°, and putting $\boldsymbol{H}$ in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane, we can get:

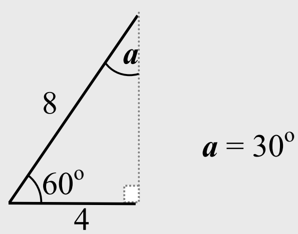
Fig. 5



Then, we get $\boldsymbol{H}\cdot\mathbf{cos}\ \mathbf{60^o} = \boldsymbol{H}\cdot\frac{\mathbf{1}}{\mathbf{2}} = \boldsymbol{A}$, since $\mathbf{cos}\ \mathbf{60^o} = \frac{1}{2}$. So we get $\boldsymbol{A} = \frac{\boldsymbol{H}}{\mathbf{2}}$.

For another instance, in Fig. 6 below, if the hypotenuse is 8, the opposite is $\boldsymbol{O}$, and the angle is 30°, that is, the angle between the hypotenuse and the adjacent is 30°,
we get $\boldsymbol{O} = \boldsymbol{H}\ \mathbf{sin}\ \theta = \mathbf{8}\ \mathbf{sin}\ \mathbf{30^o} = \mathbf{8}\cdot\frac{1}{2} = \mathbf{4}$.

Fig. 6



And if in Fig. 6 above, the adjacent is $\boldsymbol{A}$, and the angle is 60°, that is, the angle between the hypotenuse and the adjacent is 60°, we get $\boldsymbol{A} = \boldsymbol{H}\ \mathbf{cos}\ \theta = \mathbf{8}\ \mathbf{cos}\ \mathbf{60^o} = \mathbf{8}\cdot\frac{1}{2} = \mathbf{4}$.

Thus, in general, if $\boldsymbol{V}$ is a vector used as the hypotenuse in a right triangle, θ is the angle between $\boldsymbol{V}$ and the adjacent, and we need to get the adjacent, we multiply the vector $\boldsymbol{V}$ by $\mathbf{cos}\ \theta$. Then, we get $\boldsymbol{V}\cdot\mathbf{cos}\ \theta$, which is the adjacent in the right triangle. And if the adjacent is parallel to the $\boldsymbol{x}$-axis, it is the horizontal component.

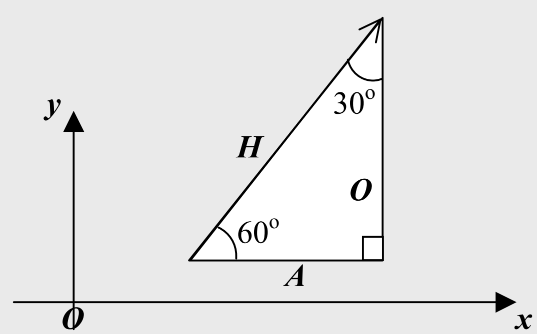
And in Fig.5 above, multiplying the hypotenuse $\boldsymbol{H}$ by $\mathbf{sin}\ \boldsymbol{\theta}$, we get $\boldsymbol{H}\cdot\mathbf{sin}\ \boldsymbol{\theta} = \boldsymbol{H}\cdot\dfrac{\boldsymbol{O}}{\boldsymbol{H}} = \boldsymbol{O}$, which is the opposite in the right triangle, and is in that case, the vertical component of the vector $\boldsymbol{H}$, because it is parallel to the $\boldsymbol{y}$-axis.

So of course, in general, assuming $\boldsymbol{V}$ is a vector used as the hypotenuse in a right triangle, $\boldsymbol{\theta}$ is the angle between the hypotenuse and the adjacent, and we need to get the opposite, we multiply the vector $\boldsymbol{V}$ by $\mathbf{sin}\ \boldsymbol{\theta}$. Then, we get $\boldsymbol{V}\cdot\mathbf{sin}\ \boldsymbol{\theta}$, which is the opposite in the right triangle. And if the opposite is parallel to the $\boldsymbol{y}$-axis, it is the vertical component.

Depending on the situation however, we need to decide which component is vertical or horizontal. So what matters is the interpretation of each component, so it depends on the situation.

Suppose for instance, in the figure below, $\boldsymbol{O}$ is parallel to a wall, a force is applied to the wall in the direction of $\boldsymbol{H}$, and the length of $\boldsymbol{H}$ indicates the amount of the force.

Fig. 7



Then, the amount of force acting on the wall vertically is $\boldsymbol{A}$, and the amount of force acting on the wall horizontally is $\boldsymbol{O}$. That is, of the force $\boldsymbol{H}$, the vertical component acting on the wall is $\boldsymbol{A}$, and the horizontal component acting on the wall is $\boldsymbol{O}$.

So assuming for instance, $\boldsymbol{H}$ is $10\dfrac{\boldsymbol{kg}\cdot\boldsymbol{m}}{\sec^2}$ acting on the wall at 30°, and referring to the figure above, we get the vertical component = $\boldsymbol{H}\cdot\mathbf{cos}\ \mathbf{60^o} = \mathbf{10}\cdot\dfrac{\mathbf{1}}{\mathbf{2}} = \mathbf{5}$, which is $\boldsymbol{A}$, and the horizontal component = $\boldsymbol{H}\cdot\mathbf{sin}\ \mathbf{60^o} = \mathbf{10}\cdot\dfrac{\sqrt{\mathbf{3}}}{\mathbf{2}} = \mathbf{5}\sqrt{\mathbf{3}}$, which is $\boldsymbol{O}$.

And checking to see if they are right, we get $A^2 + O^2 = 25 + 25{\cdot}3 = 25{\cdot}4 = 10^2 = H^2$.
Why is though, the angle not 30° but 60°?

The angle has to be between the hypotenuse and the ***adjacent***, which is crucial. So getting the adjacent wrong, we get a wrong ratio. Thus, we want to make sure which side is the adjacent determining the angle or applying trig ratios.
Examining also, the triangle above, we can notice that **sin 30°** is equal to **cos 60°**.

In fact, if two angles are complementary to each other, that is, the sum of the two is 90°, the sine of one angle equals the cosine of the other.

That is, if $\theta_1 + \theta_2 = 90^\circ$, we get $\sin \theta_1 = \cos \theta_2$.

In other words, we have $\cos (90^\circ - \theta) = \sin \theta$.

And of course, we have $\sin (90^\circ - \theta) = \cos \theta$, too.

What then, about $\tan (90^\circ - \theta)$?

It is $\cot \theta$. So we have $\tan (90^\circ - \theta) = \cot \theta$. Also, we have $\cot (90^\circ - \theta) = \tan \theta$.
Thus, if $\theta_1 + \theta_2 = 90^\circ$, we get $\tan \theta_1 = \cot \theta_2$.

And we call those equalities trig identities. We can show how we get the identities using other trig identities as $\sin (a \pm b) = \sin a \cdot \cos b \pm \cos a \cdot \sin b$.

And we can explain them using graphs, too.

By the way, if $\theta_1 + \theta_2 = 180^\circ$, the two angles are said to be *supplementary* to each other.

And in that case, too, we have some trig identities. For instance, we have these:

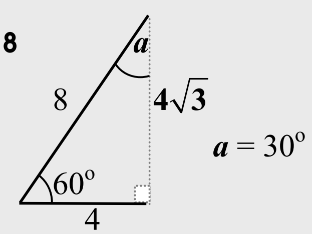
$\sin (180^\circ - \theta) = \sin \theta$, $\cos (180^\circ - \theta) = -\cos \theta$, and $\tan (180^\circ - \theta) = -\tan \theta$.

Next, let's move on to the tangent.

The tangent is the ratio of the opposite to the adjacent in a right triangle.
So assuming $\boldsymbol{O}$ is the opposite, $\boldsymbol{A}$ is the adjacent, and $\boldsymbol{\theta}$ is the angle the adjacent makes with the hypotenuse, we get $\tan\theta = \dfrac{\boldsymbol{O}}{\boldsymbol{A}}$.
Getting thus, the opposite $\boldsymbol{O}$ subject to the adjacent $\boldsymbol{A}$, we multiply the adjacent $\boldsymbol{A}$ by **tan** $\boldsymbol{\theta}$. That is, we get $\boldsymbol{A}\cdot\tan\theta = \boldsymbol{A}\cdot\dfrac{\boldsymbol{O}}{\boldsymbol{A}} = \boldsymbol{O}$.
So getting the opposite subject to the adjacent, we multiply the adjacent by the tangent.

So for instance, if in a right triangle, the adjacent $\boldsymbol{A}$ is 4, the opposite is $\boldsymbol{O}$, and the angle between the hypotenuse and the adjacent is 60°, since $\mathbf{tan\ 60^o} = \sqrt{3}$,
we get $\boldsymbol{O} = \boldsymbol{A}\ \mathbf{tan}\ \boldsymbol{\theta} = \mathbf{4\ tan\ 60^o = 4 \cdot \sqrt{3} = 4\sqrt{3}}$.

Fig. 8



What then about **the cosecant**?

The cosecant is the reciprocal of the sine, which is the ratio of the opposite to the hypotenuse, and is denoted by **csc**.

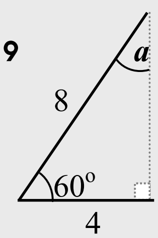
So assuming $\boldsymbol{O}$ is the opposite, $\boldsymbol{H}$ is the hypotenuse, and $\boldsymbol{\theta}$ is the angle the adjacent makes with the hypotenuse, we get $\mathbf{sin}\ \boldsymbol{\theta} = \dfrac{\boldsymbol{O}}{\boldsymbol{H}}$ and $\csc\theta = \dfrac{\boldsymbol{H}}{\boldsymbol{O}}$.

Getting thus, the hypotenuse $\boldsymbol{H}$ subject to the opposite $\boldsymbol{O}$, we multiply the opposite $\boldsymbol{O}$ by **csc** $\boldsymbol{\theta}$. That is, we get $\boldsymbol{O}\cdot\csc\theta = \boldsymbol{O}\cdot\dfrac{\boldsymbol{H}}{\boldsymbol{O}} = \boldsymbol{H}$.

So for instance, if in a right triangle, the opposite is 4, the hypotenuse is ***H***, and the angle between the hypotenuse and the adjacent is 30^o,

we get ***H* = *O* csc *θ* = 4 csc 30º = 4·2 = 8**, since $\mathbf{sin\ 30^o} = \frac{1}{2}$, and $\mathbf{csc\ 30^o} = \frac{1}{\sin 30^o} = 2$.

Fig. 9



$\boldsymbol{a} = 30^o$

And the same principle applies to the secant and the cotangent, too.

Also, we can say that once we've set the hypotenuse and the angle between the hypotenuse and a leg, we actually made a right triangle. The right triangle is defined. That's because the adjacent and the opposite are subject to the hypotenuse and the angle.

And one side is subject to another side by a trig ratio as the sine. So for instance, if ***A*** is subject to ***B*** by the sine, multiplying ***B*** by the sine, we get ***A***.

Now, the opposite is subject to the hypotenuse by the sine. So multiplying the hypotenuse by the sine, we get the opposite.
Also, the opposite is subject to the adjacent by the tangent. So multiplying the adjacent by the tangent, we get the opposite.

The adjacent is subject to the hypotenuse by the cosine. So multiplying the hypotenuse by the cosine, we get the adjacent.
Also, the adjacent is subject to the opposite by the cotangent. So multiplying the opposite by the cotangent, we get the adjacent.

The hypotenuse is subject to the opposite by the cosecant. So multiplying the opposite by the cosecant, we get the hypotenuse.
Also, the hypotenuse is subject to the adjacent by the secant. So multiplying the adjacent by the secant, we get the hypotenuse.

So in sum, assuming $\boldsymbol{H}$ is the hypotenuse, $\boldsymbol{A}$ is the adjacent, $\boldsymbol{O}$ is the opposite and $\boldsymbol{\theta}$ is the angle between $\boldsymbol{H}$ and $\boldsymbol{O}$, we get these:

$$\boldsymbol{O = H \cdot \sin\theta}, \quad \boldsymbol{O = A \cdot \tan\theta},$$

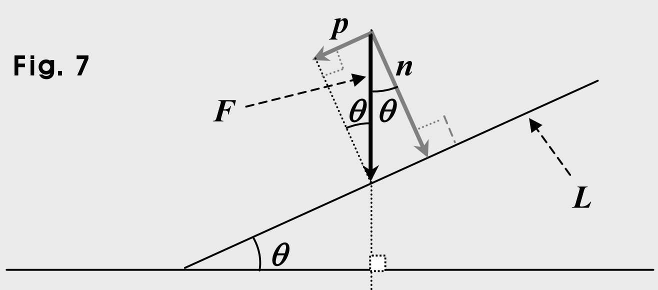
$$\boldsymbol{A = H \cdot \cos\theta}, \quad \boldsymbol{A = O \cdot \cot\theta},$$

$$\boldsymbol{H = O \cdot \csc\theta}, \quad \boldsymbol{H = A \cdot \sec\theta}.$$

No one needs to memorize any of these though.

Getting the concept, that is, understanding it conceptually, we can derive it in no time.
Let's now do another example.
So for instance, consider the force vector $\boldsymbol{F}$ as follows.

Fig. 7



The force acting on the line $\boldsymbol{L}$ can be decomposed into two components.
And of course, the two components are perpendicular to each other.

In Fig. 7 above, one component is $\boldsymbol{p}$, which is a force parallel to the line $\boldsymbol{L}$, and the other is $\boldsymbol{n}$, which is a force perpendicular to the line $\boldsymbol{L}$.

Multiplying thus, the vector $\boldsymbol{F}$ by the sine or the cosine, we can extract (find) those two components $\boldsymbol{p}$ and $\boldsymbol{n}$ from the vector $\boldsymbol{F}$. So we can get $\boldsymbol{p}$ and $\boldsymbol{n}$ the way as follows.

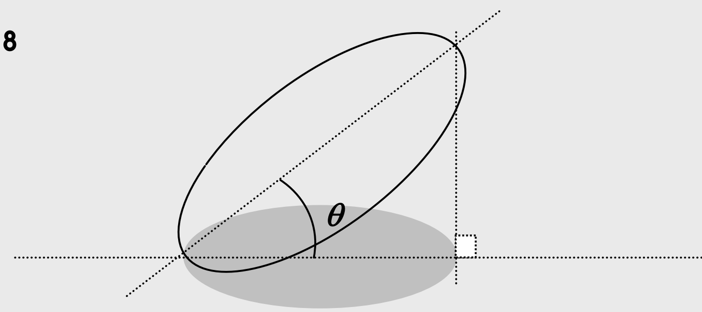
$$\boldsymbol{p = F \sin\theta}, \text{ and } \boldsymbol{n = F \cos\theta}.$$

Therefore, extracting or finding each component of a vector, we multiply the vector by the sine or the cosine.

Note that the object we can multiply by the sine or the cosine does not have to be a vector. That is, an object with such components does not have to be a vector.

Such an object can be in fact, just about anything if we can put an object in terms of its components perpendicular to each other. So it can be a line segment, a disk, a plane, etc. Besides, getting each component, we often say that we get each projection, which is like a shadow. So a projection is a component of an object with an angle. For instance, we can get a projection of a circular disk the way as follows.

Fig. 8

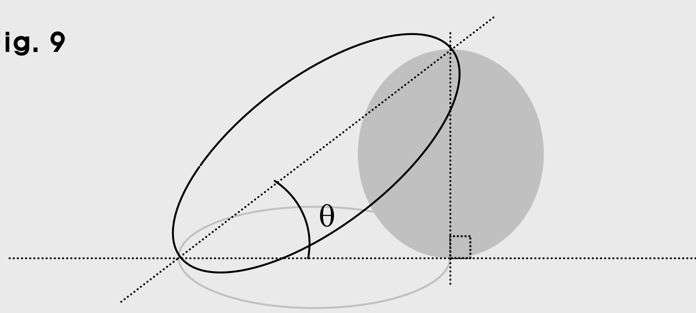


The elliptic disk in gray is the projection.
And assuming $\boldsymbol{C}$ is the area of the circular disk, $\boldsymbol{E}$ is the area of the projection, that is, the area of the elliptic disk in gray, and finding $\boldsymbol{E}$, we can get it the way as follows.

$$\boldsymbol{E = C \cos \theta}.$$

Also, we can get another projection of the circular disk the way as follows.

Fig. 9



Then, as in Fig. 9 above, the other projection is an elliptic disk in gray, too, and the two projections are perpendicular to each other.
And assuming for instance, $\boldsymbol{M}$ is the area of the other projection in Fig. 9 above, that is, the area of the other elliptic disk in gray, and finding $\boldsymbol{M}$, we can get it the way as follows.

$$\boldsymbol{M = C \sin \theta}.$$

Note: $(\sin x)(\cos x) = \sin x \cos x$, and $(\sin x)^n = \sin^n x$, so for instance, $(\sin x)^2 = \sin^2 x$.

Solving problems in math, we get to do algebra.

So solving problems with trigonometry, we need to do algebra, too.
And doing algebra, we often get to work with expressions with trig ratios as $\sin x$.
We can call such an expression a trig expression. And doing algebra with such expressions, we can say we do trig algebra.

Doing trig algebra, we manipulate trig expressions so that we can change them the way we can get to the solution.

And manipulating a trig expression, we change or alter it, convert or modify it, or break it apart, and put the pieces together to get the expression we want so that we can get to the solution. So algebra connects the problems to the solutions.

Therefore, we will get to practice some trig algebra. And we will get to know some tools called trig identities, which can help us do trig algebra easier and faster, of course.
So let's now do some algebra in trigonometry.

We may want to begins with a trig identity $\sin^2 x + \cos^2 x = 1$.
Dividing first, both sides by $\cos^2 x$, we can get this:

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \Rightarrow \left(\frac{\sin x}{\cos x}\right)^2 + 1 = \left(\frac{1}{\cos x}\right)^2 \Rightarrow \tan^2 x + 1 = \sec^2 x$$

Next, dividing by $\sin^2 x$ both sides of $\sin^2 x + \cos^2 x = 1$, we can get this:

$$\frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x} \Rightarrow 1 + \left(\frac{\cos x}{\sin x}\right)^2 = \left(\frac{1}{\sin x}\right)^2 \Rightarrow \cot^2 x + 1 = \csc^2 x$$

Also, since we have $\sin^2 x + \cos^2 x = 1$, we can get $\sin^2 x = 1 - \cos^2 x$.

So we can get $\tan^2 x \sin^2 x = \tan^2 x\,(1 - \cos^2 x) = \tan^2 x - \tan^2 x \cos^2 x$

$$= \tan^2 x - \frac{\sin^2 x}{\cos^2 x}\cos^2 x = \tan^2 x - \sin^2 x.$$

Thus, we get $\tan^2 x \sin^2 x = \tan^2 x - \sin^2 x$.

So in sum, we have these:

$\sin^2 x + \cos^2 x = 1.$

$\tan^2 x + 1 = \sec^2 x.$ $\cos^2 x + 1 = \csc^2 x.$

$\tan^2 x \sin^2 x = \tan^2 x - \sin^2 x.$

Next, doing some algebra, we can get this $\dfrac{1+\tan x}{1-\tan x} = \dfrac{1+2\sin x \cos x}{\cos^2 x - \sin^2 x}$. How?

Begining with the left hand side, we can get this:
$\dfrac{1+\tan x}{1-\tan x} = \dfrac{(1+\tan x)^2}{(1-\tan x)(1+\tan x)} = \dfrac{(1+\tan x)^2}{(1-\tan^2 x)}$. Then first, we can get this:

$1 - \tan^2 x = 1 - \dfrac{\sin^2 x}{\cos^2 x} = \dfrac{\cos^2 x - \sin^2 x}{\cos^2 x}$, so we get $\dfrac{1}{1-\tan^2 x} = \dfrac{\cos^2 x}{\cos^2 x - \sin^2 x}$.

Next, we can get $(1 + \tan x)^2 = 1 + 2\tan x + \tan^2 x$. And we know $1 + \tan^2 x = \sec^2 x$.
So we get $(1 + \tan x)^2 = \sec^2 x + 2\tan x$.

Thus, we can get $\dfrac{(1+\tan x)^2}{(1-\tan^2 x)} = \dfrac{\cos^2 x(\sec^2 x + 2\tan x)}{\cos^2 x - \sin^2 x} = \dfrac{1+2\cos x \sin x}{\cos^2 x - \sin^2 x}$, because

we have $\sec x = \dfrac{1}{\cos x}$, and $\tan x = \dfrac{\sin x}{\cos x}$. Done.

And doing some algebra, we can simpify some expressions. For instance, we can get
$\dfrac{1+\cos x}{\sec x - \tan x} - \dfrac{1-\cos x}{\sec x + \tan x} = 2(1+\tan x)$. How?

Beginning with the left hand side, we get:
$\dfrac{1+\cos x}{\sec x - \tan x} - \dfrac{1-\cos x}{\sec x + \tan x} = \dfrac{(1+\cos x)(\sec x + \tan x) - (1-\cos x)(\sec x - \tan x)}{\sec^2 x - \tan^2 x}$.

Next, looking at the denominator first, we can get this: $\sec^2 x - \tan^2 x = 1$, since we have $\tan^2 x + 1 = \sec^2 x$. Next, looking at the numerator, we can get this:

$(1 + \cos x)(\sec x + \tan x) - (1 - \cos x)(\sec x - \tan x)$
$= \sec x + \tan x + \cos x \sec x + \cos x \tan x - (\sec x - \tan x) + \cos x (\sec x - \tan x)$

$= \sec x + \tan x + 1 + \sin x - \sec x + \tan x + 1 - \sin x = 2\tan x + 2 = 2(\tan x + 1)$. Done.

And doing factorization, we can get $\cos^4 x - \sin^4 x = (\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x)$
$= \cos^2 x - \sin^2 x$, simply because we have $\cos^2 x + \sin^2 x = 1$. What then about $\sin 2x$?

At the end of the example 4 in the set of **Examples in Cosine Functions**, it was mentioned that $\sin 2x = 2 \cos x \sin x$. And we can get it from a trig identity as follows.

$$\sin (a + b) = \sin a \cos b + \cos a \sin b.$$

So we get $\sin 2x = \sin (x + x) = \sin x \cos x + \cos x \sin x = 2 \sin x \cos x$.
What then about $\sin (a - b)$?

We know $\sin (-b) = -\sin b$, and $\cos (-b) = \cos b$. So we get:
$\sin \{a + (-b)\} = \sin (a - b) = \sin a \cos (-b) + \cos a \sin (-b) = \sin a \cos b - \cos a \sin b$.

Thus, in short, we can set both this way: $\sin (a \pm b) = \sin a \cos b \pm \cos a \sin b$.
What then about $\sin 3x$?

Using the identity above, we can get $\sin 3x = \sin (2x + x) = \sin 2x \cos x + \cos 2x \sin x$.
What then about $\cos 2x$?

We can get it from another trig identity where $\cos (a + b) = \cos a \cos b - \sin a \sin b$.
So we get $\cos 2x = \cos x \cos x - \sin x \sin x = \cos^2 x - \sin^2 x \Rightarrow \cos 2x = \cos^2 x - \sin^2 x$.

And we know $\sin^2 x + \cos^2 x = 1$, so we get $\cos^2 x = 1 - \sin^2 x$, and $\sin^2 x = 1 - \cos^2 x$.
Thus, we get $\cos 2x = \cos^2 x - \sin^2 x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x$.

Now, we know $\mathbf{\sin 2}\boldsymbol{x}\mathbf{ = 2 \cos}\ \boldsymbol{x}\ \mathbf{\sin}\ \boldsymbol{x}$, and have $\mathbf{\cos 2}\boldsymbol{x}\mathbf{ = 1 - 2 \sin^2}\ \boldsymbol{x}$. So we can get this:

$\sin 2x \cos x + \cos 2x \sin x$

$= 2 \cos x \sin x \cos x + (1 - 2 \sin^2 x)\sin x$

$= 2 \sin x \cos^2 x + \sin x - 2 \sin^3 x$

$= 2 \sin x (1 - \sin^2 x) + \sin x - 2 \sin^3 x$

$= 3 \sin x - 4 \sin^3 x$.

And we have $\sin 3x = \sin 2x \cos x + \cos 2x \sin x$.
So we get $\sin 3x = 3 \sin x - 4 \sin^3 x$.   What then about $\cos 3x$?

We have $\cos (a + b) = \cos a \cos b - \sin a \sin b$.
So we get $\cos 3x = \cos (2x + x) = \cos 2x \cos x - \sin 2x \sin x$.

And we know $\sin 2x = 2 \cos x \sin x$, and $\cos 2x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x$.
So we get this:

$\cos 2x \cos x - \sin 2x \sin x = (2 \cos^2 x - 1)\cos x - 2 \cos x \sin x \sin x$

$= 2 \cos^3 x - \cos x - 2 \cos x \sin^2 x = 2 \cos^3 x - \cos x - 2 \cos x (1 - \cos^2 x)$

$= 2 \cos^3 x - \cos x - 2 \cos x + 2\cos^3 x = 4 \cos^3 x - 3\cos x$.

Thus, we get $\cos 3x = 4 \cos^3 x - 3 \cos x$.
So in sum, we have these:
$\sin 3x = 3 \sin x - 4 \sin^3 x$, and $\cos 3x = 4 \cos^3 x - 3 \cos x$.

Next, we have a trig identity where $\cos (a + b) = \cos a \cos b - \sin a \sin b$.
What then about $\cos (a - b)$?

We know: $\cos (-b) = \cos b$, and $\sin (-b) = -\sin b$.   So we can get this:
$\cos \{a + (-b)\} = \cos (a - b) = \cos a \cos (-b) - \sin a \sin (-b) = \cos a \cos b + \sin a \sin b$.

Thus, in short, we can set both this way: $\cos (a \pm b) = \sin a \cos b \mp \cos a \sin b$.

What then about **tan 2*x***?

We can get it from another trig identity where $\tan(a+b) = \dfrac{\tan a + \tan b}{1 - \tan a \tan b}$.

What then about **tan (*a* – *b*)**?

We know **tan (–*b*) = –tan *b***. So we get $\tan(a-b) = \dfrac{\tan a + \tan(-b)}{1 - \tan a \tan(-b)} = \dfrac{\tan a - \tan b}{1 + \tan a \tan b}$.

Thus, in short, we can set both this way: $\tan(a \pm b) = \dfrac{\tan a \pm \tan b}{1 \mp \tan a \tan b}$.

So we get $\tan 2x = \dfrac{\tan x + \tan x}{1 - \tan x \tan x} = \dfrac{2\tan x}{1 - \tan^2 x}$.

And in $\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$, replacing x with $\dfrac{x}{2}$,

we get $\cos x = 2\cos^2 \dfrac{x}{2} - 1 = 1 - 2\sin^2 \dfrac{x}{2}$

So we can get $\sin^2 \dfrac{x}{2} = \dfrac{1 - \cos x}{2}$, and $\cos^2 \dfrac{x}{2} = \dfrac{1 + \cos x}{2}$.

And we know $\tan x = \dfrac{\sin x}{\cos x}$. So we can get $\tan^2 \dfrac{x}{2} = \dfrac{\sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}} = \dfrac{1 - \cos x}{1 + \cos x}$.

What then about $\tan \dfrac{x}{2}$?

Since we have $\tan x = \dfrac{\sin x}{\cos x}$, we can get $\tan \dfrac{x}{2} = \dfrac{\sin \frac{x}{2}}{\cos \frac{x}{2}} = \dfrac{\sin \frac{x}{2} \cos \frac{x}{2}}{\cos^2 \frac{x}{2}}$

And we have $\sin 2x = 2\sin x \cos x$. So we get $\sin x = 2\sin \dfrac{x}{2} \cos \dfrac{x}{2}$.

Also, we have $\cos^2 \dfrac{x}{2} = \dfrac{1 + \cos x}{2}$. So we get $2\cos^2 \dfrac{x}{2} = 1 + \cos x$.

Thus, we get $\tan \dfrac{x}{2} = \dfrac{\sin \frac{x}{2} \cos \frac{x}{2}}{\cos^2 \frac{x}{2}} = \dfrac{2\sin \frac{x}{2} \cos \frac{x}{2}}{2\cos^2 \frac{x}{2}} = \dfrac{\sin x}{1 + \cos x} \Rightarrow \tan \dfrac{x}{2} = \dfrac{\sin x}{1 + \cos x}$

And from $\mathbf{sin}\ (a \pm b)$ and $\mathbf{cos}\ (a \pm b)$, we can get some useful expressions as follows.

To begin with, we have $\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$. So we get this:
$\sin(a + b) + \sin(a - b) = \sin a \cos b + \cos a \sin b + \sin a \cos b - \cos a \sin b$

$= 2\sin a \cos b$. Thus, we get $\sin a \cos b = \frac{1}{2}\{\sin(a + b) + \sin(a - b)\}$.

Next, $\sin(a + b) - \sin(a - b) = \sin a \cos b + \cos a \sin b - \sin a \cos b + \cos a \sin b$
$= 2\cos a \sin b$. Thus, we get $\sin(a + b) - \sin(a - b) = 2\cos a \sin b$.(1)
So we get $\cos a \sin b = \frac{1}{2}\{\sin(a + b) - \sin(a - b)\}$.

Next, $\cos(a + b) + \cos(a - b) = \cos a \cos b - \sin a \sin b + \cos a \cos b + \sin a \sin b$

$= 2\cos a \cos b$. (2) So we get $\cos a \cos b = \frac{1}{2}\{\cos(a + b) + \cos(a - b)\}$.
Next, $\cos(a + b) - \cos(a - b) = \cos a \cos b - \sin a \sin b - \cos a \cos b - \sin a \sin b$
$= -2 \sin a \sin b$. Thus, we get $\cos(a + b) - \cos(a - b) = -2 \sin a \sin b$. (3)
So we get $\sin a \sin b = -\frac{1}{2}\{\cos(a + b) + \cos(a - b)\}$.

Also, we can put the expressions above the way as follows.
To begin with, we have $\sin(a + b) + \sin(a - b) = 2\sin a \cos b$. (3)
And setting $x = a + b$, and $y = a - b$, we can get $a = \frac{x+y}{2}$, and $b = \frac{x-y}{2}$. So we get

$\sin(a + b) + \sin(a - b) = \sin x + \sin y$, and $2\sin a \cos b = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$. (4)

Thus from (3) and (4), we get $\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$.

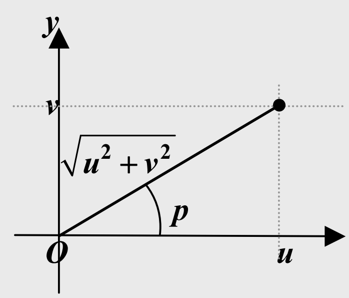
Next, from (1), we have $\sin(a + b) - \sin(a - b) = 2\cos a \sin b$, where
$a = \frac{x+y}{2}$, and $b = \frac{x-y}{2}$ So we get $\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$
Next from (2), we have $\cos(a + b) + \cos(a - b) = 2\cos a \cos b$, where
$a = \frac{x+y}{2}$, and $b = \frac{x-y}{2}$. So we get $\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$.

Next from (3), we have **cos (*a* + *b*) – cos (*a* – *b*) = –2sin *a* sin *b***, where $a = \frac{x+y}{2}$, and $b = \frac{x-y}{2}$. So we get $\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$.

And using the distance formula, i.e., Pythagorean Theorem, together with **sin (*a* ± *b*)** and **cos (*a* ± *b*)**, we can get a couple of useful expressions as follows.

Suppose first, in the ***x*-*y*** plane, ***p*** is the angle between the ***x***-axis and a line segment connecting the origin and a point **(*u*, *v*)** in the first quadrant.

Fig. 10

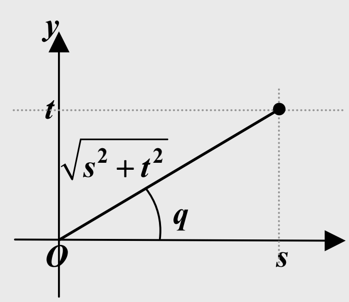


Then, we can get this:

$u\sin x + v\cos x = \sqrt{u^2+v^2}\,\sin(x+p)$, where $\cos p = \frac{u}{\sqrt{u^2+v^2}}$, and $\sin p = \frac{v}{\sqrt{u^2+v^2}}$.

And assuming next, as in Fig. 11, in the ***x*-*y*** plane, ***q*** is the angle between the ***x***-axis and a line segment connecting the origin and a point **(*s*, *t*)** in the first quadrant,

Fig. 11



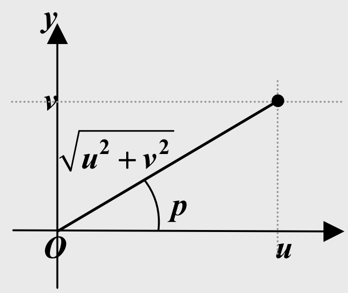
Then, we can get this:

$s\cos x + t\sin x = \sqrt{s^2+t^2}\,\cos(x-q)$, where $\cos q = \frac{s}{\sqrt{s^2+t^2}}$ and $\sin q = \frac{t}{\sqrt{s^2+t^2}}$.

Let's see now how it is the case.

To begin with, putting the point **(*u*, *v*)** in the ***x*-*y*** plane, we get this:

Fig. 12



First, we get $\cos p = \dfrac{u}{\sqrt{u^2+v^2}}$, and $\sin p = \dfrac{v}{\sqrt{u^2+v^2}}$.

Next, we can set $u \sin x + v \cos x = \sqrt{u^2+v^2}\,\left(\dfrac{u}{\sqrt{u^2+v^2}} \sin x + \dfrac{v}{\sqrt{u^2+v^2}} \cos x\right)$.

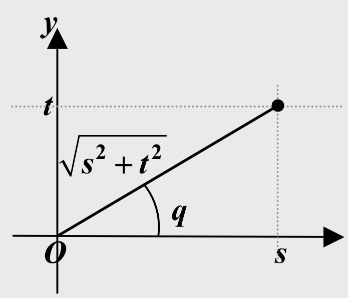
So we get $u \sin x + v \cos x = \sqrt{u^2+v^2}\,(\cos p \sin x + \sin p \cos x)$.

And we know $\sin(x+p) = \sin p \cos x + \cos p \sin x$. Thus, we get this:

$u \sin x + v \cos x = \sqrt{u^2+v^2}\,\sin(x+p)$, where $\cos p = \dfrac{u}{\sqrt{u^2+v^2}}$, and $\sin p = \dfrac{v}{\sqrt{u^2+v^2}}$.

Let's next, move on to the point **(*s*, *t*)** in the ***x*-*y*** plane.

Fig. 13



Then, first, we get $\cos q = \dfrac{s}{\sqrt{s^2+t^2}}$ and $\sin q = \dfrac{t}{\sqrt{s^2+t^2}}$.

And we can set $s \cos x + t \sin x = \sqrt{s^2+t^2}\,\left(\dfrac{s}{\sqrt{s^2+t^2}} \cos x + \dfrac{t}{\sqrt{s^2+t^2}} \sin x\right)$.

So we get $s \cos x + t \sin x = \sqrt{s^2+t^2}\,(\cos q \cos x + \sin q \sin x)$.

And we know $\cos(x - q) = \cos x \cos q + \sin x \sin q$. Thus, we get this:

$s \cos x + t \sin x = \sqrt{s^2 + t^2} \cos(x - q)$, where $\cos q = \dfrac{s}{\sqrt{s^2 + t^2}}$, and $\sin q = \dfrac{t}{\sqrt{s^2 + t^2}}$.

Of course, using $\sin(a \pm b)$, $\cos(a \pm b)$, and other trig identities, along with factorization, we can change many other trig expressions, too. For instance,

$\sin(a + b) \sin(a - b) = (\sin a \cos b + \cos a \sin b)(\sin a \cos b - \cos a \sin b)$

$= \sin^2 a \cos^2 b - \cos^2 a \sin^2 b = \sin^2 a\,(1 - \sin^2 b) - (1 - \sin^2 a)\sin^2 b$

$= \sin^2 a - \sin^2 a \sin^2 b - \sin^2 b + \sin^2 a \sin^2 b = \sin^2 a - \sin^2 b$

$= (1 - \cos^2 a) - (1 - \cos^2 b) = \cos^2 b - \cos^2 a$. So we can get this:

$$\sin(a + b) \sin(a - b) = \sin^2 a - \sin^2 b = \cos^2 b - \cos^2 a.$$

Let's next, try using the equality above.

We are going to put $\dfrac{\sin^2 a - \sin^2 b}{\sin^2(a + b)}$ in terms of $\tan a$ and $\tan b$.

Then, we get $\dfrac{\sin^2 a - \sin^2 b}{\sin^2(a + b)} = \dfrac{\sin(a + b)\sin(a - b)}{\sin^2(a + b)} = \dfrac{\sin(a - b)}{\sin(a + b)} = \dfrac{\sin a \cos b - \cos a \sin b}{\sin a \cos b + \cos a \sin b}$

$$= \frac{(\sin a \cos b - \cos a \sin b)\dfrac{1}{\cos a \cos b}}{(\sin a \cos b + \cos a \sin b)\dfrac{1}{\cos a \cos b}} = \frac{\dfrac{\sin a}{\cos a} - \dfrac{\sin b}{\cos b}}{\dfrac{\sin a}{\cos a} + \dfrac{\sin b}{\cos b}} = \frac{\tan a - \tan b}{\tan a + \tan b}.$$

And let's next, try changing $\cos(a + b) \cos(a - b)$ the way we did for the sines above.

$\cos(a + b) \cos(a - b) = (\cos a \cos b - \sin a \sin b)(\cos a \cos b + \sin a \sin b)$

$= \cos^2 a \cos^2 b - \sin^2 a \sin^2 b = \cos^2 a\,(1 - \sin^2 b) - (1 - \cos^2 a)\sin^2 b$

$= \cos^2 a - \cos^2 a \sin^2 b - \sin^2 b + \cos^2 a \sin^2 b = \cos^2 a - \sin^2 b$

$= (1 - \sin^2 a) - (1 - \cos^2 b) = \cos^2 b - \sin^2 a$. So we can get this:

$\mathbf{\cos(\mathit{a} + \mathit{b}) \cos(\mathit{a} - \mathit{b}) = \cos^2 \mathit{a} - \sin^2 \mathit{b} = \cos^2\mathit{b} - \sin^2 \mathit{a}}$.

And let's next, for another instance, see what we can get simplifying a trig expression as follows. $\mathbf{\cos \mathit{a} \sin(\mathit{b} - \mathit{c}) + \cos \mathit{b} \sin(\mathit{c} - \mathit{a}) + \cos \mathit{c} \sin(\mathit{a} - \mathit{b})}$.

$\mathbf{\cos \mathit{a} \sin(\mathit{b} - \mathit{c}) = \cos \mathit{a} (\sin \mathit{b} \cos \mathit{c} - \cos \mathit{b} \sin \mathit{c}) = \cos \mathit{a} \cos \mathit{c} \sin \mathit{b} - \cos \mathit{a} \cos \mathit{b} \sin \mathit{c}}$

$\mathbf{\cos \mathit{b} \sin(\mathit{c} - \mathit{a}) = \cos \mathit{b} (\sin \mathit{c} \cos \mathit{a} - \cos \mathit{c} \sin \mathit{a}) = \cos \mathit{a} \cos \mathit{b} \sin \mathit{c} - \cos \mathit{b} \cos \mathit{c} \sin \mathit{a}}$

$\mathbf{\cos \mathit{c} \sin(\mathit{a} - \mathit{b}) = \cos \mathit{c} (\sin \mathit{a} \cos \mathit{b} - \cos \mathit{a} \sin \mathit{b}) = \cos \mathit{b} \cos \mathit{c} \sin \mathit{a} - \cos \mathit{a} \cos \mathit{c} \sin \mathit{b}}$

So we can get $\mathbf{\cos \mathit{a} \sin(\mathit{b} - \mathit{c}) + \cos \mathit{b} \sin(\mathit{c} - \mathit{a}) + \cos \mathit{c} \sin(\mathit{a} - \mathit{b}) = 0}$.

And let's next, for another instance, see what we can get simplifying a trig expression as follows. $\mathbf{\cos^2 (\mathit{x} + \frac{\pi}{4}) + \cos^2 (\mathit{x} - \frac{\pi}{4})}$.

To begin with, we have $\mathbf{\cos(\mathit{x} + \frac{\pi}{4}) = \cos \mathit{x} \cos\frac{\pi}{4} - \sin \mathit{x} \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}(\cos \mathit{x} - \sin\ )}$.

And we have $\mathbf{\cos(\mathit{x} - \frac{\pi}{4}) = \cos \mathit{x} \cos\frac{\pi}{4} + \sin \mathit{x} \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}(\cos \mathit{x} + \sin \mathit{x})}$.

So we get these:

$\mathbf{\cos^2 (\mathit{x} + \frac{\pi}{4}) = \frac{1}{2}(\cos^2 \mathit{x} - 2\cos \mathit{x} \sin \mathit{x} + \sin^2 \mathit{x})}$.

$\mathbf{\cos^2 (\mathit{x} - \frac{\pi}{4}) = \frac{1}{2}(\cos^2 \mathit{x} + 2\cos \mathit{x} \sin \mathit{x} + \sin^2 \mathit{x})}$.

Thus, we get $\mathbf{\cos^2 (\mathit{x} + \frac{\pi}{4}) + \cos^2 (\mathit{x} - \frac{\pi}{4}) = \cos^2 \mathit{x} + \sin^2 \mathit{x} = 1}$.

What then about $\mathbf{\cos^2 (\mathit{x} + \frac{\pi}{3}) + \cos^2 (\mathit{x} - \frac{\pi}{3})}$?

We have $\mathbf{cos}\ (x + \frac{\pi}{3}) = \mathbf{cos}\ x\ \mathbf{cos}\frac{\pi}{3} - \mathbf{sin}\ x\ \mathbf{sin}\frac{\pi}{3} = \frac{1}{2}(\mathbf{cos}\ x - \sqrt{3}\ \mathbf{sin}\ x)$.

And we have $\mathbf{cos}\ (x - \frac{\pi}{3}) = \mathbf{cos}\ x\ \mathbf{cos}\frac{\pi}{3} + \mathbf{sin}\ x\ \mathbf{sin}\frac{\pi}{3} = \frac{1}{2}(\mathbf{cos}\ x + \sqrt{3}\ \mathbf{sin}\ x)$.

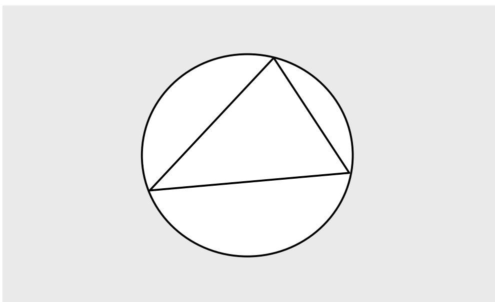
So we get these:

$\mathbf{cos}^2\ (x + \frac{\pi}{3}) = \frac{1}{4}(\mathbf{cos}^2\ x - 2\sqrt{3}\ \mathbf{cos}\ x\ \mathbf{sin}\ x + 3\mathbf{sin}^2\ x)$.

$\mathbf{cos}^2\ (x - \frac{\pi}{3}) = \frac{1}{4}(\mathbf{cos}^2\ x + 2\sqrt{3}\ \mathbf{cos}\ x\ \mathbf{sin}\ x + 3\mathbf{sin}^2\ x)$.

Thus, we get $\mathbf{cos}^2\ (x + \frac{\pi}{3}) + \mathbf{cos}^2\ (x - \frac{\pi}{3}) = \frac{1}{2}(\mathbf{cos}^2\ x + 3\mathbf{sin}^2\ x)$.

A. The Sine Rule



The sine is a trig ratio, the sine of an angle is the ratio of the opposite to the hypotenuse in a right triangle, and the angle is between the adjacent and the hypotenuse. So in short, the sine is the opposite over the hypotenuse.
What then is the sine rule?

It is often called the law of sines or the sine formula, too. We can apply the rule to a triangle. And using it, we can relate the three sides in a triangle in terms of the sines of the three angles.

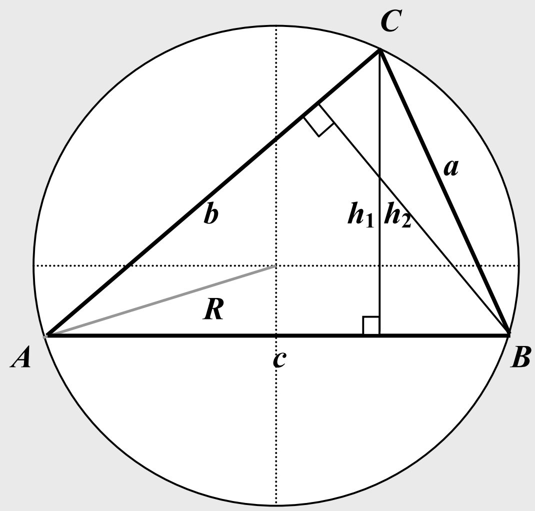
The sine rule is a system of equalities, where three ratios are equal to each other, and equals the diameter of a circle circumscribes a triangle.
Such a circle is called a circumcircle passing through all the three vertices.
And each of the three ratios is a ratio of a side to the sine of the angle facing the side.

Thus, assuming the three angles are $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$, the three sides are $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$, and the diameter is $\boldsymbol{D}$, we can put the sine rule the way as follows.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = D$$

And referring to Fig. 0 below, we can prove the rule as follows.

Fig. 0



Note that $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ are the three angles, $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are the three sides, and $2\boldsymbol{R}$ is the diameter. So assuming each of $\boldsymbol{h_1}$ and $\boldsymbol{h_2}$ is a height of the triangle $\boldsymbol{ABC}$, we can get this:

$\boldsymbol{b \sin A = h_1 = a \sin B \Rightarrow \dfrac{a}{\sin A} = \dfrac{b}{\sin B}}$, and $\boldsymbol{c \sin A = h_2 = a \sin C \Rightarrow \dfrac{a}{\sin A} = \dfrac{c}{\sin C}}$.

(And of course, we can get this, too: $\boldsymbol{b \sin C = c \sin B \Rightarrow \dfrac{b}{\sin B} = \dfrac{c}{\sin C}}$.)

So we get $\boldsymbol{\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}}$.

However, the equality above does not show the diameter, that is, $\boldsymbol{D}$, which is $\boldsymbol{2R}$.

Besides, the proof above does not cover all possible cases where the triangle can be shaped. It covers acute triangles only, where all the three angles are between 0 and 90°.

The sine rule covers in fact, any triangle, which can be therefore, an acute triangle, an obtuse triangle where one angle is between 90° and 180°, and even a right triangle, too.

So we may want to see the full proof, which covers triangles of all kinds, of course.

And more importantly, going through the full proof, we get to cover many tools we can use solving problems not only in trigonometry but other areas, too. In fact, what's really important is that going through the full proof taking each and every step it takes, you can grow your math skills. So let's now go through the full proof.

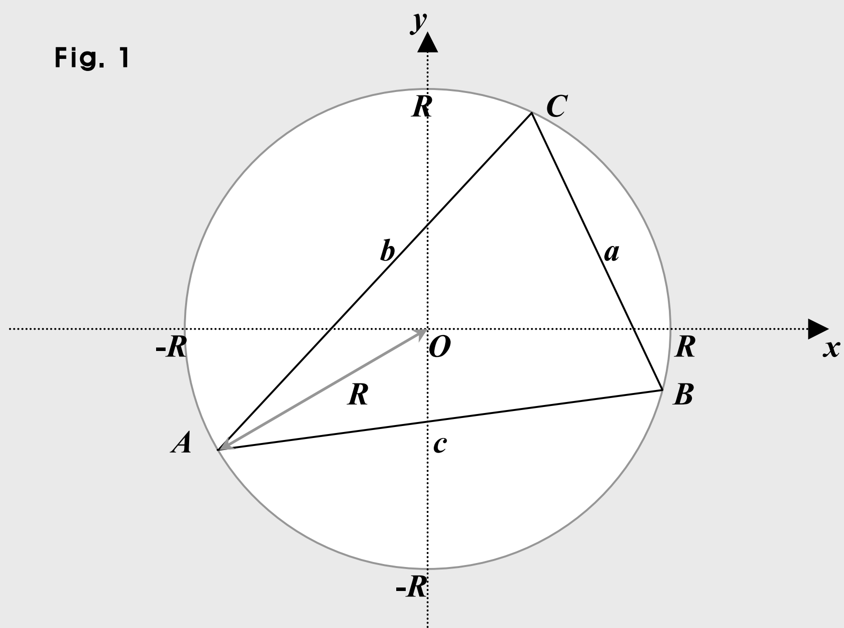
Now, beginning with the definition of the sine rule, we can put it the way as follows.

Assuming $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ are the three angles in a triangle inscribed in a circle of radius $\boldsymbol{R}$, $\boldsymbol{a}$ is the side facing the angle $\boldsymbol{A}$, $\boldsymbol{b}$ is the side facing the angle $\boldsymbol{B}$, and $\boldsymbol{c}$ is the side opposite to the angle $\boldsymbol{C}$, we can get this:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R,$$ which is called the sine rule or the sine formula.

Next, assuming $\boldsymbol{T}$ is a triangle $\boldsymbol{ABC}$, $\boldsymbol{U}$ is the circumcircle, and putting both in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane, we can put them the way as follows.

Fig. 1



Quite often though, it is simply put this way, too: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$,

which is just less the diameter, $\boldsymbol{2R}$.

And in the formula above, note that one of the three angles can be obtuse as well as acute or 90°. That is, we can have either of $\boldsymbol{0 < A < \pi}$, $\boldsymbol{0 < B < \pi}$, and $\boldsymbol{0 < C < \pi}$.

And we are now going to see the proof, that is, how it is the case.

To begin with, no matter what triangle it may be, it can be inscribed in a circle.

In other words, every triangle has its circumscribed circle, called a circumcircle, which is a circle passing through all the vertices of the triangle.

That's because three line segments can make one triangle only. Four line segments though, can make more than one quadrangle, and in fact, can make infinitely many quadrangles. And the same is true for the other polygons, too. So not every polygon has a circumcircle.

And next, we want the proof to cover all triangles. We can do so producing each proof in each of three different cases where the triangle $\boldsymbol{T}$ can be.

One is a case where the angle $\boldsymbol{C}$ is an acute angle, that is: $\mathbf{0} < \boldsymbol{C} < \frac{\pi}{2}$, and the other two angles are acute, too. That is, each of all the three angles is between 0 and 90°.

Another is a case where the angle $\boldsymbol{C}$ is an obtuse angle, that is, $\frac{\pi}{2} < \boldsymbol{C} < \pi$.

And the other is a case where the angle $\boldsymbol{C}$ is a right angle, that is, $\boldsymbol{C} = \frac{\pi}{2}$.

So let's now begin with the case where $\mathbf{0} < \boldsymbol{C} < \frac{\pi}{2}$, that is, $\mathbf{0} < \boldsymbol{C} < \mathbf{90^{o}}$.

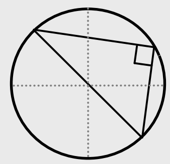
Then, to begin with, basically, what are we working on now?

It is a trig ratio, and in this case, is the sine.   Where then is the trig ratio from?

It is from a right triangle. So it's a good idea to form a right triangle. We can form one using a property of a circle. And the property is **Idea 1** as follows.

Idea 1: Connecting three points in a circle where two of the three are endpoints in the diameter, we can get a right triangle, where an angle is a right angle.

Fig. 2



(And we are going to see the proof shortly after this full proof.)

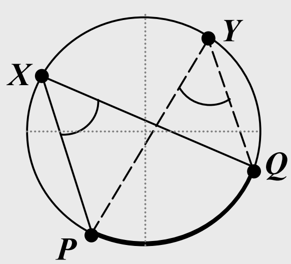
So using **Idea 1**, we can form a right triangle, and should be able to get the sine of each angle in the triangle T. Where then do we put a diameter in the circumcircle U?

We may want to put it so that we can use one of the three sides of the triangle T forming a right triangle. Also, we want to keep in mind that we want to come up with the sine of every angle in the triangle T.
How then can we actually come up with the sine of each angle in the triangle T?

We have another idea, which is **Idea 2** as follows:

Idea 2: Suppose X, P and Q are three points in a circle, and X is outside the arc PQ.

Fig. 3



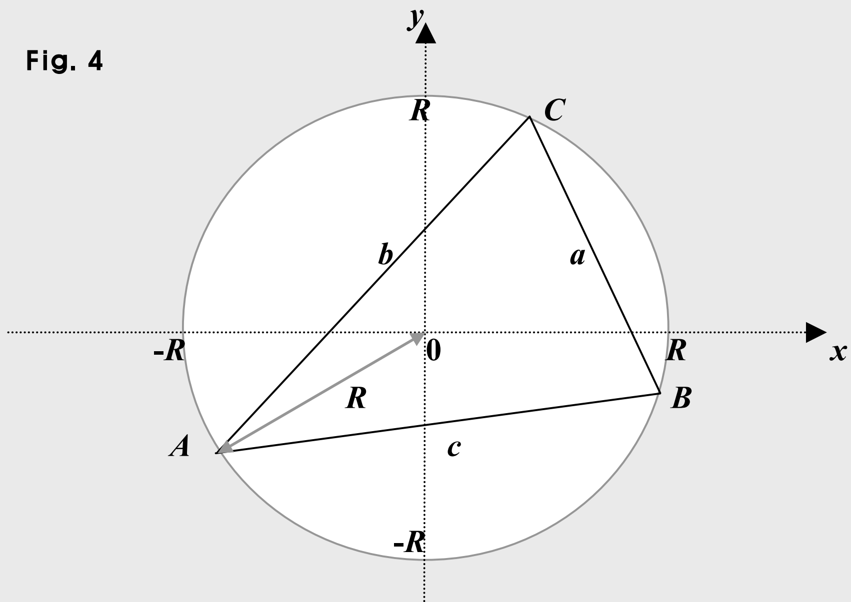
$$\angle PXQ = \angle PYQ$$

Then, no matter where X may be in the circle, the angle PXQ is the same.
So if Y is another point outside the arc, the angle PYQ equals the angle PXQ.

(And we will get to see the proof right after the end of this full proof.)

So keeping in mind **Idea 1** and **Idea 2** above, we may want to get back to now, the triangle T and the circumcircle U, and try putting a diameter in it.

Fig. 4

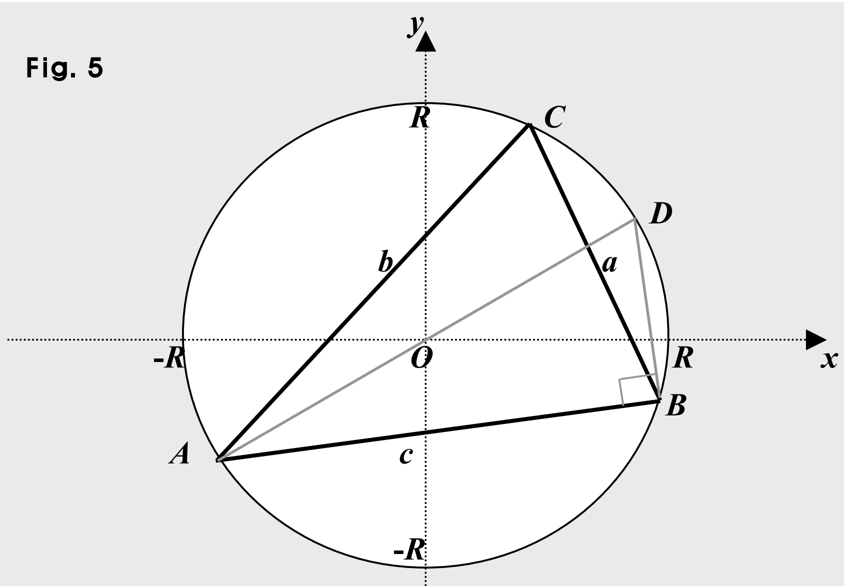


Suppose first, we use the side ***c*** to come up with one of the three ratios in the sine rule. What ratio then should we come up with?

It will be the ratio of the side ***c*** to the sine of the angle ***C***, that is, $\frac{\boldsymbol{c}}{\mathbf{sin}\,\boldsymbol{C}}$.

So we may want to put a diameter in the circle so that one of its endpoints can be a vertex in the triangle ***T***. And using the vertex ***A*** as the endpoint, we can get this:

Fig. 5



Next, assuming the line segment $\overline{AD}$ passes through the center, we get $\overline{AD} = 2R$.

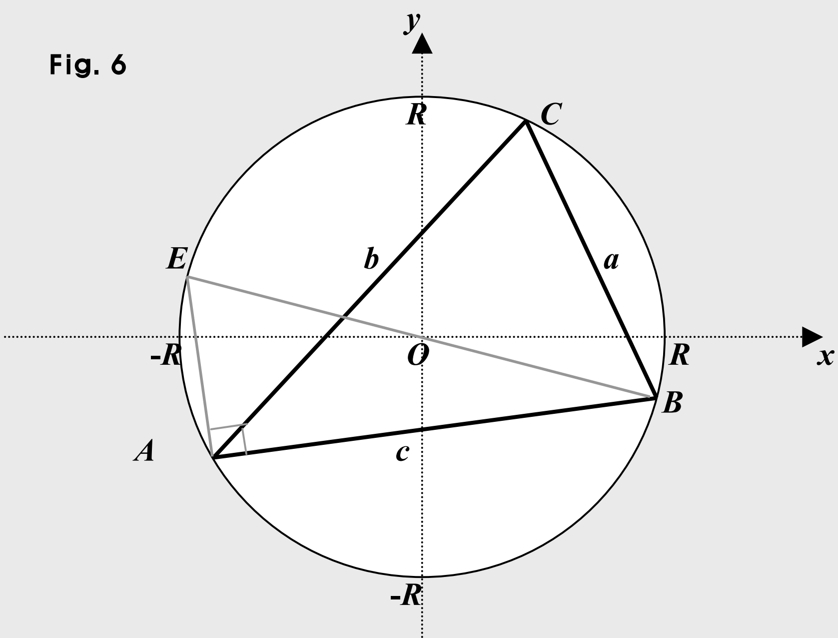
So taking the sine of the angle D, that is, taking $\sin D$, we get $\sin D = \dfrac{c}{2R}$.

Next, using **Idea 2** above, we can say that the two angles C and D are the same.

So we get $\sin C = \sin D$. Thus, we get $\sin C = \dfrac{c}{2R} \Rightarrow 2R = \dfrac{c}{\sin C}$.

And using **Idea 2** again, we can get the same using B as the endpoint, too:

Fig. 6



So first, assuming the line segment $\overline{EB}$ passes through the center, we get $\overline{EB} = 2R$.

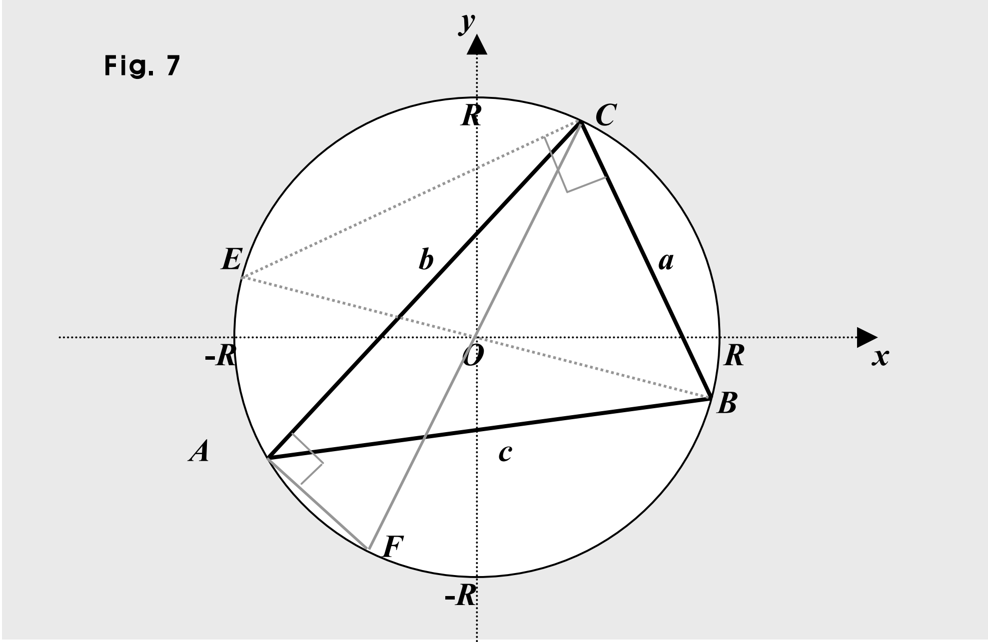
Thus, taking $\sin E$, we get $\sin E = \dfrac{c}{2R}$.

Next, using **Idea 2** again, we can say that the two angles C and E are the same.

So we get $\sin C = \sin E$.

Next, using each of C and B as an endpoint of a diameter in the circle, we can get this:

Fig. 7



So next, Assuming $\overline{CF} = 2R$, and taking $\sin F$, we get $\sin F = \dfrac{b}{2R}$.

And assuming $\overline{EB} = 2R$, and taking $\sin E$, we get $\sin E = \dfrac{a}{2R}$.

Next, using **Idea 2** again, we can say that two angles B and F are equal,

so we get $\sin B = \sin F$, which is $\dfrac{b}{2R}$; thus, we get $\sin B = \dfrac{b}{2R}$.

Also, we can say that two angles A and E are equal,

so we get $\sin A = \sin E$, which is $\dfrac{a}{2R}$; thus, we get $\sin A = \dfrac{a}{2R}$.

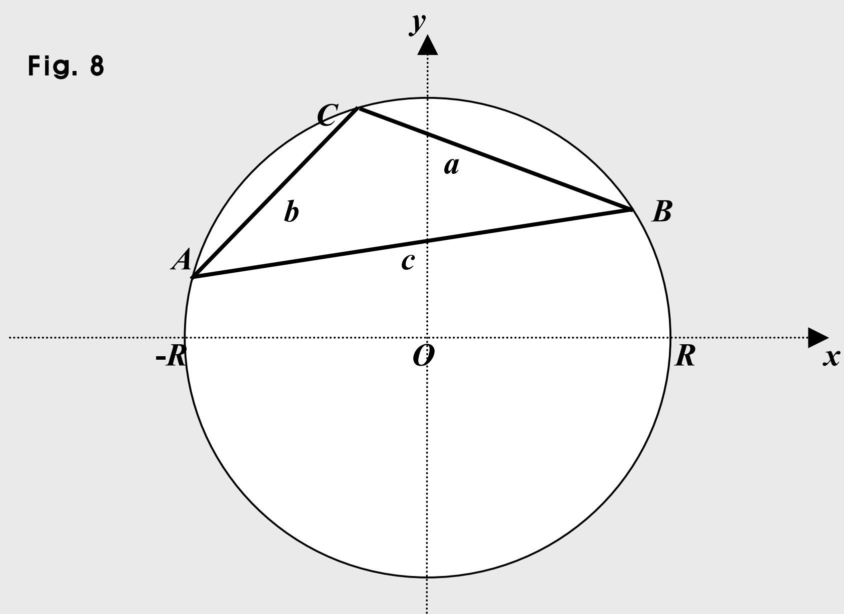
So we get $\sin B = \dfrac{b}{2R} \Rightarrow 2R = \dfrac{b}{\sin B}$, and $\sin A = \dfrac{a}{2R} \Rightarrow 2R = \dfrac{a}{\sin A}$.

And thus, putting threads together, we get $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$.

Let's next, move on to the case where $\frac{\pi}{2} < C < \pi$, that is, $90^{\circ} < C < 180^{\circ}$.

So one angle in a triangle is an obtuse angle. That is, the triangle ***T*** is an obtuse triangle. Putting thus, in the ***x-y*** plane, the obtuse triangle ***T***, along with its circumcircle ***U***, we can get this:

Fig. 8



As in the case above, we are basically, working on a trig ratio, which in this case, is the sine, which is from a right triangle. So it's again a good idea to begin with forming a right triangle.

And we can form one using **Idea 1**, which is saying that forming a triangle using one point in a circle and two endpoints in the diameter, we can get a right triangle.

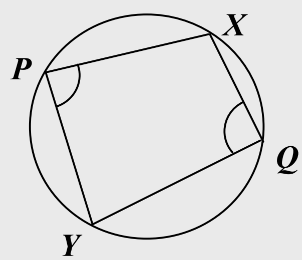
Thus, using **Idea 1** again, we can form a right triangle, and should be able to get the trig ratios, that is, the sines. Where then do we put a diameter in the circumcircle ***U***?

We may want to put it so that we can use one of the sides in the triangle ***T*** forming a right triangle. And also, we want to keep in mind that we want to come up with the sines of all the three angles in the triangle ***T***.

We can actually come up with each sine using two more ideas, and one is as follows.

Idea 3: Assuming as in Fig. 9 below, $\boldsymbol{P}$ and $\boldsymbol{Q}$ are two vertex angles facing each other in a quadrangle that has a circumcircle, we get $\boldsymbol{P} + \boldsymbol{Q} = \pi$.

Fig. 9



So we get $\boldsymbol{X} + \boldsymbol{Y} = \pi$, too.

And in fact, we have $\boldsymbol{X} + \boldsymbol{Y} + \boldsymbol{P} + \boldsymbol{Q} = \mathbf{2}\pi$.

And the other idea is in fact, a trig identity where $\mathbf{sin}\ (\pi - \theta) = \mathbf{sin}\ \theta$.

In other words, assuming $\theta_1 + \theta_2 = \pi$, we get $\mathbf{sin}\ \theta_1 = \mathbf{sin}\ \theta_2$.
So referring to the quadrangle above, we get $\mathbf{sin}\ \boldsymbol{P} = \mathbf{sin}\ \boldsymbol{Q}$.

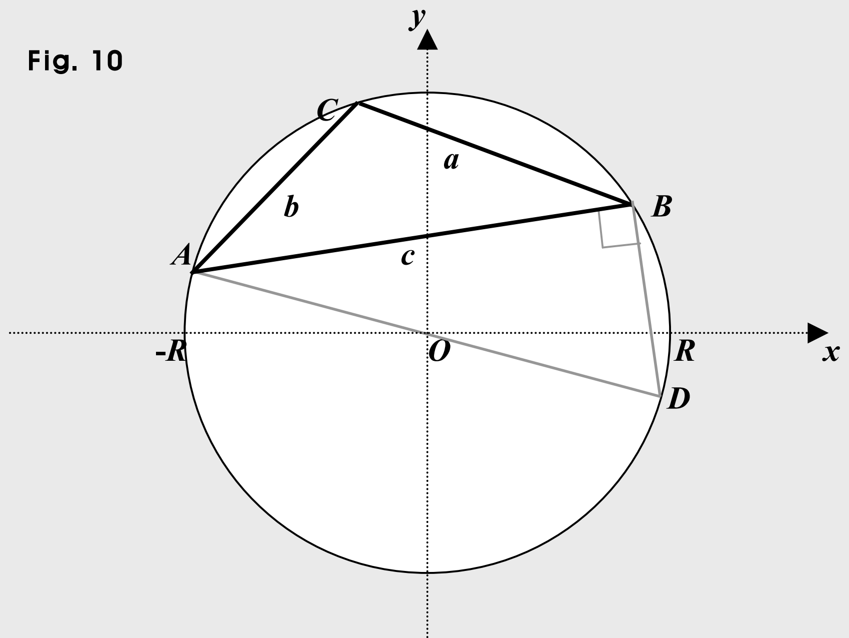
So keeping in mind the ideas and the identity above, we may want to get back to now, the triangle $\boldsymbol{T}$ and the circumcircle $\boldsymbol{U}$, and try putting a diameter in it.

Assuming first, we use the side $\boldsymbol{c}$ to come up with one of the ratios in the equality we want to prove, what ratio should we come up with?

It will be the ratio of the side $\boldsymbol{c}$ to the sine of the angle $\boldsymbol{C}$, that is $\dfrac{\boldsymbol{c}}{\mathbf{sin}\,\boldsymbol{C}}$.

So we may want to put a diameter in the circumcircle $\boldsymbol{U}$ so that one of its endpoints can be a vertex in the triangle $\boldsymbol{T}$. And using the vertex $\boldsymbol{A}$ as the endpoint, we can get this:

Fig. 10



So next, assuming the line segment $\overline{AD}$ passes through the center, we get $\overline{AD} = 2R$.

Thus, taking **sin *D***, we get $\sin D = \dfrac{c}{2R}$.

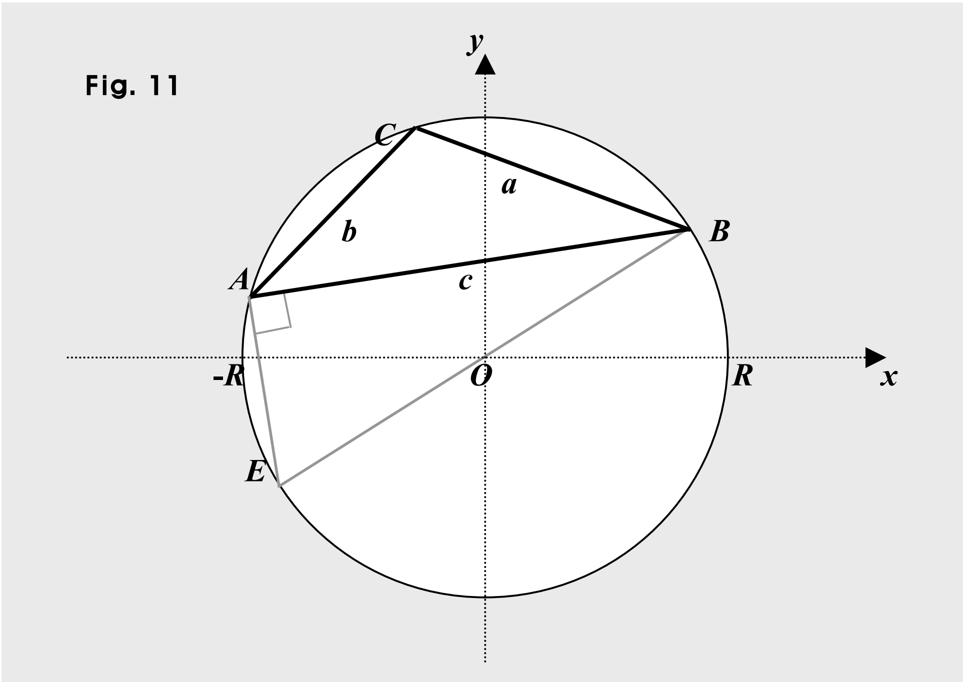
Next, using **Idea 3** that says the sum of two vertex angles facing each other in a quadrangle that has a circumcircle is π, we get $C + D = \pi$.

So next, using the trig identity above, we get $\sin C = \sin D$.

Thus, we get $\sin C = \dfrac{c}{2R} \Rightarrow 2R = \dfrac{c}{\sin C}$.

And using again **Idea 1** that says connecting three points in a circle where two of the three are endpoints in the diameter, we can get a right triangle, we can get the same also, using the vertex ***B*** as the endpoint of a diameter. So let's try it now for practice.

Using this, the vertex ***B*** as the endpoint of a diameter, we can put a right triangle the way as follows.



Then first, assuming $\overline{EB} = 2R$, and taking **sin *E***, we get $\sin E = \dfrac{c}{2R}$.

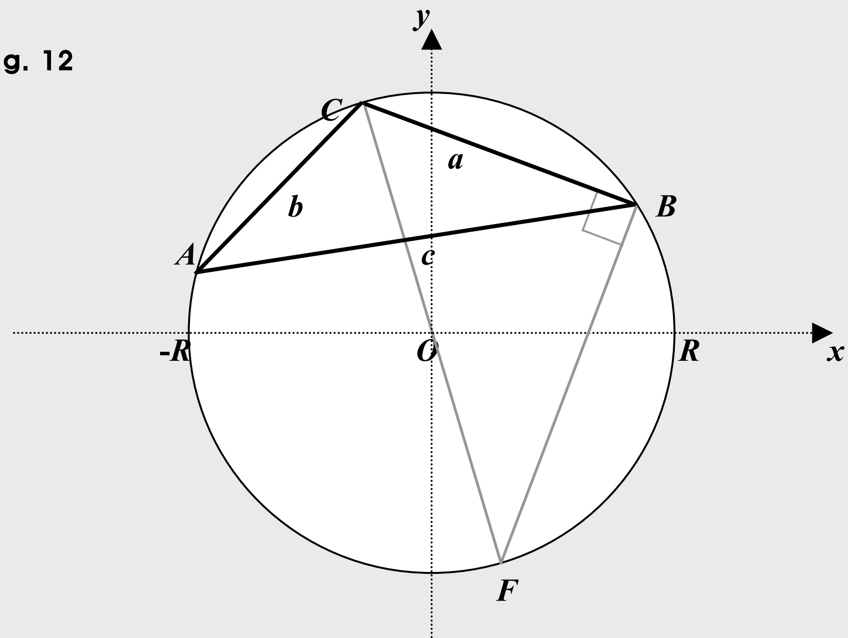
And next, using **Idea 3** that says the sum of two vertex angles facing each other in a quadrangle that has a circumcircle is π,, we get $C + E = \pi$.

So next, using the identity $\sin(\pi - \theta) = \sin\theta$, we get $\sin C = \sin E$.

Thus, we get $\sin C = \dfrac{c}{2R} \Rightarrow 2R = \dfrac{c}{\sin C}$.

Next, using the vertex ***C*** as an endpoint of a diameter in the circle ***U***, we can put a right triangle the way as follows:

Fig. 12

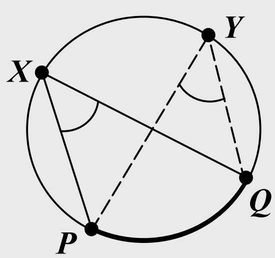


So next, assuming $\overline{CF} = 2R$, and taking **sin *F***, we get: $\sin F = \dfrac{a}{2R}$

And we have **Idea 2** as follows.

Suppose ***X***, ***P*** and ***Q*** are three points in a circle, and ***X*** is outside the arc ***PQ***.

Fig. 13



Then, no matter where the point ***X*** may be, the angle ***PXQ*** is the same.

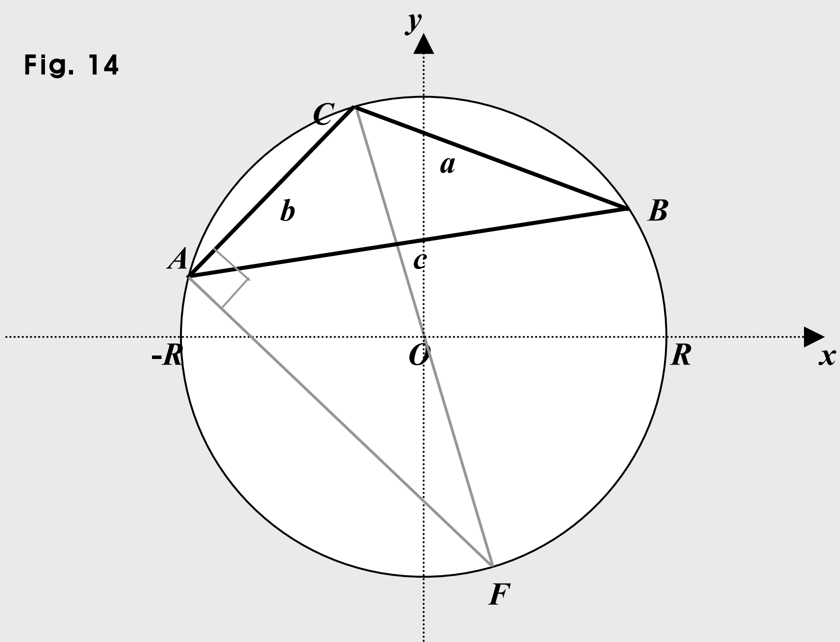
So we can say that the angle ***PXQ*** equals the angle ***PYQ***.

Thus, getting back to Fig. 12 above, and using **Idea 2**, we can say that two angles ***A*** and ***F*** are equal, so we get **sin *A* = sin *F***.

Thus, we get $\sin F = \dfrac{a}{2R} = \sin A \Rightarrow 2R = \dfrac{a}{\sin A}$. So now we need to get this: $\dfrac{b}{\sin B}$.

Again, using the vertex $\boldsymbol{C}$ as an endpoint of a diameter in the circle $\boldsymbol{U}$, we can get this:

Fig. 14



Using **Idea 2** we have just used above, we can say that two angles $\boldsymbol{B}$ and $\boldsymbol{F}$ are equal.

So next, assuming again, $\overline{\boldsymbol{CF}} = \mathbf{2}\boldsymbol{R}$, and taking $\sin \boldsymbol{F}$, we get this time $\sin \boldsymbol{F} = \dfrac{\boldsymbol{b}}{2\boldsymbol{R}}$.

Then, since $\angle \boldsymbol{B} = \angle \boldsymbol{F}$, we get $\sin \boldsymbol{B} = \sin \boldsymbol{F}$, which is $\dfrac{\boldsymbol{b}}{2\boldsymbol{R}}$.

So we get: $\sin \boldsymbol{B} = \dfrac{\boldsymbol{b}}{2\boldsymbol{R}} \Rightarrow 2\boldsymbol{R} = \dfrac{\boldsymbol{b}}{\sin \boldsymbol{B}}$.

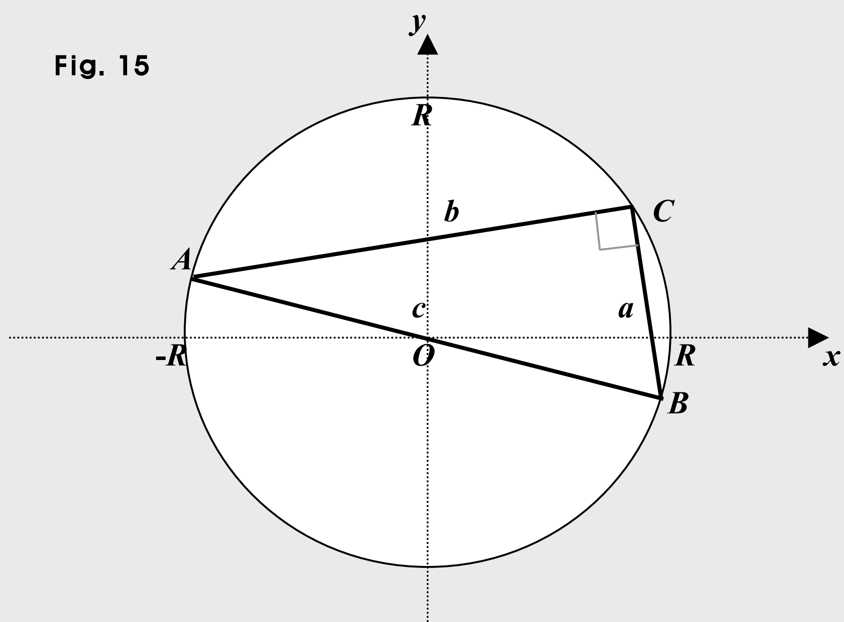
Thus, putting threads together, we get $\dfrac{\boldsymbol{a}}{\sin \boldsymbol{A}} = \dfrac{\boldsymbol{b}}{\sin \boldsymbol{B}} = \dfrac{\boldsymbol{c}}{\sin \boldsymbol{C}} = 2\boldsymbol{R}$.

Let's next, move on to the last case where $\boldsymbol{C} = \dfrac{\pi}{2}$, which is 90°.

That is, the triangle $\boldsymbol{T}$ is a right triangle.

So putting in the $\boldsymbol{x}$-$\boldsymbol{y}$ plane the right triangle $\boldsymbol{T}$, along with its circumcircle $\boldsymbol{U}$, we can put it the way as follows.

Fig. 15



Then first, since $C = \frac{\pi}{2}$, we get $\mathbf{\sin C = \sin\frac{\pi}{2} = 1 \Rightarrow \sin C = 1}$.

And assuming the line segment $\overline{AB}$ passes through the center, we get $\overline{AB} = 2R$.

So we get $\mathbf{2R = c}$. And doing some algebra on it, we can get this:

$\mathbf{2R = c \Rightarrow 2R = \frac{c}{1} = \frac{c}{\sin C}}$ since we have $\mathbf{\sin C = 1}$. Thus, we can get $\mathbf{2R = \frac{c}{\sin C}}$.

Next, taking simply the sines of the angles A and B,
we get $\mathbf{\sin A = \frac{a}{c} \Rightarrow c = \frac{a}{\sin A}}$, and $\mathbf{\sin B = \frac{b}{c} \Rightarrow c = \frac{b}{\sin B}}$.

And we know $\mathbf{c = 2R}$. and $\mathbf{2R = \frac{c}{\sin C}}$.

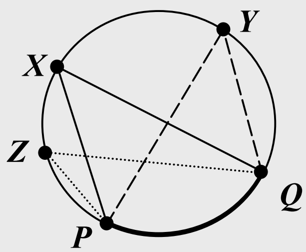
So we get $\mathbf{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R}$.

Let's next, move on to the proofs of the ideas we used covering the proof above.

To begin with, we have **Idea 1** as follows.

Idea 2: Suppose ***X***, ***P*** and ***Q*** are three points in a circle, and ***X*** is outside the arc ***PQ***.

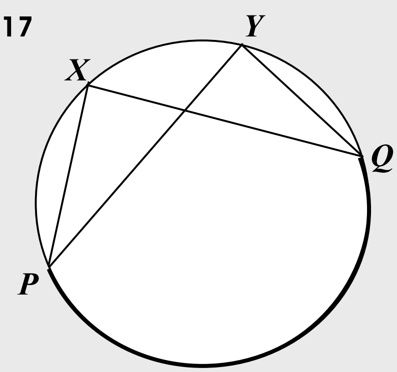
Fig. 16



Then, no matter where the point ***X*** may be, the angle ***PXQ*** is the same. If for instance, therefore, ***Y*** and ***Z*** are two other points outside the arc, but are in the circle, we get $\angle PZQ = \angle PXQ = \angle PYQ$.

So now, to see how it is the case, let's take a closer look.

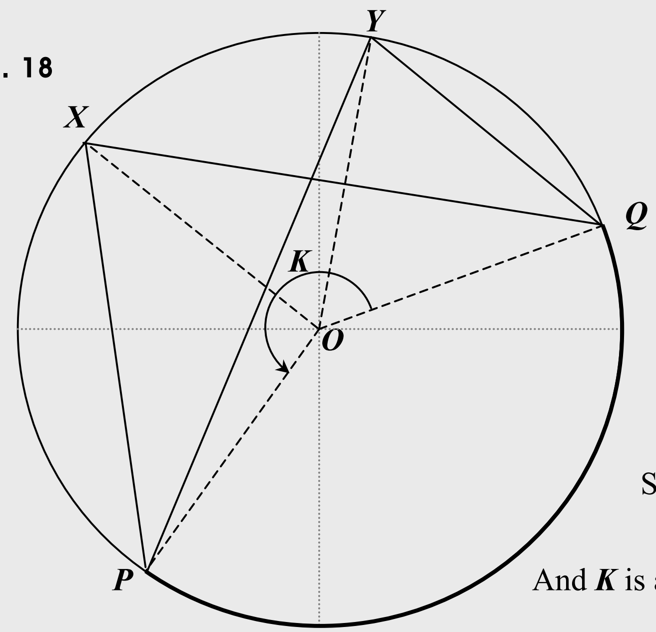
Fig. 17



Where then do we want to begin?

Putting some radii in it, we can see better what we can do about it. And also, it's a good idea to add labels as necessary. So doing the suggestions above, we can get this:

Fig. 18



So $\boldsymbol{O}$ is the center of the circle.

And $\boldsymbol{K}$ is an angle between $\boldsymbol{OQ}$ and $\boldsymbol{OP}$.

Then, first, from the figure above, we can readily get three facts as follows:

Fact A:

$\Delta \boldsymbol{QOY}$, $\Delta \boldsymbol{POY}$, $\Delta \boldsymbol{QOX}$, and $\Delta \boldsymbol{POX}$ are isosceles triangles, since $\boldsymbol{OX}$, $\boldsymbol{OY}$, $\boldsymbol{OP}$, and $\boldsymbol{OQ}$ are the radii, and thus, are the same in magnitude.

Fact B: The sum of all the angles in $\Delta \boldsymbol{QOY}$ and $\Delta \boldsymbol{POY}$ is 2π, which is the sum of all the angles in $\Delta \boldsymbol{QOX}$ and $\Delta \boldsymbol{POX}$, for the sum of all the three angles in a triangle is π.

And **Fact C**:

$\angle \boldsymbol{QOY} + \angle \boldsymbol{POY} = \angle \boldsymbol{QOX} + \angle \boldsymbol{POX} = \angle \boldsymbol{K}$.

Then first, from **Fact B** and **Fact C**, we can get another fact, **Fact D** as follows.

Fact D: $\angle \boldsymbol{OPY} + \angle \boldsymbol{PYQ} + \angle \boldsymbol{YQO} = \angle \boldsymbol{OPX} + \angle \boldsymbol{PXQ} + \angle \boldsymbol{XQO}$.

That's because $\angle \boldsymbol{OPY} + \angle \boldsymbol{PYQ} + \angle \boldsymbol{YQO} = \mathbf{2\pi} - \boldsymbol{K} = \angle \boldsymbol{OPX} + \angle \boldsymbol{PXQ} + \angle \boldsymbol{XQO}$.

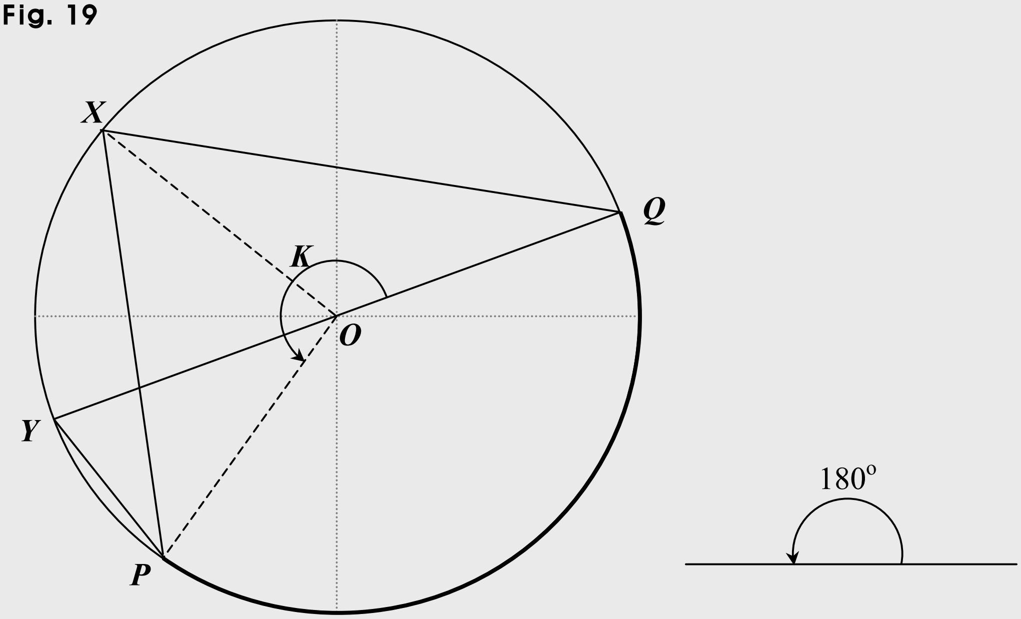
Next, from **Fact A**, we can get another fact, **Fact E** as follows.

Fact E: $\angle YQO + \angle OPY = \angle PYQ$, and $\angle XQO + \angle OPX = \angle PXQ$. It's because $\angle YQO = \angle QYO$, $\angle OPY = \angle OYP$, $\angle XQO = \angle QXO$, and $\angle OPX = \angle OXP$.

So next, putting **Fact E** into **Fact D**, we can get $2\angle PYQ = 2\angle PXQ$.
And therefore, we get $\angle PYQ = \angle PXQ$. So the proof is done.

What if though, the point Y is put the way as follows?

Fig. 19



That is, QY is not just a line segment connecting Q and Y but a diameter, too, in the circle. Then, first, from the figure above, we can get readily three facts as follows.

Fact F: ΔPOY, ΔQOX, and ΔPOX are isosceles triangles,
since OX, OY, OP, and OQ are the radii, and thus, share the same length.

Fact G: The sum of the angle in the line segment YQ and all the angles in ΔPOY is 2π, which is the sum of all the angles in ΔQOX and ΔPOX, because the angle in a line segment YQ is 180°, which is π.

So we get **Fact H** as follows.

Fact H: $\angle QOY + \angle POY = 180^\circ + \angle POY = \angle QOX + \angle POX$.

Then first, from **Fact G** and **Fact H**, we can get another fact as follows.

Fact I: $\angle OPY + \angle PYO = \angle OPX + \angle PXQ + \angle XQO$. How?

First, from **Fact G**, we can get $2\pi - (\angle OPX + \angle PXQ + \angle XQO) = K$.
Also, we can get $2\pi - (\angle OPY + \angle PYO) = K$.

So we get $2\pi - (\angle OPY + \angle PYO) = 2\pi - (\angle OPX + \angle PXQ + \angle XQO)$.

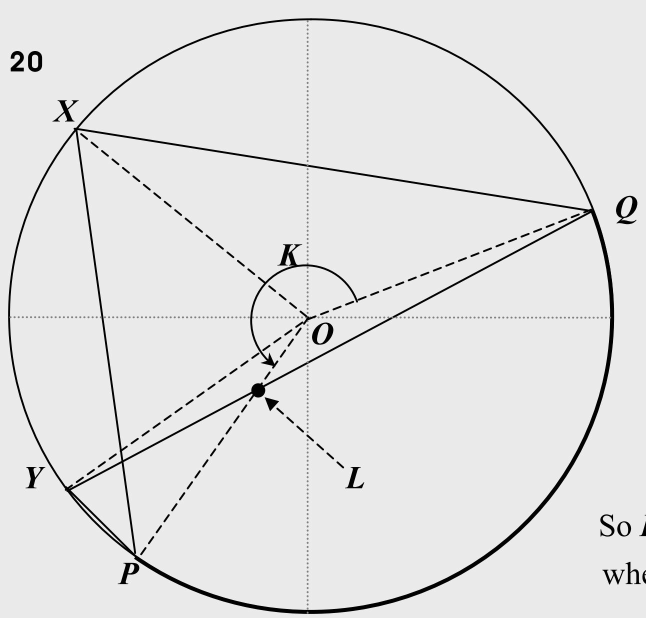
Thus, we get $\angle OPY + \angle PYO = \angle OPX + \angle PXQ + \angle XQO$.

Next, from **Fact F**, we can get another fact as follows.

Fact J: $\angle OPY = \angle PYO$, and $\angle XQO + \angle OPX = \angle PXQ$.

So next, putting **Fact J** into **Fact I**, we get $2\angle PYO = 2\angle PXQ \Rightarrow \angle PYO = \angle PXQ$.
Thus, the proof is done.
What if this time though, the point Y is put the way as follows?

Fig. 20



So L is the point where YQ meets OP.

Then, to begin with, from the figure above, we can readily get four facts as follows.

Fact K: $\Delta \boldsymbol{QOY}$, $\Delta \boldsymbol{POY}$, $\Delta \boldsymbol{QOX}$, and $\Delta \boldsymbol{POX}$ are isosceles triangles, since $\boldsymbol{OX}$, $\boldsymbol{OY}$, $\boldsymbol{OP}$, and $\boldsymbol{OQ}$ are the radii, and thus, share the same length.

Fact L: The sum of all the angles in $\Delta \boldsymbol{QOX}$ and $\Delta \boldsymbol{POX}$ is 2π, since the sum of all the three angles in a triangle is π.

Fact M: We have $\angle \boldsymbol{PYQ} + \angle \boldsymbol{OPY} = \angle \boldsymbol{YLO}$.

And **Fact N**: We have $\angle \boldsymbol{LOQ} + \angle \boldsymbol{OQL} = \angle \boldsymbol{YLO}$.

Then first, from **Fact K**, we get $\angle \boldsymbol{OYP} = \angle \boldsymbol{OPY}$, since $\Delta \boldsymbol{POY}$ is an isosceles triangle.

So we get $\angle \boldsymbol{OPY} = \angle \boldsymbol{PYQ} + \angle \boldsymbol{OYL}$. (1)

Thus, from **Fact M** and (1) we get this:

$\angle \boldsymbol{YLO} = \angle \boldsymbol{PYQ} + \angle \boldsymbol{OPY} = \angle \boldsymbol{PYQ} + \angle \boldsymbol{PYQ} + \angle \boldsymbol{OYL} = \boldsymbol{2}\angle \boldsymbol{PYQ} + \angle \boldsymbol{OYL}$

$\Rightarrow \angle \boldsymbol{YLO} = \boldsymbol{2}\angle \boldsymbol{PYQ} + \angle \boldsymbol{OYL}$.

So next, from **Fact N**, we get $\boldsymbol{2}\angle \boldsymbol{PYQ} + \angle \boldsymbol{OYL} = \angle \boldsymbol{LOQ} + \angle \boldsymbol{OQL}$. (2)

Also, from **Fact K**, we get $\angle \boldsymbol{OYL} = \angle \boldsymbol{OQL}$ (3), since $\Delta \boldsymbol{QOY}$ is an isosceles triangle.

So from (2) and (3), we get $\boldsymbol{2}\angle \boldsymbol{PYQ} + \angle \boldsymbol{OYL} = \angle \boldsymbol{LOQ} + \angle \boldsymbol{OYL}$.
Thus, we get another fact as follows.

Fact O: $\boldsymbol{2}\angle \boldsymbol{PYQ} = \angle \boldsymbol{LOQ}$.

And also, from **Fact L**, we can get another fact as follows.

Fact P: $\angle \boldsymbol{LOQ} = \angle \boldsymbol{OPX} + \angle \boldsymbol{PXQ} + \angle \boldsymbol{XQO}$.   How though?

First, we know the sum of all the angles in $\Delta \boldsymbol{QOX}$ and $\Delta \boldsymbol{POX}$ is 2π.

So we get this:
$\mathbf{2\pi - (\angle \boldsymbol{OPX} + \angle \boldsymbol{PXQ} + \angle \boldsymbol{XQO}) = \boldsymbol{K} \Rightarrow 2\pi - \boldsymbol{K} = \angle \boldsymbol{OPX} + \angle \boldsymbol{PXQ} + \angle \boldsymbol{XQO}}$.

And also, we have $\mathbf{2\pi - \boldsymbol{K} = \angle \boldsymbol{LOQ}}$.

So we get $\angle \boldsymbol{LOQ} = \angle \boldsymbol{OPX} + \angle \boldsymbol{PXQ} + \angle \boldsymbol{XQO}$, which is **Fact P**.

Next, from **Fact K**, we get $\angle \boldsymbol{OPX} = \angle \boldsymbol{OXP}$, and $\angle \boldsymbol{XQO} = \angle \boldsymbol{OXQ}$.

And we have $\angle \boldsymbol{OXP} + \angle \boldsymbol{OXQ} = \angle \boldsymbol{PXQ}$, so we get $\angle \boldsymbol{OPX} + \angle \boldsymbol{XQO} = \angle \boldsymbol{PXQ}$.

Thus, we get $\angle \boldsymbol{OPX} + \angle \boldsymbol{PXQ} + \angle \boldsymbol{XQO} = \mathbf{2}\angle \boldsymbol{PXQ}$. (4)

And we have **Fact P**: $\angle \boldsymbol{LOQ} = \angle \boldsymbol{OPX} + \angle \boldsymbol{PXQ} + \angle \boldsymbol{XQO}$.

So from (4), we get $\angle \boldsymbol{LOQ} = \mathbf{2}\angle \boldsymbol{PXQ}$.

And also, we have **Fact O**: $\mathbf{2}\angle \boldsymbol{PYQ} = \angle \boldsymbol{LOQ}$.

So we get $\mathbf{2}\angle \boldsymbol{PYQ} = \mathbf{2}\angle \boldsymbol{PXQ}$.

Thus, we get $\angle \boldsymbol{PYQ} = \angle \boldsymbol{PXQ}$.

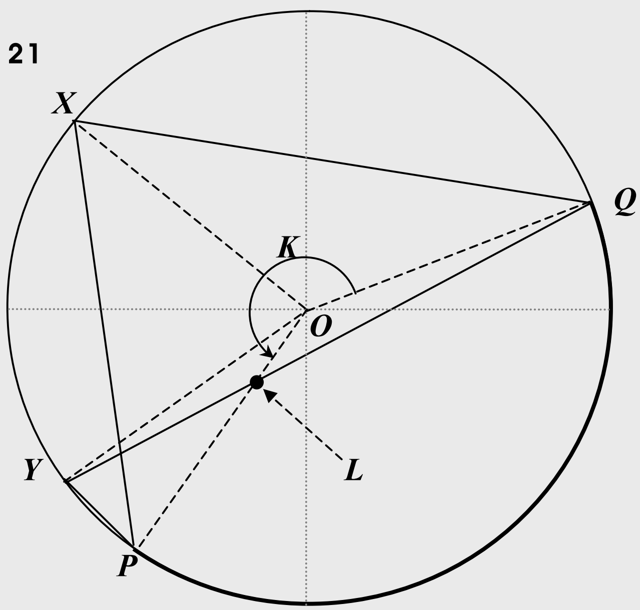
So the proof is done.

Besides, in the processes above, we have found an important fact we can use solving many problems. And the fact is as follows.

First, we know in Fig. 21 below, the point ***O*** is the center of the circle.
So we call the angle ***POQ*** a central angle.
More specifically, $\angle$***POQ*** is called the central angle for the arc ***PQ***.

And also, we can call the angle ***PXQ*** an angle of circumference.
More specifically, $\angle$***PXQ*** can be called the angle of circumference for the arc ***PQ***.

Fig. 21

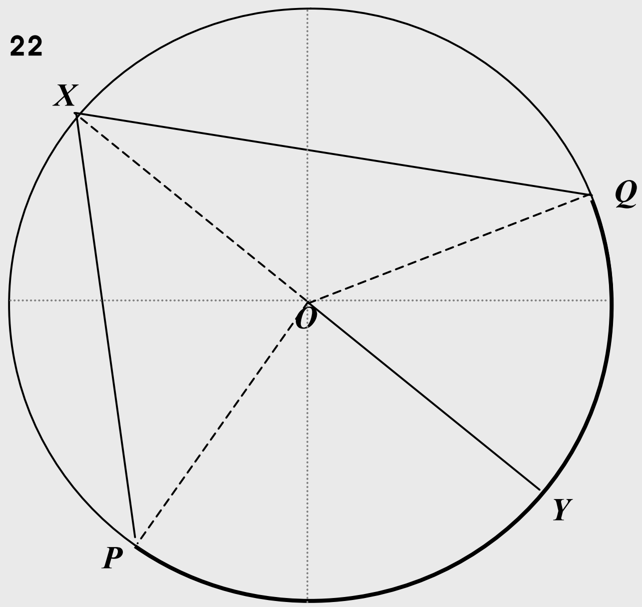


Now, the important fact is that the angle of circumference for the arc ***PQ*** is half the central angle for the arc ***PQ***.

So we have $\angle LOQ = \angle POQ = 2\angle PXQ$, and $\angle POQ = 2\angle PYQ$.

And we can simply prove the fact above the way as follows.

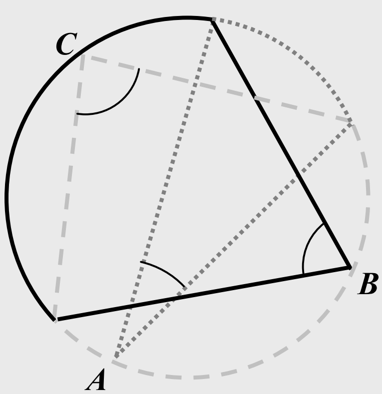
Fig. 22



First, we know the two triangles $\Delta \boldsymbol{PXO}$ and $\Delta \boldsymbol{QXO}$ are isosceles triangles.
So we get $\angle \boldsymbol{POY} = \angle \boldsymbol{OPX} + \angle \boldsymbol{OXP} = \mathbf{2}\angle \boldsymbol{OXP}$.
Also, we get $\angle \boldsymbol{QOY} = \angle \boldsymbol{OQX} + \angle \boldsymbol{OXQ} = \mathbf{2}\angle \boldsymbol{OXQ}$.
Thus, we get $\angle \boldsymbol{POQ} = \angle \boldsymbol{POY} + \angle \boldsymbol{QOY} = \mathbf{2}\angle \boldsymbol{OXP} + \mathbf{2}\angle \boldsymbol{OXQ} = \mathbf{2}(\angle \boldsymbol{OXP} + \angle \boldsymbol{OXQ})$.
And we have $\angle \boldsymbol{PXQ} = \angle \boldsymbol{OXP} + \angle \boldsymbol{OXQ}$. So we get $\angle \boldsymbol{POQ} = \mathbf{2}\angle \boldsymbol{PXQ}$.

And we can notice that if a circle is partitioned into many arcs, the sum of all the angles of circumference for all the arcs is 180°.

Fig. 23

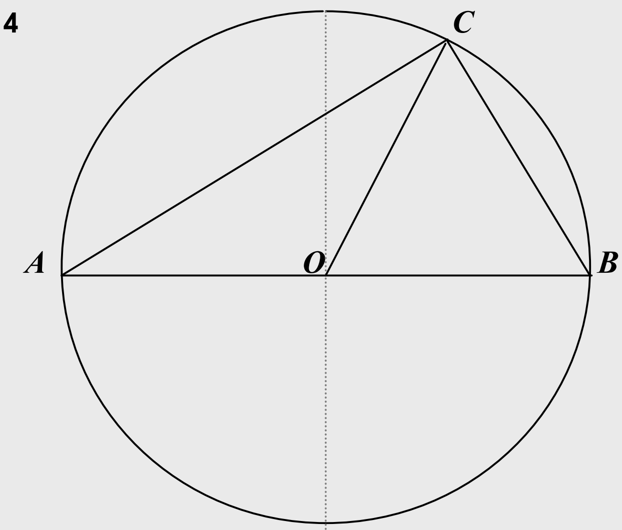


So the circle on the left is partitioned into 3 arcs.
One is solid, another is dotted, and the other is dashed.
And the sum of the three angles $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ is 180°.
That is, we get $\boldsymbol{A} + \boldsymbol{B} + \boldsymbol{C} = \pi$.

And let's now move on to the next fact. And the fact is saying that the angle of circumference for a half circle is a right angle, that is, 90°.

So assuming $\boldsymbol{O}$ is the center of a circle, $\boldsymbol{AB}$ is a diameter, and $\boldsymbol{C}$ is an angle of circumference for a half circle, that is, the arc $\boldsymbol{AB}$, we can put them the way as follows.

Fig. 24



Then, we can begin with the two facts as follows.

Fact Q: $\Delta \boldsymbol{AOC}$ and $\Delta \boldsymbol{COB}$ are isosceles triangles,
since $\boldsymbol{OA}$, $\boldsymbol{OC}$, and $\boldsymbol{OB}$ are the radii.

Fact R: $\boldsymbol{\angle CAO + \angle ACO = \angle COB}$, $\boldsymbol{\angle BCO + \angle CBO = \angle COA}$,
and $\boldsymbol{\angle COB + \angle COA} = \pi$.

Then first, from **Fact Q**, we can get $\boldsymbol{\angle CAO = \angle ACO}$ (1), and $\boldsymbol{\angle BCO = \angle CBO}$. (2)

So from (1), we get $\boldsymbol{\angle CAO + \angle ACO = \angle COB \Rightarrow \angle COB = 2\angle ACO}$.

Also from (2), we get $\boldsymbol{\angle BCO + \angle CBO = \angle COA \Rightarrow \angle COA = 2\angle BCO}$.

So we get $\boldsymbol{\angle COB + \angle COA = 2\angle ACO + 2\angle BCO = 2(\angle ACO + \angle BCO) = 2\angle ACB} = \pi$.

Thus, we get $\boldsymbol{\angle ACB} = \dfrac{\pi}{2}$, which is 90°.

How though, do we get $\boldsymbol{\angle CAO + \angle ACO = \angle COB}$?

We know that angle in the line segment $\boldsymbol{AB}$ is π.
So first, we get $\angle \boldsymbol{COA} + \angle \boldsymbol{COB} = \pi$.

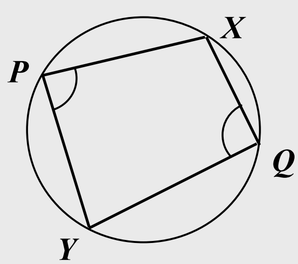
Next, we have $\angle \boldsymbol{CAO} + \angle \boldsymbol{ACO} + \angle \boldsymbol{COA} = \pi$, since the sum of all the three angles in a triangle is π. So we get $\angle \boldsymbol{CAO} + \angle \boldsymbol{ACO} + \angle \boldsymbol{COA} = \angle \boldsymbol{COA} + \angle \boldsymbol{COB}$.

Thus, we get $\angle \boldsymbol{CAO} + \angle \boldsymbol{ACO} = \angle \boldsymbol{COB}$.

And moving on to the next fact, **Fact 4** covered earlier, we have this:

Fact 4: Assuming $\boldsymbol{P}$ and $\boldsymbol{Q}$ are two vertex-angles facing each other in a quadrangle that has a circumcircle, we get $\boldsymbol{P} + \boldsymbol{Q} = \pi$.

Fig. 25

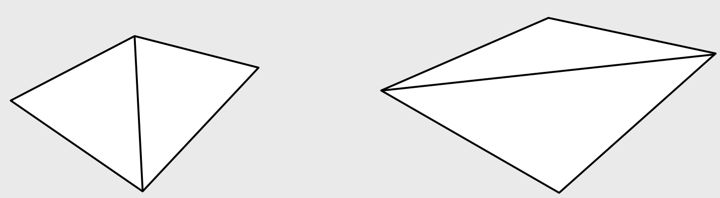


So we get $\boldsymbol{X} + \boldsymbol{Y} = \pi$, too.

And in fact, we have $\boldsymbol{X} + \boldsymbol{Y} + \boldsymbol{P} + \boldsymbol{Q} = 2\pi$.

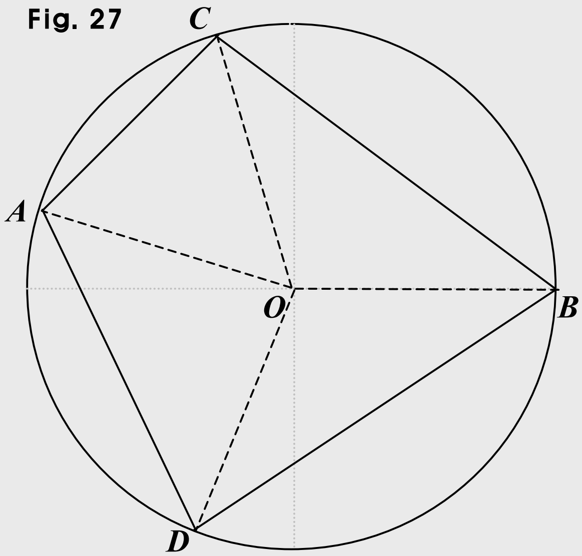
That's because a quadrangle can be made of two triangles:

Fig. 26



And we want to get the proof of the fact that $\boldsymbol{P} + \boldsymbol{Q} = \pi$. So let's now take a closer look.

Fig. 27



To begin with, we can partition a quadrangle into triangles.

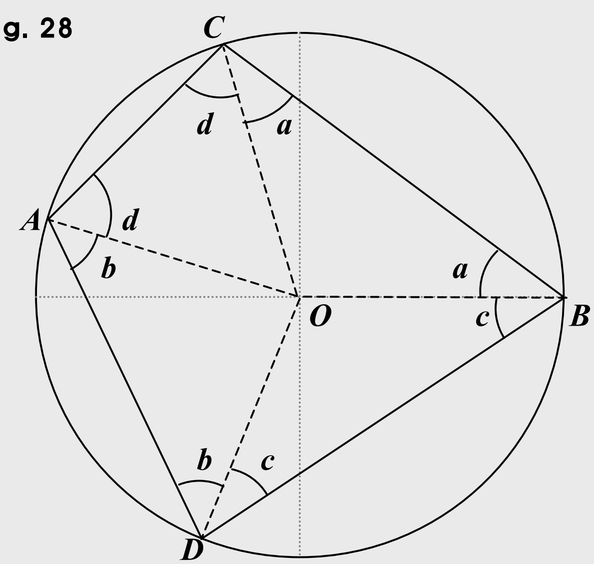
And we are working with a circle, too.

And what determines a circle is the radius and the center.
So we may want to try using the radii and the center partitioning the quadrangle.

Then first, we can notice that the quadrangle is made of isosceles triangles, which are ΔOBC, ΔOCA, ΔOAD, and ΔODB.

In an isosceles triangle, at least two angles are the same. So we can use such a property. That is, labeling or specifying those angles, we can see better what we can do.

Fig. 28



We can notice that the angle ***C*** is made of ***a*** and ***d***, and that the angle ***D*** is made of ***b*** and ***c***, which means, the two angles altogether are made of ***a***, ***b***, ***c***, and ***d***, which are all kinds of angles there.

And what we want to show is that in a quadrangle with its circumcircle, the sum of the two angles facing each other is π.

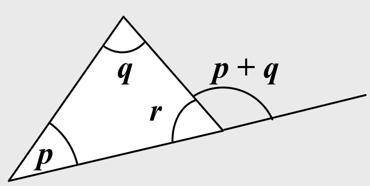
So we can reasonably expect that if we can put together all the angles around the center of the circle, that is, the point ***O***, the sum of all the angles can be 2π.

How then, can we put them together around the center?

We can use the fact that all the radii radiate from the center, and the fact that all the equal sides in all the isosceles triangles are the radii.
Also, we have a fact that a sum of two internal angles is the same as an external angle in a triangle.

More specifically, as in Fig. 29 below, in a triangle, the sum of two internal angles is equal to the external angle supplement to the other internal angle.

Fig. 29

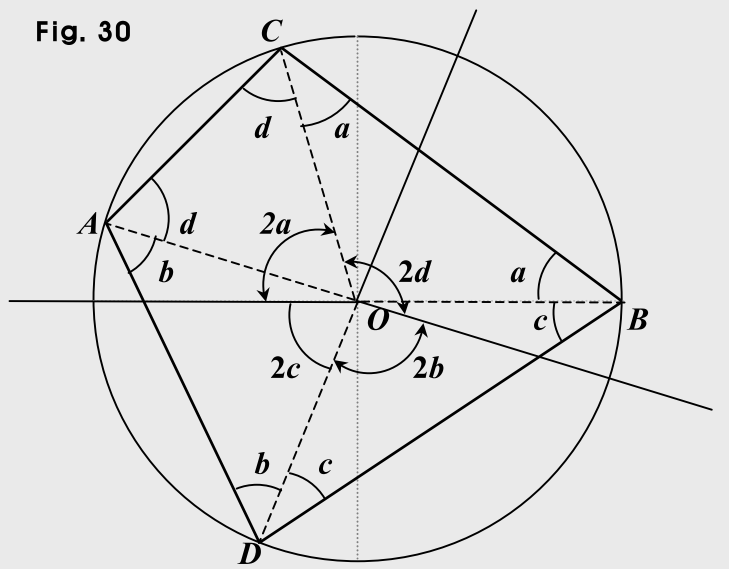


What is a supplement angle?

In Fig. 29 above, the sum $\boldsymbol{p + q}$ is supplement to $\boldsymbol{r}$, and vice versa, so $\boldsymbol{r}$ is supplement to $\boldsymbol{p + q}$, too, because $\boldsymbol{p + q + r = \pi}$.

So using the fact above, and extending one of the two equal sides in each of the isosceles triangles, we can put together all the angles around the center.

Fig. 30



Thus, we get $\mathbf{2a + 2b + 2c + 2d = 2(a + b + c + d) = 2\pi}$.

So we get $\boldsymbol{a + b + c + d} = \pi$.

We have $\boldsymbol{A = b + d}$, $\boldsymbol{B = a + c}$, $\boldsymbol{C = a + d}$, and $\boldsymbol{D = b + c}$.

Thus, we get $\boldsymbol{A + B = a + b + c + d} = \pi$, and $\boldsymbol{C + D = a + b + c + d} = \pi$.

And finally, the other fact is in fact, a trig identity where $\mathbf{\sin\,(\pi - \theta) = \sin\,\theta}$.

In other words, assuming $\theta_1 + \theta_2 = \pi$, we get $\mathbf{\sin\,\theta_1 = \sin\,\theta_2}$.
So referring to the quadrangle in Fig. 30 above, we get $\mathbf{\sin}\,\boldsymbol{A} = \mathbf{\sin}\,\boldsymbol{B}$, since $\boldsymbol{A + B} = \pi$.
Let's now produce the proof.

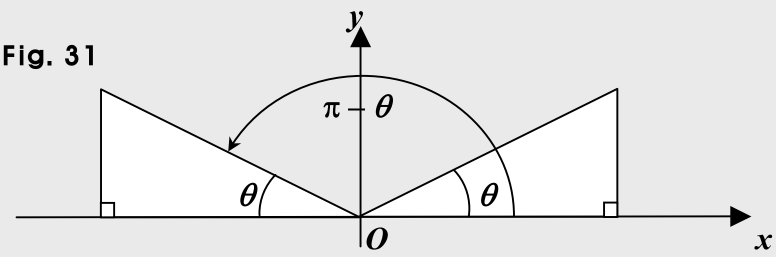
To produce the proof, we can use another trig identity, which is as follows.

$$\mathbf{\sin\,(a \pm b) = \sin\,a\,\cos\,b \pm \cos\,a\,\sin\,b.}$$

First, we have $\sin \pi = 0$ and $\cos \pi = -1$.
So we get $\sin(\pi - \theta) = \sin \pi \cos \theta - \cos \pi \sin \theta = 0 \cdot \cos \theta - (-1) \cdot \sin \theta = \sin \theta$.

And we can do the proof graphically, too.
We are now doing coordinate trigonometry.

Fig. 31



So to begin with, we can see that in each triangle, the opposite is the same.

Next, in coordinate trigonometry, using the triangle on the left, we can get $\sin(\pi - \theta)$.
And also, using the triangle on the right, we can get $\sin \theta$.

We know the sine is the ratio of the opposite to the hypotenuse, that is, the opposite over the hypotenuse. And both triangles have the same hypotenuse and the same opposite.

So we get $\sin(\pi - \theta) = \sin \theta$.
And we have $(\pi - \theta) + \theta = \pi$.
Thus, if $A + B = \pi$, we get $\sin A = \sin B$.

And using the sine rule, we can come up with an expression showing the ratio between the three sides in terms of the sines of all the three angles.

That is to say that since we have $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$,

we get $a : b : c = \sin A : \sin B : \sin C$. And of course, we can get these, too:

$a : b = \sin A : \sin B$, $b : c = \sin B : \sin C$, and $c : a = \sin C : \sin A$.

That is to say that we have this: $\frac{\boldsymbol{a}}{\boldsymbol{b}} = \frac{\sin \boldsymbol{A}}{\sin \boldsymbol{B}}$, $\frac{\boldsymbol{b}}{\boldsymbol{c}} = \frac{\sin \boldsymbol{B}}{\sin \boldsymbol{C}}$, and $\frac{\boldsymbol{c}}{\boldsymbol{a}} = \frac{\sin \boldsymbol{C}}{\sin \boldsymbol{A}}$. Why though?

We can get $\boldsymbol{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R \Rightarrow \frac{a}{\sin A} = 2R \Rightarrow a = 2R \sin A}$.

And by the same token, we can get this:
$\boldsymbol{\frac{b}{\sin B} = 2R \Rightarrow b = 2R \sin B}$, and $\boldsymbol{\frac{c}{\sin C} = 2R \Rightarrow c = 2R \sin C}$.

So we get $\boldsymbol{a : b : c = 2R \sin A : 2R \sin B : 2R \sin C = \sin A : \sin B : \sin C}$.

Thus, we get $\boldsymbol{a : b : c = \sin A : \sin B : \sin C}$.

So we can say that in a triangle, the ratio between the sides is the same as the ratio between the sines of the angles facing the sides.

B. The Cosine Rule

The cosine is a trig ratio, and the cosine of an angle is the ratio of the adjacent to the hypotenuse in a right triangle, where the hypotenuse and the adjacent makes the angle.

So in short, the cosine is the adjacent over the hypotenuse in a right triangle.
What then is the cosine rule?

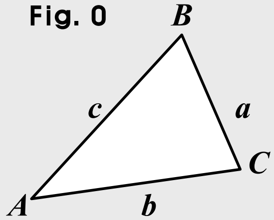
It is also, often called the law of cosines, and we can call it the cosine formula, too.

And as stated above, we get the trig ratio called the cosine from a right triangle. Using the cosine rule though, we can apply it to any triangle, and thus, not a right triangle only.

And by means of the cosine rule, we can express a particular side in a triangle in terms of the other two sides and the cosine of the angle facing the particular side.

So for instance, assuming the three angles are $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$, the three sides are $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$, and expressing the side $\boldsymbol{a}$ using the cosine rule, we can put it the way as follows.

Fig. 0



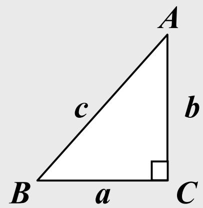
$$\boldsymbol{a^2 = b^2 + c^2 - 2bc \cos A}.$$

And the same is true for the other sides, too:

$$\boldsymbol{b^2 = c^2 + a^2 - 2ca \cos B}.$$
$$\boldsymbol{c^2 = a^2 + b^2 - 2ab \cos C}.$$

And assuming in particular, the angle $\boldsymbol{C}$ is 90°, we get $\boldsymbol{c^2 = a^2 + b^2}$.

Fig. 1



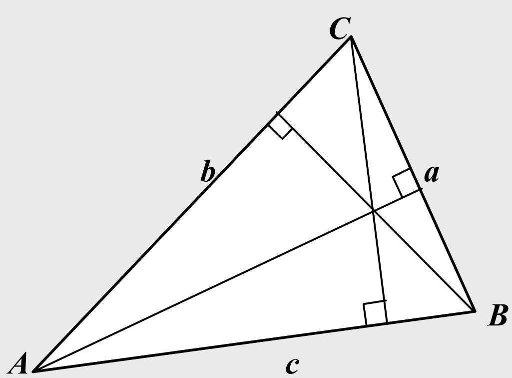
And of course, the same is true for the other angles, too.

So we get $\boldsymbol{a^2 = b^2 + c^2}$ if $\boldsymbol{A}$ is $\frac{\pi}{2}$, and $\boldsymbol{b^2 = c^2 + a^2}$ if $\boldsymbol{B}$ is 90°.

As in the case of the sine rule, in the cosine rule also, one of the three angles can be obtuse as well as acute and 90°.
That is, we can have either of three cases as follows. $\boldsymbol{0 < A < \pi}$, $\boldsymbol{0 < B < \pi}$, and $\boldsymbol{0 < C < \pi}$.

Let's now get the proof.
The proof is rather easier than the one for the sine rule.
So to begin with, assuming $\boldsymbol{T}$ is an *acute* triangle, we can put it the way as follows.

Fig. 2



And we know multiplying the hypotenuse by the cosine, we get the adjacent.

That is, $\textbf{hypotenuse} \cdot \textbf{cos}\ \boldsymbol{\theta} = \textbf{hypotenuse} \cdot \dfrac{\textbf{adjacent}}{\textbf{hypotenuse}} = \textbf{adjacent}$.

So first, we can get these:

$$a = b \cos C + c \cos B \quad (0)$$
$$b = c \cos A + a \cos C \quad (1)$$
$$c = a \cos B + b \cos A \quad (2)$$

Next, multiplying (0) by a, we get $a^2 = ab \cos C + ac \cos B$ (3)

Multiplying (1) by b, we get $b^2 = bc \cos A + ab \cos C$ (4)

Multiplying (2) by c, we get $c^2 = ac \cos B + bc \cos A$ (5)

Next, adding together (4) and (5), we get this:

$b^2 + c^2 = bc \cos A + ab \cos C + ac \cos B + bc \cos A = ab \cos C + ac \cos B + 2bc \cos A$.

And we have $a^2 = ab \cos C + ac \cos B$ (3)

So we get $b^2 + c^2 = a^2 + 2bc \cos A$. Thus, we get $a^2 = b^2 + c^2 - 2bc \cos A$.

Next, adding together (5) and (3), we get this:

$c^2 + a^2 = ac \cos B + bc \cos A + ab \cos C + ac \cos B = ab \cos C + bc \cos A + 2ac \cos B$.

And we have $b^2 = bc \cos A + ab \cos C$ (4)

So we get $c^2 + a^2 = b^2 + 2ac \cos B$. Thus, we get $b^2 = c^2 + a^2 - 2ca \cos B$.

Next, adding together (3) and (4), we get this:

$a^2 + b^2 = ab \cos C + ac \cos B + bc \cos A + ab \cos C = ac \cos B + bc \cos A + 2ab \cos C$.

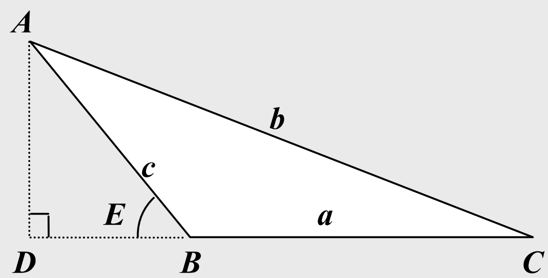
And we have $c^2 = ac \cos B + bc \cos A$ (5)

So we get $a^2 + b^2 = c^2 + 2ab \cos C$. Thus, we get $c^2 = a^2 + b^2 - 2ab \cos C$.

The proof is done.

Let's next, move on to the case where the triangle $\boldsymbol{T}$ is an *obtuse* triangle.
Then first, we can put it the way as follows.

Fig. 3

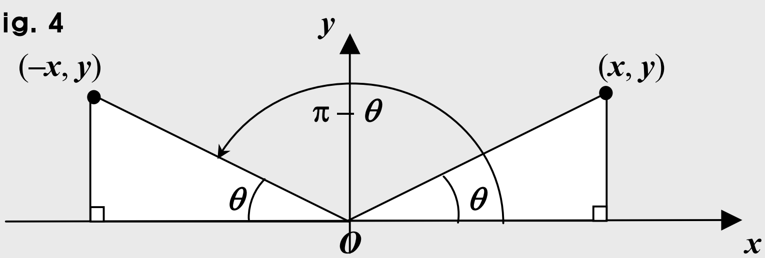


Then first, we get $\boldsymbol{a} = \overline{\boldsymbol{CD}} - \overline{\boldsymbol{BD}} = \boldsymbol{b}\ \mathbf{cos}\ \boldsymbol{C} - \boldsymbol{c}\ \mathbf{cos}\ \boldsymbol{E} = \boldsymbol{b}\ \mathbf{cos}\ \boldsymbol{C} - \boldsymbol{c}\ \mathbf{cos}\ (\pi - \boldsymbol{B})$. (6)

And we have a trig identity where $\mathbf{cos}\ (\pi - \theta) = -\mathbf{cos}\ \theta$.   How?

We can see how it is the case graphically as follows.
We are now doing coordinate trigonometry.

Fig. 4



So in Fig. 4 above, since both triangles are the same, assuming the hypotenuse of each triangle is 1, and is the vector turning on the $\boldsymbol{x}$-$\boldsymbol{y}$ plane about the origin $\boldsymbol{O}$,
we can get $\mathbf{cos}\ (\pi - \theta) = -\boldsymbol{x}$, and $\mathbf{cos}\ \theta = \boldsymbol{x}$, since the adjacent is the $\boldsymbol{x}$-coordinate at the terminal point of the vector.

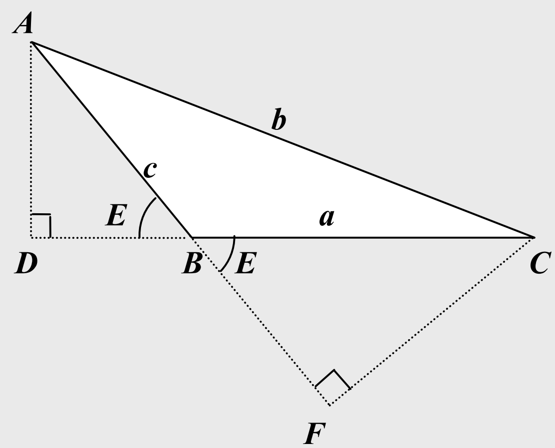
So we get $\mathbf{cos}\ (\pi - \theta) = -\mathbf{cos}\ \theta$.
And we have these, too $\mathbf{sin}\ (\pi - \theta) = \mathbf{sin}\ \theta$, and $\mathbf{tan}\ (\pi - \theta) = -\mathbf{tan}\ \theta$.

So now, from (6) above, we can get $\boldsymbol{a} = \boldsymbol{b}\ \mathbf{cos}\ \boldsymbol{C} - \boldsymbol{c}\ \mathbf{cos}\ (\pi - \boldsymbol{B}) = \boldsymbol{b}\ \mathbf{cos}\ \boldsymbol{C} + \boldsymbol{c}\ \mathbf{cos}\ \boldsymbol{B}$.

Thus, we get $\boldsymbol{a} = \boldsymbol{b}\ \mathbf{cos}\ \boldsymbol{C} + \boldsymbol{c}\ \mathbf{cos}\ \boldsymbol{B}$.

Let's next, move on to the case of $\boldsymbol{c}$ putting the triangle the way as follows.

Fig. 5



Then first, we get $\boldsymbol{c = \overline{AF} - \overline{BF} = b \cos A - a \cos E = b \cos A - a \cos (\pi - B)}$.

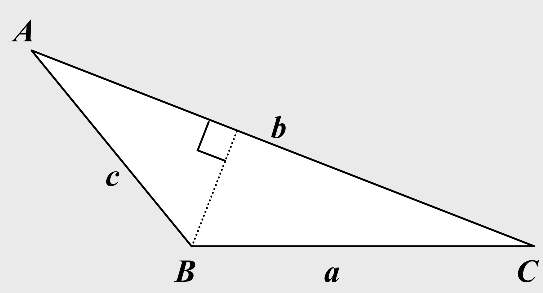
And we have a trig identity where $\boldsymbol{\cos(\pi - \theta) = -\cos\theta}$.

So we get $\boldsymbol{c = b \cos A - a \cos(\pi - B) = b \cos A + a \cos B}$.

Thus, we get $\boldsymbol{c = b \cos A + a \cos B}$.

Let's next, move on to the case of $\boldsymbol{b}$ putting the triangle the way as follows.

Fig. 6



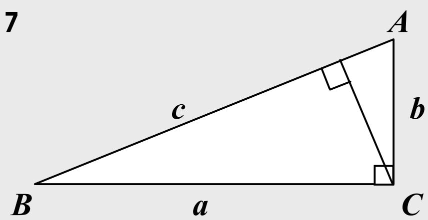
Then, first, we can simply get $\boldsymbol{b = c \cos A + a \cos C}$.

And doing the same in the case where $\boldsymbol{T}$ is an acute triangle, we can get this:

$\boldsymbol{a^2 = b^2 + c^2 - 2bc \cos A}$, $\quad \boldsymbol{b^2 = c^2 + a^2 - 2ca \cos B}$, and $\boldsymbol{c^2 = a^2 + b^2 - 2ab \cos C}$.

And let's next, move on to the case where the triangle $\boldsymbol{T}$ is a right triangle.
Then first, we can put $\boldsymbol{T}$ the way as follows.

Fig. 7



Then first, we get $\boldsymbol{a = c \cos B = 0 + c \cos B = b{\cdot}0 + c \cos B = b \cos C + c \cos B}$,

because $\boldsymbol{C} = \dfrac{\pi}{2}$, and $\boldsymbol{\cos \dfrac{\pi}{2} = 0}$. So we get $\boldsymbol{a = b \cos C + c \cos B}$.

Next, we get $\boldsymbol{b = c \cos A = c \cos B + 0 = c \cos A + a{\cdot}0 = c \cos A + a \cos C}$, because

$\boldsymbol{C} = \dfrac{\pi}{2}$, and $\boldsymbol{\cos \dfrac{\pi}{2} = 0}$. So we get $\boldsymbol{b = c \cos A + a \cos C}$.

Also, we can simply get $\boldsymbol{c = a \cos B + b \cos A}$.

And doing the same in the case where $\boldsymbol{T}$ is an acute triangle, we can get this:

$\boldsymbol{a^2 = b^2 + c^2 - 2bc \cos A}$, $\boldsymbol{b^2 = c^2 + a^2 - 2ca \cos B}$, and $\boldsymbol{c^2 = a^2 + b^2 - 2ab \cos C}$.

C. Areas of Triangles

To begin with, what is a triangle?

Tri means three, so literally, a triangle is made of three angles.

And next, a triangle is made of three sides, and is a polygon.
What is a polygon though?

It is a 2-D object closed, and is made of three or more line segments.

Usually though, we just call each line segment a side. And in basic math, we normally work with polygons said to be simple and convex. In each side in such a polygon, each endpoint is an endpoint of one of the other sides. And a line passing through such a polygon can cross up to two sides in the polygon, so it cannot cross three or more sides.

And among such polygons, we have triangles, quadrangles as rectangles, squares, rhombuses, parallelograms, and trapezoids, and others as pentagons, hexagons, etc.

So a triangle is a polygon made of three sides, in each of which, each endpoint is an endpoint of one of the other two sides. Thus, triangles are the simplest polygons.

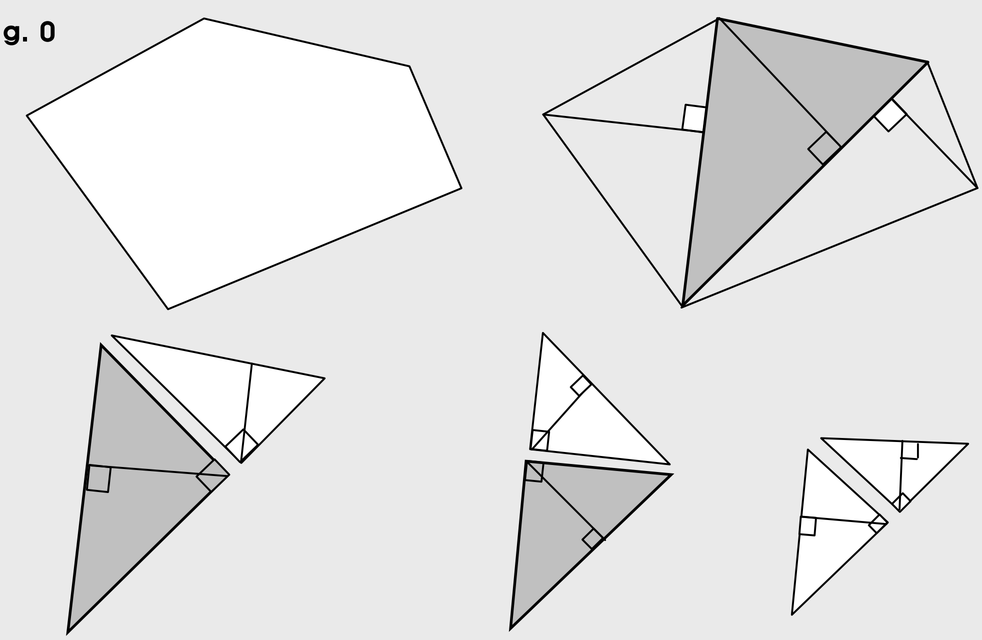
No matter what polygon it may be though, it can be made of triangles. We can put it this way, too: a polygon can be partitioned into triangles. Thus, triangles are not only the simplest but the most basic polygons, too.

And we can say also, a triangle is made of many other triangles, too, in each of which, one angle is a right angle, and thus, is 90°, which is $\frac{\pi}{2}$ in radians. So they are called right triangles.

So no matter what triangle it may be, it can be partitioned into two right triangles at a time. And of course, we can partition a right triangle into two other right triangles at a time, too.

Thus, it can be said that every polygon is made of right triangles.

Fig. 0

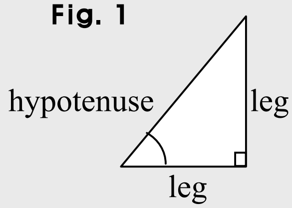


And we can keep partitioning a right triangle into two other right triangles.

So right triangles are fundamental triangles, and thus, are important. And they are very much so. It is often the case we can't do much without right triangles solving problems.

And we can do a lot using right triangles. Right triangles do much not only in geometry but in algebra, too. It is often vital that we use right triangles *right*. Thus, we want to know them very well, and use them very well, too.

Fig. 1

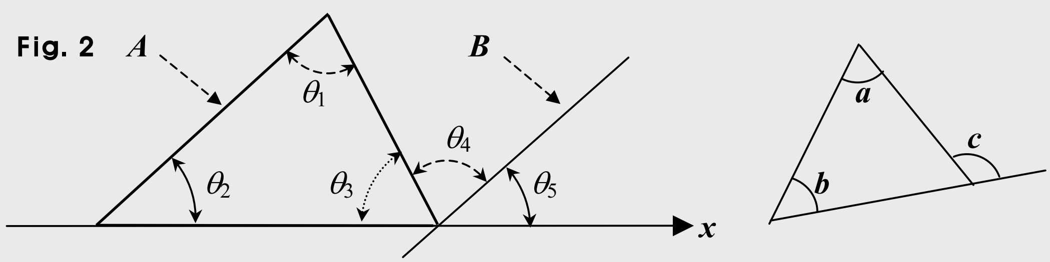


A right triangle is made of three sides, called a hypotenuse and two legs. And the two legs are perpendicular to each other, and thus, make 90°. So the hypotenuse is facing the angle 90°, and thus, is opposite to the right angle.

And the sum of the two angles adjacent to the hypotenuse is 90°, because the sum of all the three angles in a triangle is 180°. How is it 180° though?

To begin with, putting a triangle on a line called ***x***, we can put it the way below:

Fig. 2



And assuming next, the line segment ***A*** is parallel to the line ***B*** above, we can say that the two angles $\boldsymbol{\theta_2}$ and $\boldsymbol{\theta_5}$ are corresponding angles, and thus, are equal, and the two angles $\boldsymbol{\theta_1}$ and $\boldsymbol{\theta_4}$ are alternate angles, and thus, are equal, too.

So we get: $\boldsymbol{\theta_1 + \theta_2 + \theta_3 = \theta_3 + \theta_4 + \theta_5}$, which is 180°, that is, π.

Besides, we can say that in a triangle, the sum of two internal angles is the same as the external angle *supplementary* to the other internal angle. What do we mean by though, angles supplementary to each other?

If two angles are supplementary to each other, their sum is 180°. So in the figure above, we can say that: $\boldsymbol{\theta_3}$ is supplementary to the sum: $\boldsymbol{\theta_4 + \theta_5}$, because: $\boldsymbol{\theta_3 + \theta_4 + \theta_5 = 180^o}$.

Thus, in the figure above, we get: $\boldsymbol{a + b = c}$.
So if an angle is a supplement to another angle, the sum of the two angles is 180°.

By the way, if two angles are complementary to each other, their sum is 90^o, a right angle. So if in a triangle, one angle is a complement to another, the other is a right angle, since the sum of the three angles is 180^o. Thus, such a triangle is a right triangle.

And next, from a right triangle, we can get an important tool, which is the distance formula, often called the Pythagorean theorem, too. What distance then is it?

It is the distance between two points, and the distance is the length of the line segment connecting the two points. What two points though?

The two points can be in a 2-D space called a plane as the ***x-y*** plane, or in a 3-D space as the ***x-y-z*** space, where $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$ axes are perpendicular to each other.

So for instance, assuming first, $\boldsymbol{d}$ is the distance between two points $\boldsymbol{(x_1, y_1)}$ and $\boldsymbol{(x_2, y_2)}$, and using the theorem, we get this:

$$\boldsymbol{d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2},$$

which equals $\boldsymbol{(x_1 - x_2)^2 + (y_1 - y_2)^2}$, too, of course.

And next, assuming $\boldsymbol{D}$ is the distance between two points $\boldsymbol{(x_1, y_1, z_1)}$ and $\boldsymbol{(x_2, y_2, z_2)}$, which are in a 3-D space, of course, and using the theorem again, we get this:

$$\boldsymbol{D^2 = d^2 + (z_2 - z_1)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

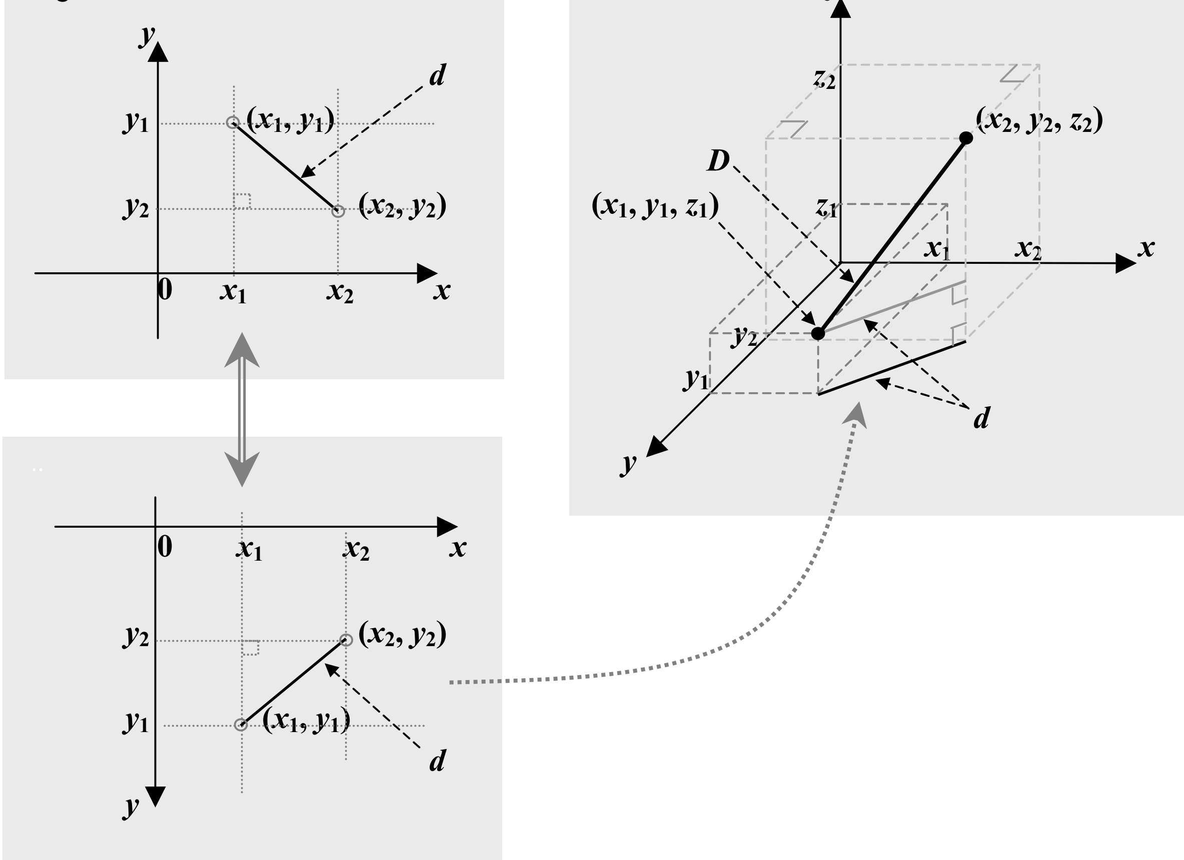
And of course, given the two points $\boldsymbol{(x_1, y_1, z_1)}$ and $\boldsymbol{(x_2, y_2, z_2)}$, we can get the distance $\boldsymbol{D}$ at once the way as follows, too.

$$\boldsymbol{D^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2},$$

which equals $\boldsymbol{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$.

And putting in graphs all the distance ideas above, we can put them the way as follows

Fig. 3



What then abut the area of a triangle?

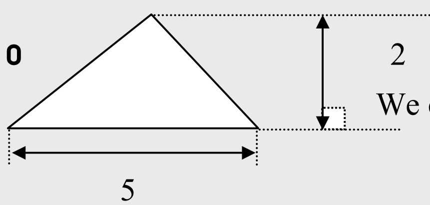
It seems quite easy to find the area of a triangle.   How do we find it though?

We can find it finding the area of a parallelogram, and then, taking half the area.

So the area of the parallelogram is of course, twice the area of the triangle we want to get. And finding the area of a parallelogram, we take the product of the length and the width or the height.

So let's now, for instance, find the area of a triangle as follows.

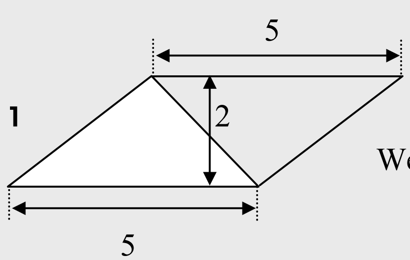
Fig. 0



We can call 5 the base, and call 2 the height.

Then first, we make a parallelogram twice as big as the triangle above. And making such a parallelogram, we add to the triangle the same triangle the way as follows.

Fig. 1

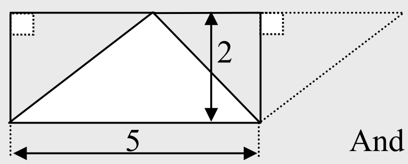


We can call 5 the length, and call 2 the width.

And taking the area of the parallelogram above, we simply get $5 \cdot 2 = 10$.   Why though?

Cutting either triangle into two right triangles, and then, moving one the way below, we get a rectangle that has the same area the parallelogram has.

Fig. 2



And the area of the rectangle is: 5·2.

So the area of the triangle is $\frac{5 \cdot 2}{2} = 5$.

Thus, taking the area of a triangle, we just take half the product of its base and height.

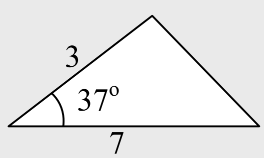
What if though, we don't know the base or the height?

We can't get the area using the idea above.
If we know however, two sides and an angle, we can still get the area.
What sides and what angle though?

The angle known is the angle facing the side unknown.
That is to say that the angle is between the two sides known.

So for instance, we can get the area of the triangle as follows.

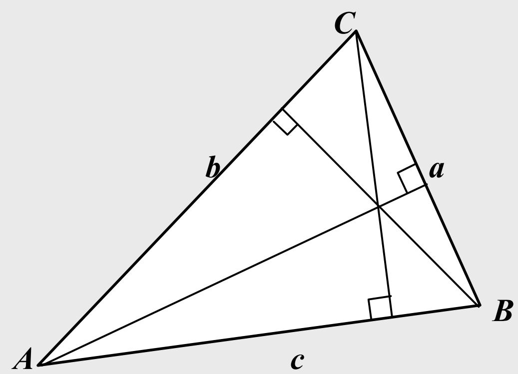
Fig. 3



And the area is $\frac{3 \cdot 7 \cdot \sin 37^{\circ}}{2} = \frac{21 \cdot 0.6018...}{2} \approx \frac{12.6381}{2} = 6.31905.$

Thus, assuming $\boldsymbol{S}$ is the area of the triangle in Fig. 4, we can get $\boldsymbol{S}$ the way as follows.

Fig. 4



Then, we get $\boldsymbol{S} = \dfrac{\boldsymbol{ab}\sin \boldsymbol{C}}{2} = \dfrac{\boldsymbol{bc}\sin \boldsymbol{A}}{2} = \dfrac{\boldsymbol{ca}\sin \boldsymbol{B}}{2}$. How?

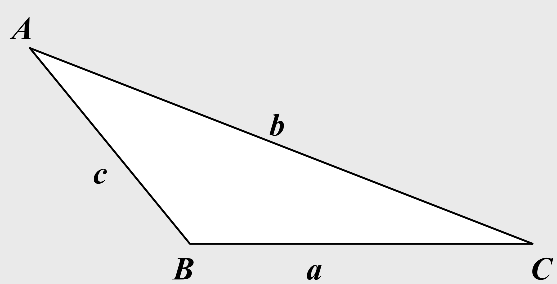
Assuming first, $\boldsymbol{a}$ is the base, we can take $\boldsymbol{b}$ **sin** $\boldsymbol{C}$ as the height, and get $\boldsymbol{S} = \dfrac{\boldsymbol{ab}\sin \boldsymbol{C}}{2}$.

Assuming next, $\boldsymbol{b}$ is the base, we can take $\boldsymbol{c}$ **sin** $\boldsymbol{A}$ as the height, and get $\boldsymbol{S} = \dfrac{\boldsymbol{bc}\sin \boldsymbol{A}}{2}$.

And assuming, $\boldsymbol{c}$ is the base, we can take $\boldsymbol{a}$ **sin** $\boldsymbol{B}$ as the height, and get: $\boldsymbol{S} = \dfrac{\boldsymbol{ca}\sin \boldsymbol{B}}{2}$.

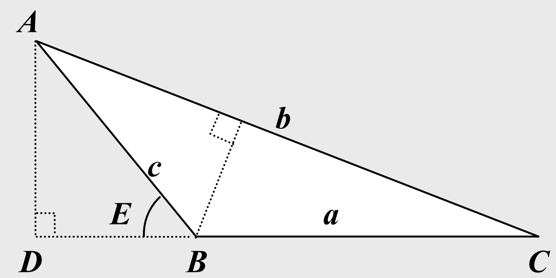
What if the triangle is obtuse as follows?

Fig. 5



We can put the triangle above the way as follows.

Fig. 6



Assuming first, $\boldsymbol{a}$ is the base, we can take $\boldsymbol{b}$ **sin** $\boldsymbol{C}$ as the height, and get $S = \dfrac{ab \sin C}{2}$.

Assuming again, $\boldsymbol{a}$ is the base, we can take as the height $\boldsymbol{c}$ **sin** $\boldsymbol{E}$, which equals however, $\boldsymbol{c}$ **sin** **(π –** $\boldsymbol{B}$**)**, which equals also, $\boldsymbol{c}$ **sin** $\boldsymbol{B}$.

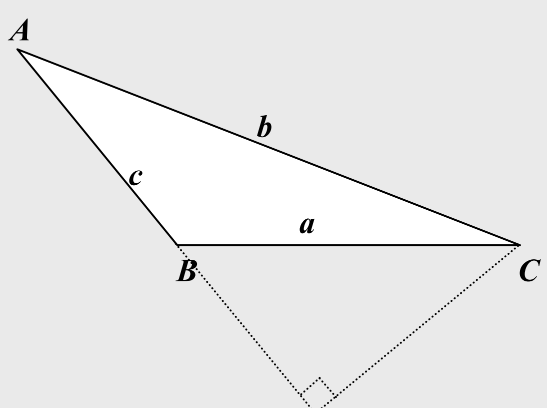
So we get: $S = \dfrac{ac \sin B}{2} = \dfrac{ca \sin B}{2}$.

And assuming $\boldsymbol{b}$ is the base, we can take $\boldsymbol{c}$ **sin** $\boldsymbol{A}$ as the height, and get $S = \dfrac{bc \sin A}{2}$.

Can we not then assume that $\boldsymbol{c}$ is the base?

Of course, we don't have to. We can put it this way, too:

Fig. 7

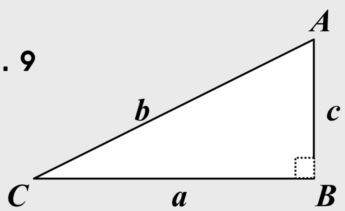


That is to say that assuming $\boldsymbol{c}$ is the base, we can take $\boldsymbol{b}$ **sin** $\boldsymbol{A}$ as the height.

Then, we get $S = \dfrac{bc \sin A}{2}$.

What if the triangle is a right triangle as follows?

Fig. 9



Assuming first, $\boldsymbol{a}$ is the base, we can take $\boldsymbol{c}$ **sin** $\boldsymbol{B}$ as the height. Why?

We know $\boldsymbol{B}$ is $\frac{\pi}{2}$, which is 90°. So we get **sin** $\boldsymbol{B} = \mathbf{1}$. Thus, we get $\boldsymbol{S} = \frac{\boldsymbol{ac}\sin \boldsymbol{B}}{2}$.

And we can put it the way as follows, too.

Assuming $\boldsymbol{c}$ is the base, we can take $\boldsymbol{a}$ **sin** $\boldsymbol{B}$ as the height, since we have **sin** $\boldsymbol{B} = \mathbf{1}$. Thus, we get $\boldsymbol{S} = \frac{\boldsymbol{ac}\sin \boldsymbol{B}}{2}$.

Assuming $\boldsymbol{a}$ is the base, we can take $\boldsymbol{c}$ **sin** $\boldsymbol{B}$ as the height.

And assuming again, $\boldsymbol{a}$ is the base, we can also. take $\boldsymbol{b}$ **sin** $\boldsymbol{C}$ as the height, which is $\boldsymbol{c}$, and thus, we get $\boldsymbol{S} = \frac{\boldsymbol{ab}\sin \boldsymbol{C}}{2}$. What then about using **sin** $\boldsymbol{A}$?

Assuming $\boldsymbol{c}$ is the base, we can take $\boldsymbol{b}$ **sin** $\boldsymbol{A}$ as the height, which is $\boldsymbol{a}$, and therefore, we get $\boldsymbol{S} = \frac{\boldsymbol{bc}\sin \boldsymbol{A}}{2}$.

So in short, given the angle between two sides known, we can use $\boldsymbol{S} = \frac{\boldsymbol{bc}\sin \boldsymbol{A}}{2}$, where $\boldsymbol{A}$ is the angle between $\boldsymbol{b}$ and $\boldsymbol{c}$, of course.

What if we are given three sides only? That is, three sides with no angle are given.

We can get the area using the three sides only.

That's because three line segments makes one triangle only. So given the three line segments, we should be able to find the area of the triangle. Some one did in fact, find it. It is called Heron's formula, named after Heron of Alexandria, and is as follows.

Assuming $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are three sides in a triangle, and $\boldsymbol{S}$ is the area of it,

we get $\boldsymbol{S} = \sqrt{\boldsymbol{t(t-a)(t-b)(t-c)}}$ where $\boldsymbol{2t = a + b + c}$, that is, $\boldsymbol{t} = \dfrac{\boldsymbol{a+b+c}}{2}$.

And another formula was found by a Chinese mathematician, too, and is as follows.

$$\boldsymbol{S = \frac{1}{2}\sqrt{a^2c^2 - \left(\frac{a^2+c^2-b^2}{2}\right)^2}},$$

which is thus, equivalent to the one above, but has been found independently from the one above.

And what's more important to us is of course, how the formula can be made.
So let's see now, how it is the case.

It is in fact, from the formula as follows. $\boldsymbol{S} = \dfrac{\boldsymbol{bc}\sin \boldsymbol{A}}{2}$.

So using the formula above, we can put $\boldsymbol{S}$ in terms of the three sides $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ only.
We don't get it though, using the formula only, of course.

Getting the formula, we use a tool called a trig identity and another tool called the cosine rule, which is $\boldsymbol{a^2 = b^2 + c^2 - 2bc \cos A}$.

And the trig identity is $\boldsymbol{\sin^2 A + \cos^2 A = 1}$ for any angle $\boldsymbol{A}$, of course.
And of course, we'll use $\boldsymbol{S} = \dfrac{\boldsymbol{bc}\sin \boldsymbol{A}}{2}$, too.

So using all the three tools above, we can put $\boldsymbol{S}$ in terms of the three sides $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$.
Now, we have these: $\boldsymbol{S} = \dfrac{\boldsymbol{bc}\sin \boldsymbol{A}}{2}$, $\boldsymbol{\sin^2 A + \cos^2 A = 1}$, and $\boldsymbol{a^2 = b^2 + c^2 - 2bc \cos A}$.
So we've got to mix and knead them now.

Thus, kneading the mixture of the three above, we get to come up with the one called Heron's formula. So what we are going to do is algebra.

After all, we get to put somehow **sin** $\boldsymbol{A}$ in terms of $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ only.

To begin with, we can see that the third tool has the three sides.
We can put **cos** $\boldsymbol{A}$ in terms of $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$.
How then can we put **sin** $\boldsymbol{A}$ in terms of $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ only?

There is a bridge between **sin** $\boldsymbol{A}$ and the three sides. What then is the bridge?

It is this tool: $\mathbf{sin}^2 A + \mathbf{cos}^2 A = 1$.

So what we do is that we solve first, for **cos** $\boldsymbol{A}$ the equation $a^2 = b^2 + c^2 - 2bc \cos A$, and then, put it into the bridge above, isn't it?

We do solve it for **cos** $\boldsymbol{A}$, but it's not a good idea to put it directly into the bridge, because the subsequent calculations will be quite messy.
How then can we avoid such a mess?

We have lots of tools to work with in math.
Among those, we have a tool called a factorization, called factoring, too.

And the tool is this: $x^2 - y^2 = (x + y)(x - y)$.

So to begin with, we get $a^2 = b^2 + c^2 - 2bc \cos A \Rightarrow \cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

Next, we get $\sin^2 A + \cos^2 A = 1 \Rightarrow \sin^2 A = 1 - \cos^2 A = (1 + \cos A)(1 - \cos A)$.

And we have this, too: $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$.

So next, we get $\sin^2 A = (1 + \cos A)(1 - \cos A) =$

$$\left(1 + \frac{b^2 + c^2 - a^2}{2bc}\right)\left(1 - \frac{b^2 + c^2 - a^2}{2bc}\right) = \left(\frac{2bc + b^2 + c^2 - a^2}{2bc}\right)\left(\frac{2bc - b^2 - c^2 + a^2}{2bc}\right),$$

which is not quite simple. We can notice this, though:

$\mathbf{2}bc + b^2 + c^2 = (b + c)^2$, and $\mathbf{2}bc - b^2 - c^2 = -(b^2 - \mathbf{2}bc + c^2) = -(b - c)^2$.

So we get $\sin^2 A = \left(\frac{(b+c)^2 - a^2}{2bc}\right)\left(\frac{a^2 - (b-c)^2}{2bc}\right) = \frac{1}{4b^2c^2}\{(b + c)^2 - a^2\}\{a^2 - (b - c)^2\}$.

Meanwhile,
$(b + c)^2 - a^2 = (b + c + a)(b + c - a)$, and $a^2 - (b - c)^2 = (a + b - c)(a - b + c)$.

So we get $\sin^2 A = \frac{1}{4b^2c^2}(a + b + c)(b + c - a)(a - b + c)(a + b - c)$. Now, what?

In the formula we are proving now, we have $2t = a + b + c$.
So it is time to do substitutions. And doing substitutions, we can get this:

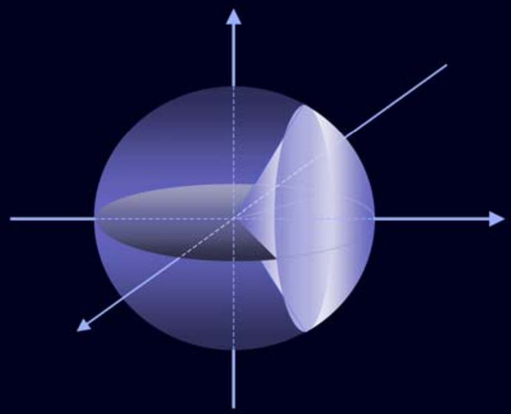
$b + c - a = 2t - 2a$, $a - b + c = 2t - 2b$, and $a + b - c = 2t - 2c$.

So we get this:
$(a + b + c)(b + c - a)\ (a - b + c)(a + b - c) = 2t(2t - 2a)(2t - 2b)(2t - 2c)$
$= 2^4 t(t - a)(t - b)(t - c)$.

Thus, we get $\sin^2 A = \frac{2^4}{4b^2c^2}\, t(t - a)(t - b)(t - c) = \frac{2^2}{b^2c^2}\, t(t - a)(t - b)(t - c)$.

So we get $\sin A = \frac{2}{bc}\sqrt{t(t-a)(t-b)(t-c)}$, because $0 < A < \pi \Rightarrow \sin A > 0$, since the angle A is an angle in a triangle.

Now, we have $S = \frac{bc \sin A}{2} = \frac{bc}{2}\sin A$. So we get $S = \frac{bc}{2}\,\frac{2}{bc}\sqrt{t(t-a)(t-b)(t-c)}$.
Thus, we get $S = \sqrt{t(t-a)(t-b)(t-c)}$ where $t = \frac{1}{2}(a + b + c)$.



삼각함수

Trigonometry

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